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Ye.O. Moldagali[1], **K.N. Ospanov**[1*]
[1]L.N. Gumilyov Eurasian National University, Kazakhstan, Astana
*e-mail: kordan.ospanov@gmail.com

ON THE CORRECTNESS OF ONE CLASS OF FOURTH-ORDER SINGULAR DIFFERENTIAL EQUATIONS

The differential equation of fourth-order with variable smooth coefficients, given on the real axis, is investigated. The coefficients of the equation can be unbounded, and the sign of the potential (the lowest coefficient) is not defined. The case is considered that the intermediate term containing the second derivative of the desired function, as an operator, does not obey the operator of the equation. It is known that fourth-order differential equations with variable coefficients are used in various problems of mathematical physics. The simplest biharmonic equation with minor terms is important for its applications in the theory of elasticity, the mechanics of elastic plates and the slow flow of viscous liquids. However, it is a very special case of a general equation with variable coefficients. Some problems of stochastic analysis, oscillation theory, biology and mathematical finance lead to the equation we are studying. In addition, fourth–order singular differential equations are often used as regularizers in the study of lower-order equations, for example, reaction-diffusion equations. In the paper, sufficient solvability conditions of the equation and the maximal regularity inequality of a strong generalized solution are obtained. The restrictions are formulated in terms of the coefficients themselves and rather weak. They do not contain conditions for any derivatives of coefficients and represent relations between the orders of growth of coefficients of different orders at infinity. The weight estimation of the norms of a solution established by us allows us to additionally apply the methods of the theory of functional spaces to the study of further properties of the solution, for example, the approximation of its elements by finite-dimensional spaces.

Key words: Fourth-order differential equation, Generalized solution, Norm, Maximal regularity, Unbounded coefficient.

Е.Ө. Молдағали[1], К.Н. Оспанов[1*]
[1]Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Қазақстан, Астана қ.
*e-mail: kordan.ospanov@gmail.com

ТӨРТІНШІ РЕТТІ СИНГУЛЯРЛЫ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕРДІҢ БІР КЛАСЫНЫҢ КОРРЕКТІЛІГІ ТУРАЛЫ

Нақты сандар осінде берілген айнымалы тегіс коэффициенттері бар төртінші ретті дифференциалдық теңдеу зерттеледі. Теңдеудің коэффициенттері шенелмеген функциялар болуы мүмкін, ал потенциалдың (кіші коэффициент) таңбасы белгісіз. Ізделінді функцияның екінші ретті туындысын қамтитын аралық мүше оператор мағынасында берілген теңдеудің операторына бағынбайтын жағдай қарастырылады. Айнымалы коэффициенттері бар төртінші ретті дифференциалдық теңдеулер математикалық физиканың әртүрлі есептерінде қолданылатыны белгілі. Кіші мүшелері бар қарапайым бигармоникалық теңдеу серпімділік теориясында, серпімді пластиналар механикасында же тұтқыр сұйықтықтардың баяу ағысында маңызды. Алайда, бұлар - айнымалы коэффициенттері бар жалпы теңдеудің мейлінше дербес жағдайлары болып табылады. Біз зерттейтін теңдеуге стохастикалық талдаудың, тербеліс теориясының, биологияның және математикалық қаржының кейбір есептері алып келеді.

Сонымен қатар, төртінші ретті сингулярлық дифференциалдық теңдеулер әдетте реакция – диффузия теңдеулері сияқты төменгі ретті теңдеулерді зерттеуде регуляризатор ретінде қолданылады. Жұмыста теңдеудің шешілуінің жеткілікті шарттары және әлді жалпыланған шешімнің максималды регулярлық бағасы алынды. Шектеулер коэффициенттердің өздерінің терминінде келтірілген және айтарлықтай әлсіз. Оларда коэффициенттердің туындылары бойынша ешбір шарттар жоқ және шексіздіктегі әртүрлі ретті коэффициенттердің өсу жылдамдығы арасындағы қатынастарды білдіреді. Шешімнің салмақты нормаларының біз дәлелдеген бағасы, шешімнің басқа да қасиеттерін, мысалы, оның ақырлы өлшемді кеңістіктердің элементтерімен жуықталуын зерттеу үшін функционалдық кеңістіктер теориясының әдістерін қосымша қолдануға мүмкіндік береді.

Түйін сөздер: төртінші ретті дифференциалдық теңдеу, жалпыланған шешім, норма, максималды регулярлық, шектеусіз коэффициент.

Е.О. Молдагали[1], К.Н. Оспанов[1*]
[1]Евразийский национальный университет имени Л.Н. Гумилева, Казахстан, г. Астана
[*]e-mail: kordan.ospanov@gmail.com

О КОРРЕКТНОСТИ ОДНОГО КЛАССА СИНГУЛЯРНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ЧЕТВЕРТОГО ПОРЯДКА

Исследуется дифференциальное уравнение четвертого порядка с переменными гладкими коэффициентами, заданное на действительной оси. Коэффициенты уравнения могут быть не ограниченными функциями, а потенциал (младший коэффициент) незнакоопределен. Рассматривается случай, когда промежуточный член, содержащий вторую производную искомой функции, как оператор не подчиняется оператору уравнения. Известно, что дифференциальные уравнения четвертого порядка с переменными коэффициентами используются в разных задачах математической физики. Простейшее бигармоническое уравнение с младшими членами важно своими приложениями в теории упругости, механике упругих пластин и медленном течении вязких жидкостей. Однако, оно является весьма частным случаем общего уравнения с переменными коэффициентами. К изучаемому нами уравнению приводят некоторые задачи стохастического анализа, теории колебании, биологии и математических финансов. Помимо этого, сингулярные дифференциальные уравнения четвертого порядка часто используются как регуляризаторы при изучений уравнений более низких порядков, например, уравнений реакций – диффузий. В работе получены достаточные условия разрешимости уравнения и оценка максимальной регулярности сильного обобщенного решения. Ограничения сформулированы в терминах самих коэффициентов и являются достаточно слабыми. Они не содержат условия на какие-либо производные от коэффициентов и представляют собой отношения между порядками ростов коэффициентов разных порядков на бесконечности. Установленная нами весовая оценка норм решения позволяет дополнительно применить методы теории функциональных пространств к изучению дальнейших свойств решения, например, приближаемость его элементами конечномерных пространств.
Ключевые слова: дифференциальное уравнение четвертого порядка, обобщенное решение, норма, оценка максимальной регулярности, неограниченный коэффициент.

1 Introduction

The work considers the fourth order equation

$$L_0 y = -y^{(4)} + r\left(x\right) y^{(2)} + s\left(x\right) y^{(1)} + q\left(x\right) y = F\left(x\right), \tag{1}$$

where $x \in R = \left(-\infty, \infty\right), F\left(x\right) \in L_2\left(R\right)$. In what follows, we will assume throughout that $r\left(x\right)$ is positive and twice continuously differentiable, $s\left(x\right)$ is once continuously differentiable, and

$q(x)$ is a continuous function. We denote by $C_0^{(4)}(R)$ the set of four times continuously differentiable and compactly supported functions. The operator $L_0 y = -y^{(4)} + r(x)\, y^{(2)} + s(x)\, y^{(1)} + q(x)\, y$ defined on $C_0^{(4)}(R)$ is closable by the $L_2(R)$ norm. Let L be this closure. A solution to equation (1) is an element $y \in D(L)$ that satisfies the equality $Ly = F$.

A wide class of mathematical physics problems lead to the differential equations of fourth-order with variable coefficients. The simplest biharmonic equation with minor terms is important for its applications in the theory of elasticity, the mechanics of elastic plates and the slow flow of viscous liquids. However, it and other equations for bending elastic - isotropic, orthotropic and anisotropic - plates of constant or variable thickness are very special cases of the general equation with variable coefficients. Theoretically, fourth–order differential equations are also often used as regularizers in studies of lower-order equations, in particular reaction-diffusion equations [1]. Boundary value problems for fourth-order differential equations have been well studied [2, 3]. In contrast, equation (1) is given in a non-compact domain, and its coefficients may not be bounded in the vicinity of an infinitely distant point. Fourth-order differential equations given in an non-compact domain describe a number of problems of oscillation theory, stochastic analysis, biology and mathematical finance [4, 5]. The solvability of such equations and the smoothness properties of the solution depend on the behavior of the coefficients and their mutual influence near infinity.

If $r = s = 0$, then (1) has the form

$$-y^{(4)} + q(x)\, y = F(x). \qquad (2)$$

The weakest conditions for the unique solvability of equation (2) were found in [6], in the $q \geq \delta > 0$ case. It is also shown there that when additional oscillation conditions of q are met, the solution satisfies inequality

$$\left\|y^{(4)}\right\|_2 + \|qy\|_2 \leq C \|F\|_2 .$$

However, if r is a non-zero function that rapidly increases at infinity, and q is not constant in sign, then the method of [6] is not applicable to equation (1). Because operators $r\frac{d^2}{dx^2}$ and $s\frac{d}{dx}$ may not obey $\frac{d^4}{dx^4} + q(x)\, E$ (E is the identity operator).

In this work, we will establish the sufficient conditions for the unique solvability of equation (1), as well as estimates for the weighted norms of the solution. Such problems for some classes of second- and high-order differential equations were considered in [7, 8, 9]. Some spectral applications of weighted estimates for differential equations given in an non-compact domain are demonstrated in [10].

We introduce the following notation for continuous functions $\rho(t)$ and $v(t) \neq 0$:

$$\alpha_{\rho,v,k}(x) = \sup_{x>0} \left(\int_0^x |\rho(t)|^q \, dt \right)^{\frac{1}{q}} \left(\int_x^\infty t^{(k-1)q'} |v(t)|^{-q'} dt \right)^{\frac{1}{q'}},$$

$$\beta_{\rho,v,k}(x) = \sup_{x<0} \left(\int_x^0 |\rho(s)|^q \, ds \right)^{\frac{1}{q}} \left(\int_{-\infty}^x s^{(k-1)q'} |v(s)|^{-q'} ds \right)^{\frac{1}{q'}},$$

$$\gamma_{\rho,v,k} = \max\left(\alpha_{\rho,v,k}(x), \beta_{\rho,v,k}(x)\right).$$

where $1 < q < \infty$, $q^{-1} + \left(q'\right)^{-1} = 1$, and k is a natural number.

Lemma 1 [11]. Let $1 < q < \infty$, and functions $\rho(t)$ and $v(t)$ satisfy the relation $\gamma_{\rho,v,k} < \infty$. Then for any $f \in C_0^{(k)}(R)$ the following inequality holds

$$\|\rho f\|_q \leq C \left\| v f^{(k)} \right\|_q , \tag{3}$$

moreover, for the smallest constant C in (3), the estimate

$$C \leq \frac{q^{\frac{1}{q}} \left(q'\right)^{\frac{1}{q'}}}{(k-1)!} \gamma_{\rho,v,k}$$

is valid, where $q^{-1} + \left(q'\right)^{-1} = 1$.

This lemma can also be deduced from the results of [12] and [13].

2 Research method

The work uses the perturbation theorem of a closed linear operator and the weighted integral inequality of Hardy type. First, a detailed study of a binomial differential equation consisting of the differential part of the equation under study is carried out. The correctness and maximum regularity of the generalized solution is proved. Then the main result is displayed by scaling.

3 On the properties of a two-term differential equation

Consider an operator $l_0 y = -y^{(4)} + r(x) y^{(2)}$ with domain $D(l_0) = C_0^{(4)}(R)$. Let l be its closure in L_2.

Lemma 2. Let

$$r \geq \delta > 0. \tag{4}$$

Then for each $y \in C_0^{(4)}(R)$ the following estimate is valid:

$$\left\| \sqrt{r} y^{(2)} \right\|_2 \leq \left\| \frac{l_0 y}{\sqrt{r}} \right\|_2 . \tag{5}$$

Proof. Let $y \in C_0^{(4)}(R)$. Then the scalar product $A = \left(l_0 y, y^{(2)}\right)$ makes sense. Since y is the function with compact support, integrating by parts we have

$$A = -\int_{-\infty}^{\infty} y^{(4)}(x)\, y^{(2)}(x)\, dx + \int_{-\infty}^{\infty} r(x) \left[y^{(2)}(x)\right]^2 dx = \left\| y^{(3)} \right\|_2^2 + \left\| \sqrt{r} y^{(2)} \right\|_2^2 .$$

On the other hand, taking into account (4) and Holder's inequality, we obtain

$$\left\| \sqrt{r} y^{(2)} \right\|_2^2 \leq A \leq \left(\int_{-\infty}^{\infty} |l_0 y|^2 \frac{1}{r(x)} dx \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} r(x) \left| y^{(2)} \right|^2 dx \right)^{\frac{1}{2}} .$$

This implies inequality (5). If condition (4) is satisfied, the right-hand side of (5) is finite. The lemma is proven.

Lemma 3. Let the function $r(x)$ satisfy conditions (4) and $\gamma_{1,\sqrt{r},2} < \infty$. Then the operator l is invertible and for each $y \in D(l)$ the following inequality

$$\left\| \sqrt{r} y^{(2)} \right\|_2 + \|y\|_2 \leq C \|ly\|_2 \tag{6}$$

holds.

Proof. Let $y \in C_0^{(2)}(R)$. According to condition $\gamma_{1,\sqrt{r},2} < \infty$, estimate (5) and Lemma 1, the following inequalities are true:

$$\|y\|_2 \leq 2\gamma_{1,\sqrt{r},2} \left\| \sqrt{r} y^{(2)} \right\|_2 \leq 2\gamma_{1,\sqrt{r},2} \left\| \frac{l_0 y}{\sqrt{r}} \right\|_2 .$$

From here, taking into account (4), we have

$$\left\| \sqrt{r} y^{(2)} \right\|_2 \leq \frac{1}{\sqrt{\delta}} \|l_0 y\|_2 \tag{7}$$

and

$$\|y\|_2 \leq \frac{2\gamma_{1,\sqrt{r},2}}{\sqrt{\delta}} \|l_0 y\|_2 . \tag{8}$$

Estimates (7) and (8) are also valid for each $y \in D(l)$. Adding them up, we get (6). The lemma is proven.

Consider the following equation

$$ly = -y^{(4)} + r(x) y^{(2)} = f(x) . \tag{9}$$

A solution to (9) is a function $y \in D(l)$ that satisfies the equality $ly = f$.

Lemma 4. Let the conditions of Lemma 3 be satisfied for $r(x)$. Then the solution y to equation (9) is unique.

Proof. Let y and z be two different solutions to equation (9). If we denote $v = y - z$, then $v \in D(l)$ and $lv = 0$. By inequality (6), $\|v\| = 0$, that is $y = z$. The lemma is proven.

Let the conditions of Lemma 3 are satisfied and $y \in D(l)$. We denote $z = y^{(2)}$ and $Lz = -z'' + r(x) z$.

Lemma 5. Let the conditions of Lemma 3 be fulfilled for $r(x)$. Then $D(L) \subseteq L_2(R)$ and the operator L is closed.

Proof. If $z \in D(L)$, then $z = y^{(2)}$, $y \in D(l)$. Taking into account condition (4), from inequality (6) we obtain $y^{(2)} \in L_2(R)$. Therefore, $D(L)$ is a subset of $L_2(R)$. It remains to show that L is closed. Since l is the closure of l_0, for $y \in D(l)$ there exists a sequence $\{y_n\}_{n=1}^{\infty} \subseteq C_0^{(4)}(R)$ and the following relations are satisfied

$$\|y_n - y\|_2 \to 0, \|l_0 y_n - ly\|_2 \to 0 \, (n \to \infty) . \tag{10}$$

Then, using inequality (6), we obtain

$$\left\| y_n^{(2)} - y^{(2)} \right\|_2 \to 0 \, (n \to \infty) . \tag{11}$$

If we denote $z_n = y_n^{(2)}$ $n \in N$, then $z_n \in C_0^{(2)}(R)$, and by (10) and (11) we have $\|z_n - z\|_2 \to 0, \|Lz_n - Lz\|_2 \to 0 \, (n \to \infty)$, which means that L is closed. The lemma is proven.

Let us rewrite equation (9) as

$$Lz=-z^{''}+rz=f. \tag{12}$$

Lemma 6. Let the function $r(x)$ be the same as in Lemma 3. Then for any $f(x)\in L_2(R)$, equation (9) has a solution.

Proof. By Lemmas 3 and 4, the operator l is invertible, and the set $R(l)$ is closed. By definition of the solution and Lemma 5, it is enough for us to show that $R(l)=L_2(R)$. According to our notation, $R(l)=R(L)$. Therefore it is enough to show that $R(L)=L_2(R)$.

Suppose that $R(L)\neq L_2(R)$. Then, according to the theory of Hilbert space, there is a non-zero element $w\in L_2(R)$, $w\notin R(L)$, such that $(w,Lz)=0$ for any $z\in D(L)$. Hence we have

$$(L^*w,z)=0, w\in D(L^*), \forall z\in D(L), \tag{13}$$

where L^* is the adjoint operator to L.

By assumption, the set $D(L)$ is dense in $L_2(R)$. Then from (13) it follows

$$L^*w=-w^{''}+rw=0.$$

By condition, $r>0$. Then, as is well known, the Sturm–Liouville operator L is self-adjoint: $D(L^*)=D(L)$ and $Lw=L^*w=0$. Therefore, from inequality (6) we have $w=0$. The lemma is proven.

4 About the second order differential operator

Consider the following operator

$$L_{0\lambda}z=-z^{''}+(r+\lambda)z, D(L_{0\lambda})=C_0^{(2)}(R),$$

where $\lambda\in R_+=[0,+\infty)$. We denote by L_λ the closure of $L_{0\lambda}$ in the space $L_2(R)$.

Definition 1. If for each $z\in D(L_\lambda)$ the following inequality

$$\|z''\|_2+\|rz\|_2+\lambda\|z\|_2\leq C\|L_\lambda z\|_2, \tag{14}$$

is true, then the operator L_λ is called separable in the space $L_2(R)$.

The following lemma is well-known [6].

Lemma 7. Let the coefficient r satisfy the conditions of Lemma 3 and condition

$$\sup_{x,\eta\in R,|x-\eta|\leq 1}\frac{r(x)}{r(\eta)}<\infty. \tag{15}$$

Then the operator L_λ is separable in $L_2(R)$.

From (14), taking into account the notation $y^{(2)}=z$, we obtain that the solution y of equation (9) satisfies the following estimate

$$\left\|y^{(4)}\right\|_2+\left\|ry^{(2)}\right\|_2+\|y\|_2\leq\|f\|_2. \tag{16}$$

Remark 1. Lemma 7 remains valid if condition (4) in Lemma 3 is replaced by $r(x) \geq 1$. Indeed, if we make the substitution $t = \sqrt{\delta}x$ and denote $(x) = y\left(\frac{1}{\sqrt{\delta}}t\right) = \widehat{y}(t)$ and $r(x) = \widehat{r}(t)$, then the operator $ly = y^{(4)} + r(x)\, y^{(2)}$ will take the following form

$$\widehat{l}\widehat{y}(t) = -\widehat{y}^{(4)}(t) + \frac{1}{\delta}\widehat{r}(t)\,\widehat{y}^{(2)}(t),$$

where $\frac{1}{\delta}\widehat{r}(t) \geq 1$.

5 Main result and its proof

Theorem 1. Suppose that for functions $r(x) \geq 1, s(x), q(x)$are satisfy the conditions $\gamma_{1,\sqrt{r},2} < \infty$, $\gamma_{q,r,2} < \infty$ and $\gamma_{s,r,1} < \infty$. Then for each $f \in L_2(R)$, equation (1) has a unique solution. If relation (15) and the following equalities

$$\lim_{a\to+\infty} \widetilde{\gamma}_{q,r,2}(a) = \lim_{a\to+\infty} \max\left(\sup_{x>0} \alpha_{q,r,2}\left(a^{-1}x\right), \sup_{x<0} \beta_{q,r,2}(a^{-1}x)\right) = 0, \tag{17}$$

and

$$\lim_{a\to+\infty} \widetilde{\gamma}_{s,r,1}(a) = \lim_{a\to+\infty} \max\left(\sup_{x>0} \alpha_{s,r,1}\left(a^{-1}x\right), \sup_{x<0} \beta_{s,r,1}(a^{-1}x)\right) = 0, \tag{18}$$

are also satisfied, then the inequality

$$\left\|y^{(4)}\right\|_2 + \left\|ry^{(2)}\right\|_2 + \left\|sy'\right\|_2 + \left\|qy\right\|_2 \leq C\left\|f\right\|_2 \tag{19}$$

is true for solution y.

Proof. In (1) we introduce a new variable $t = ax (a > 0)$ and denote

$$\widetilde{y}(t) = y\left(a^{-1}t\right), \widetilde{r}(t) = r\left(a^{-1}t\right), \widetilde{q}(t) = q\left(a^{-1}t\right), \widetilde{s}(t) = s\left(a^{-1}t\right), \widetilde{F}(t) = a^{-4}F\left(a^{-1}t\right).$$

Then since
$y'\left(a^{-1}t\right) = a\widetilde{y}'(t)$, $y''\left(a^{-1}t\right) = a^2\widetilde{y}''(t)$, $y^{(3)}\left(a^{-1}t\right) = a^3\widetilde{y}^{(3)}(t)$, $y^{(4)}\left(a^{-1}t\right) = a^4\widetilde{y}^{(4)}(t)$,
the equation (1) takes the following form:

$$\widetilde{L}_{0a}\widetilde{y} = -\widetilde{y}^{(4)}(t) + a^{-2}\widetilde{r}(t)\,\widetilde{y}^{(2)}(t) + a^{-3}\widetilde{s}(t)\,\widetilde{y}'(t) + a^{-4}\widetilde{q}(t)\,\widetilde{y}(t) = \widetilde{F}(t). \tag{20}$$

Let's denote the closure of the differential operator $l_{0a}\widetilde{y} = -\widetilde{y}^{(4)}(t) + a^{-2}\widetilde{r}(t)\,\widetilde{y}^{(2)}(t)$, $\widetilde{y} \in C_0^{(4)}(R)$, in the space $L_2(R)$ by l_a. By condition $\widetilde{r}(t) \geq 1$, then $1 \geq \widetilde{r}^{-1} \geq \widetilde{r}^{-2}$. According to estimate (3), we obtain

$$\left\|a^{-1}\sqrt{\widetilde{r}}\widetilde{y}^{(2)}\right\|_2 \leq a\left\|l_a\widetilde{y}\right\|_2. \tag{21}$$

By condition $r(x) \geq 1$, we have

$$\gamma_{1,r,2} = max\left(\sup_{x>0}\left(\int_0^x dt\right)^{\frac{1}{2}}\left(\int_x^\infty t^2 r^{-2}(t)\,dt\right)^{\frac{1}{2}}, \sup_{x<0}\left(\int_x^0 ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^x s^2 r^{-2}(s)\,ds\right)^{\frac{1}{2}}\right) \leq$$

$$\max\left(\sup_{x>0}\left(\int_0^x dt\right)^{\frac{1}{2}}\left(\int_x^\infty t^2\left(\sqrt{r(t)}\right)^{-2}dt\right)^{\frac{1}{2}},\sup_{x<0}\left(\int_x^0 ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^x s^2\left(\sqrt{r(s)}\right)^{-2}ds\right)^{\frac{1}{2}}\right)=$$
$$=\gamma_{1,\sqrt{r},2}. \tag{22}$$

Moreover, it is easy to show that

$$a^{-3}\gamma_{1,a^{-1}\sqrt{\widetilde{r}},2}=\gamma_{1,\sqrt{r},2}<\infty. \tag{23}$$

Taking into account (22) and inequalities (21) and (23), we have

$$\|\widetilde{y}\|_2\leq 2\gamma_{1,a^{-1}\sqrt{\widetilde{r}},2}\left\|a^{-1}\sqrt{\widetilde{r}}\widetilde{y}^{(2)}\right\|_2\leq 2a\gamma_{1,a^{-1}\sqrt{\widetilde{r}},2}\|l_a\widetilde{y}\|_2.$$

Consequently, the conditions of Lemma 3 are satisfied for the operator l_a. By Lemma 6, l_a continuously invertible. According to inequality (16) in Lemma 7, for any $\widetilde{y}\in D(l_a)$ the estimate

$$\left\|\widetilde{y}^{(4)}\right\|_2+\left\|a^{-2}\widetilde{r}\widetilde{y}^{(2)}\right\|_2+\|\lambda\widetilde{y}\|_2\leq C_a\|l_a\widetilde{y}\|_2. \tag{24}$$

is valid. It is easy to verify that $\gamma_{a^{-4}\widetilde{q},a^{-2}\widetilde{r},2}=\gamma_{q,r,2}<\infty$. According to Lemma 1,

$$\left\|a^{-4}\widetilde{q}\widetilde{y}\right\|_2\leq 2\gamma_{a^{-4}\widetilde{q},a^{-2}\widetilde{r},2}\left\|a^{-2}\widetilde{r}(t)\,\widetilde{y}^{(2)}\right\|_2\leq 2\gamma_{a^{-4}\widetilde{q},a^{-2}\widetilde{r},2}C_a\|l_a\widetilde{y}\|_2. \tag{25}$$

Making the replacement $x=a^{-1}\tau$, we get that $\gamma_{a^{-4}\widetilde{q},a^{-2}\widetilde{r},2}=\widetilde{\gamma}_{a^{-4}\widetilde{q},a^{-2}\widetilde{r},2}(a)$. Let's show that

$$\lim_{a\to\infty}\widetilde{\gamma}_{a^{-4}\widetilde{q},a^{-2}\widetilde{r},2}(a)=0.$$

Let $\widetilde{y}\in D(l_a)$. Since l_a is a closed operator, there exists a sequence $\{\widetilde{y}_n\}\subseteq C_0^{(4)}(R)$ such that $\|\widetilde{y}_n-\widetilde{y}\|_2\to 0$, $\|l_a\widetilde{y}_n-l_a\widetilde{y}\|_2\to 0(n\to\infty)$. We choose the sequence $\{\widetilde{y}_n\}$ so that the support of $\widetilde{y}_n$ belongs in $[-N_0,N_0]$, where $N_0\to\infty\,(n\to\infty)$.

Suppose

$$\widetilde{q}_{N_0}(t)=\begin{cases}q(t),t\in[-N_0,N_0]\\0,t\notin[-N_0,N_0]\end{cases},\widetilde{r}_{N_0}(t)=\begin{cases}r(t),t\in[-N_0,N_0]\\0,t\notin[-N_0,N_0]\end{cases}.$$

From (25) for every $\widetilde{y}\in C_0^{(4)}[-N_0,N_0]$ follows the estimate

$$\left\|a^{-4}\widetilde{q}_{N_0}\widetilde{y}\right\|_{L_2[-N_0,N_0]}\leq 2\widetilde{\gamma}_{\widetilde{q}_{N_0},\widetilde{r}_{N_0},2}(a)\left\|a^{-2}\widetilde{r}_{N_0}\widetilde{y}^{(2)}\right\|_{L_2[-N_0,N_0]}.$$

Further

$$\alpha_{\widetilde{q}_{N_0},\widetilde{r}_{N_0},2}(a)=\sup_{x>0}\left(\int_0^{a^{-1}x}\widetilde{q}^2_{N_0}(t)\,dt\right)^{\frac{1}{2}}\left(\int_{a^{-1}x}^{N_0}\frac{t^2}{\widetilde{r}^2_{N_0}(t)}dt\right)^{\frac{1}{2}}=$$

$$=\sup_{0<x\leq N_0}\left(\int_0^{a^{-1}x}q^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_{a^{-1}x}^{N_0}\frac{t^2}{r^2(t)}dt\right)^{\frac{1}{2}}\leq\alpha_{q,r,2}<\infty.$$

Therefore

$$\lim_{a\to\infty}\sup_{0<x\le N_0}\left(\int_0^{a^{-1}x}q^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_{a^{-1}x}^{N_0}\frac{t^2}{r^2(t)}dt\right)^{\frac{1}{2}}=$$

$$=\sup_{0<x\le N_0}\lim_{a\to\infty}\left(\int_0^{a^{-1}x}q^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_{a^{-1}x}^{N_0}\frac{t^2}{r^2(t)}dt\right)^{\frac{1}{2}}=0.$$

Likewise

$$\lim_{a\to\infty}\sup_{-N_0\le x<0}\left(\int_{a^{-1}x}^{0}q^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_{-N_0}^{a^{-1}x}\frac{t^2}{r^2(t)}dt\right)^{\frac{1}{2}}=0.$$

By condition (17), there is $a_0>0$ such that inequality $2C_a\gamma_{a^{-4}\widetilde{q},a^{-2}\widetilde{r},2}(a)\le\frac{1}{4}$ holds for all $a\ge a_0$. From inequality (25) we obtain

$$\left\|a^{-4}\widetilde{q}\widetilde{y}\right\|_2\le\frac{1}{4}\left\|l_a\widetilde{y}\right\|_2\quad(a\ge a_0). \tag{26}$$

Further, since $\gamma_{a^{-3}\widetilde{s},a^{-2}\widetilde{r},1}=\gamma_{s,r,1}$, we have $\gamma_{a^{-3}\widetilde{s},a^{-2}\widetilde{r},1}<\infty$. Therefore

$$\left\|a^{-3}\widetilde{s}\widetilde{y}'\right\|_2\le 2\gamma_{a^{-3}\widetilde{s},a^{-2}\widetilde{r},1}\left\|a^{-2}\widetilde{r}(t)\,\widetilde{y}^{(2)}\right\|_2\le 2C_a\gamma_{a^{-3}\widetilde{s},a^{-2}\widetilde{r},1}\left\|l_a\widetilde{y}\right\|_2. \tag{27}$$

If we assume $x=a^{-1}\tau$, then the equality $\gamma_{a^{-3}\widetilde{s},a^{-2}\widetilde{r},1}=\widetilde{\gamma}_{a^{-3}\widetilde{s},a^{-2}\widetilde{r},1}(a)$ is true.

Let's show that

$$\lim_{a\to\infty}\widetilde{\gamma}_{a^{-3}\widetilde{s},a^{-2}\widetilde{r},1}(a)=0.$$

Suppose

$$\widetilde{s}_{N_0}(t)=\begin{cases}s(t),t\in[-N_0,N_0]\\0,t\notin[-N_0,N_0]\end{cases},\widetilde{r}_{N_0}(t)=\begin{cases}r(t),t\in[-N_0,N_0]\\0,t\notin[-N_0,N_0]\end{cases}.$$

Then from (27) for each $\widetilde{y}\in C_0^{(4)}[-N_0,N_0]$ it follows that

$$\left\|a^{-3}\widetilde{s}_{N_0}\widetilde{y}'\right\|_{L_2[-N_0,N_0]}\le 2\widetilde{\gamma}_{\widetilde{s}_{N_0},\widetilde{r}_{N_0},1}(a)\left\|a^{-2}\widetilde{r}_{N_0}\widetilde{y}^{(2)}\right\|_{L_2[-N_0,N_0]}.$$

Further

$$\alpha_{\widetilde{s}_{N_0},\widetilde{r}_{N_0},1}(a)=\sup_{x>0}\left(\int_0^{a^{-1}x}\widetilde{s}^2_{N_0}(t)\,dt\right)^{\frac{1}{2}}\left(\int_{a^{-1}x}^{N_0}\frac{1}{\widetilde{r}^2_{N_0}(t)}dt\right)^{\frac{1}{2}}=$$

$$=\sup_{0<x\le N_0}\left(\int_0^{a^{-1}x}s^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_{a^{-1}x}^{N_0}\frac{1}{r^2(t)}dt\right)^{\frac{1}{2}}\le\alpha_{s,r,1}<\infty.$$

Therefore

$$\lim_{a\to\infty}\sup_{0<x\le N_0}\left(\int_0^{a^{-1}x}s^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_{a^{-1}x}^{N_0}\frac{1}{r^2(t)}dt\right)^{\frac{1}{2}}=?$$

$$= \sup_{0<x\leq N_0} \lim_{a\to\infty} \left(\int_0^{a^{-1}x} s^2(t)\,dt \right)^{\frac{1}{2}} \left(\int_{a^{-1}x}^{N_0} \frac{1}{r^2(t)} dt \right)^{\frac{1}{2}} = 0.$$

Likewise

$$\lim_{a\to\infty} \sup_{-N_0\leq x<0} \left(\int_{a^{-1}x}^{0} s^2(t)\,dt \right)^{\frac{1}{2}} \left(\int_{-N_0}^{a^{-1}x} \frac{1}{r^2(t)} dt \right)^{\frac{1}{2}} = 0.$$

By condition (18), there exists $a_1 > 0$ such that inequality $2C_a\gamma_{a^{-3}\widetilde{s},a^{-2}\widetilde{r},1}\leq\frac{1}{4}$ holds for any $a\geq a_1$. From inequality (27) we obtain

$$\left\| a^{-3}\widetilde{s}\widetilde{y}' \right\|_2 \leq \frac{1}{4} \left\| l_a\widetilde{y} \right\|_2.$$

Therefore, if we choose the parameter a so that $a\geq a_0 + a_1$ is satisfied, then

$$\left\| a^{-3}\widetilde{s}\widetilde{y}' \right\|_2 + \left\| a^{-4}\widetilde{q}\widetilde{y} \right\|_2 \leq \frac{1}{2} \left\| l_a\widetilde{y} \right\|_2. \tag{28}$$

Then, by the small perturbation theorem, the closure $\widetilde{L}_a$ of the operator $\widetilde{L}_{0a}\widetilde{y} = l_a\widetilde{y}+a^{-3}\widetilde{s}\widetilde{y}' + a^{-4}\widetilde{q}(t)\,\widetilde{y}(t)$ in $L_2(R)$ is invertible, and the inverse operator $\widetilde{L}_a^{-1}$ is defined on the entire space $L_2(R)$. This means that for each $\widetilde{F}(t)\in L_2(R)$, equation (20) has a unique solution $\widetilde{y}$.

Further

$$\left\| l_a\widetilde{y} \right\|_2 = \left\| l_a\widetilde{y} + a^{-3}\widetilde{s}\widetilde{y}' + a^{-4}\widetilde{q}\widetilde{y} - a^{-3}\widetilde{s}\widetilde{y}' - a^{-4}\widetilde{q}\widetilde{y} \right\|_2 \leq \left\| \widetilde{L}_a\widetilde{y} \right\|_2 + \frac{1}{2} \left\| l_a\widetilde{y} \right\|_2,$$

therefore

$$\left\| l_a\widetilde{y} \right\|_2 \leq 2 \left\| \widetilde{L}_a\widetilde{y} \right\|_2. \tag{29}$$

Since

$$\left\| \widetilde{L}_a\widetilde{y} \right\|_2 = \left\| l_a\widetilde{y} + a^{-3}\widetilde{s}\widetilde{y}' + a^{-4}\widetilde{q}\widetilde{y} \right\|_2 \leq \left\| l_a\widetilde{y} \right\|_2 + \left\| a^{-3}\widetilde{s}\widetilde{y}' \right\|_2 + \left\| a^{-4}\widetilde{q}\widetilde{y} \right\|_2 \leq$$

$$\leq \left\| l_a\widetilde{y} \right\|_2 + \frac{1}{2} \left\| l_a\widetilde{y} \right\|_2 = \frac{3}{2} \left\| l_a\widetilde{y} \right\|_2,$$

we have

$$\left\| l_a\widetilde{y} \right\|_2 \geq \frac{2}{3} \left\| \widetilde{L}_a\widetilde{y} \right\|_2.$$

Thus,

$$\frac{2}{3} \left\| \widetilde{L}_a\widetilde{y} \right\|_2 \leq \left\| l_a\widetilde{y} \right\|_2 \leq 2 \left\| \widetilde{L}_a\widetilde{y} \right\|_2.$$

From inequalities (24), (28) and (29) we obtain the estimate

$$\left\| \widetilde{y}^{(4)}(t) \right\|_2 \left\| \widetilde{y}^{(4)}(t) \right\|_2 + \left\| a^{-2}\widetilde{r}\widetilde{y}^{(2)} \right\|_2 + \left\| a^{-3}\widetilde{s}\widetilde{y}' \right\|_2 + \left\| a^{-4}\widetilde{q}\widetilde{y} \right\|_2 \leq 2 \left(C_a + \frac{1}{2} \right) \left\| \widetilde{L}_a\widetilde{y} \right\|_2$$

or

$$\left\|a^{-4}y^{(4)}\left(a^{-1}t\right)\right\|_2+\left\|a^{-4}r\left(a^{-1}t\right)y^{(2)}\left(a^{-1}t\right)\right\|_2+\left\|a^{-4}s\left(a^{-1}t\right)y'^{\left(a^{-1}t\right)}\right\|_2+$$

$$+\left\|a^{-4}q\left(a^{-1}t\right)y\left(a^{-1}t\right)\right\|_2\leq 2\left(C_a+\frac{1}{2}\right)\left\|a^{-4}F\left(a^{-1}t\right)\right\|_2.$$

If in this inequality we change to the variable x by replacing $t=ax$, we obtain estimate (19). The theorem has been proven.

6 Conclusion

In the paper, sufficient conditions for the solvability of a differential equation of fourth-order with unbounded smooth coefficients on the real axis of the equation and an estimate of the maximal regularity of a strong generalized solution are obtained. The restrictions are formulated in terms of the coefficients themselves and do not contain conditions on any derivatives of the coefficients. In general, they represent the relationships between growths of coefficients of different orders at infinity. The weighted estimate of the norms of the solution that we have established allows us to additionally apply the methods of the theory of function spaces to the study of further properties of the solution, for example, its approximability by elements of finite-dimensional spaces.

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Information about authors:

Yerkebulan Omirgaliuly Moldagali - PhD student of the Department of Fundamental Mathematics of L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, email: Yerka2998@gmail.com)

Kordan Nauryzkhanovich Ospanov (corresponding author) - Doctor of Physical and Mathematical Sciences, Professor of the Department of Fundamental Mathematics of L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, email: kordan.ospanov@gmail.com).

Авторлар туралы мәлімет:

Молдағали Еркебұлан Өміргалиұлы - Л.Н. Гумилев атындағы Еуразия ұлттық университетінің Іргелі математика кафедрасының докторанты (Астана қ., Қазақстан, электрондық пошта: Yerka2998@gmail.com)

Оспанов Қордан Наурызханұлы - Физика - математика ғылымдарының докторы, Л.Н. Гумилев атындағы Еуразия ұлттық университетінің Іргелі математика кафедрасының профессоры (Астана қ., Қазақстан, электрондық пошта: kordan.ospanov@gmail.com).

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