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APPLICATION OF THE POTENTIAL THEORY TO SOLVING A MIXED PROBLEM FOR A MULTIDIMENSIONAL ELLIPTIC EQUATION WITH ONE SINGULAR COEFFICIENT

Potential theory has played a paramount role in both analysis and computation for boundary value problems for elliptic equations. In the middle of the last century, a potential theory was constructed for a two-dimensional elliptic equation with one singular coefficient. In the study of potentials, the properties of the fundamental solutions of the given equation are essentially and fruitfully used. At the present time, fundamental solutions of a multidimensional elliptic equation with one degeneration line are already known. In this paper, we investigate the double- and simple-layer potentials for this kind of elliptic equations. Results from potential theory allow us to represent the solution of the boundary value problems in integral equation form. By using some properties of Gaussian hypergeometric function, we prove limiting theorems and derive integral equations concerning a densities of the double- and simple-layer potentials. The obtained results are applied to find a solution of the mixed problem for the multidimensional singular elliptic equation in the half-space.

Key words: multidimensional elliptic equations with singular coefficient; fundamental solution; Gauss hypergeometric function; potential theory; mixed problem; Green's function.

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Бір сингулярлық коэффициенті бар көп өлшемді эллиптикалық теңдеу үшін аралас есепті шешуге потенциал теориясын қолдану

Потенциал теориясы эллиптикалық теңдеулер үшін шеттік есептерді талдауда да, шешуде де маңызды рөл атқарады. Өткен ғасырдың ортасында бір сингулярлық коэффициенті бар екі өлшемді эллиптикалық теңдеудің потенциалдық теориясы құрылды. Потенциалдарды зерттеу кезінде осы теңдеудің іргелі шешімдерінің қасиеттері айтарлықтай қолданылады. Қазіргі уақытта бір сингулярлық сызығы бар көп өлшемді эллиптикалық теңдеудің іргелі шешімдері белгілі. Бұл мақалада біз эллиптикалық теңдеулердің осы түрі үшін қос және жай қабатты потенциалдарын қарастырамыз. Потенциал теориясының нәтижелері шеттік есептердің шешімін интегралдық теңдеулер түрінде ұсынуға мүмкіндік береді. Гаусстың гипергеометриялық функциясының кейбір қасиеттерін қолдана отырып, шекті теоремалар дәлелденеді және қос және жай қабатты потенциалдардың тығыздығына қатысты интегралдық теңдеулер шығарылады. Алынған нәтижелер жарты кеңістіктегі көп өлшемді сингулярлық эллиптикалық теңдеу үшін аралас есепті шешуге қолданылады.

Түйін сөздер: көп өлшемді сингулярлық эллиптикалық теңдеулер; іргелі шешім; Гаусстың гипергеометриялық функциясы; потенциал теориясы; аралас есеп; Грин функциясы

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Применение теории потенциала к решению смешанной задачи для многомерного эллиптического уравнения с одним сингулярным коэффициентом

Теория потенциала играет первостепенную роль как в анализе, так и в решении краевых задач для эллиптических уравнений. В середине прошлого века была построена теория потенциала для двумерного эллиптического уравнения с одним сингулярным коэффициентом. При изучении потенциалов существенно и плодотворно используются свойства фундаментальных решений данного уравнения. В настоящее время уже известны фундаментальные решения многомерного эллиптического уравнения с одной линией вырождения. В данной статье мы исследуем потенциалы двойного и простого слоев для такого рода эллиптических уравнений. Результаты теории потенциала позволяют представить решение краевых задач в виде интегральных уравнений. Используя некоторые свойства гипергеометрической функции Гаусса, доказываются предельные теоремы и выводятся интегральные уравнения, касающиеся плотностей потенциалов двойного и простого слоев. Полученные результаты применяются к решению смешанной задачи для многомерного сингулярного эллиптического уравнения в полупространстве.
Ключевые слова: многомерные сингулярные эллиптические уравнения; фундаментальное решение; гипергеометрическая функция Гаусса; теория потенциала; смешанная задача; функция Грина.

1 Introduction and preliminaries

Potentials play an important role in solving boundary value problems for elliptic equations since the separation of variables and the Green's functions allow one to obtain explicit solutions only for simplest domains. On one hand, the reducing of boundary value problems by means of double- and simple-layer potentials to integral equations is convenient for theoretical studies on solvability and uniqueness of solutions to boundary value problems. On the other hand, this gives an opportunity for an effective numerical solving of boundary value problems for domains of complicated shapes.

The double- and simple-layer potentials play an important role in solving boundary value problems for elliptic equations. For example, the representation of the solution of the Dirichlet problem for the Laplace equation is sought as a double-layer potential with unknown density and an application of certain property leads to a Fredholm equation of the second kind for determining the density function (see [1] and [2]).

Interest in the potential theory for the singular elliptic equation has increased significantly after Gellerstedt's papers [3,4]. In works [5] and [6], the potential theory was exposed for the simplest degenerating elliptic equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{2\alpha}{x_1}\frac{\partial u}{\partial x_1}=0,\ 0<2\alpha<1 \tag{1}$$

in the domain, bounded in the half-plane $x_1 > 0$. An exposition of the results on the potential theory for the two-dimensional singular elliptic equation (1) together with references to the original literature are to be found in the monograph by Smirnov [7], which is the standard work on the subject. This work also contains an extensive bibliography of all relevant papers up to 1966; the list of references given in the present work is largely supplementary to Smirnov's bibliography.

In the paper [8] for an elliptic equation with two singular coefficients

$$u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y = 0,\ \ 0 < 2\alpha,\ \ 2\beta < 1 \tag{2}$$

the constructed potential theory is applied to the solution of the Dirichlet problem in the domain bounded in the first quarter $\{(x, y) : x > 0,\ y > 0\}$ of the xOy-plane.

In 2020, the first works appeared in which a potential theory was constructed for a singular elliptic equation when the dimension of the equation exceeds two. In the works [9] and [10], the Dirichlet and Holmgren problems were solved, respectively, by the method of potentials for an elliptic equation with the singular coefficient

$$H_\alpha^m(u) \equiv \sum_{i=1}^{m} u_{x_i x_i} + \frac{2\alpha}{x_1}u_{x_1} = 0\ \ (0 < 2\alpha < 1,\ \ m > 2) \tag{3}$$

in the domain bounded in the subset (half-space) of the Euclidean space $R_m^+ = \{x = (x_1, ..., x_m) \in R_m : x_1 > 0\}$. This line of research adjoins the works [11] and [12].

However, only two works are devoted to the study of the mixed problem for a singular elliptic equation by the potential method: in [13] was considered a two-dimensional singular elliptic equation (1), and the work [14] is devoted to solving the mixed problem for the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x = 0,\ \ 0 < 2\alpha < 1$$

in a domain bounded in a half-space $x > 0$.

In the present work we shall give the potential theory for the multidimensional elliptic equations with one singular coefficient and apply this theory to the finding a regular solution of the mixed problem for the equation (3)

Naturally, in solving the problem posed for the equation (3), an important role is played the fundamental solution of this equation. One of the fundamental solutions of the equation (3) has the form [15]:

$$q(\xi, x) = kr^{2\alpha - m}\xi_1^{1-2\alpha}x_1^{1-2\alpha}F\left(\frac{m}{2} - \alpha, 1 - \alpha; 2 - 2\alpha; \zeta\right), \tag{4}$$

where

$$k = \frac{4^{-\alpha}\Gamma(1-\alpha)}{\pi^{m/2}\Gamma(2-2\alpha)}\Gamma\left(\frac{m}{2} - \alpha\right),\ \ \xi = (\xi_1, \xi_2, \cdots, \xi_m),\ \ \zeta = 1 - \frac{r_1^2}{r^2},$$

$$r^2 = \sum_{i=1}^{m}(\xi_i - x_i)^2, \quad r_1^2 = (\xi_1 + x_1)^2 + \sum_{i=2}^{m}(\xi_i - x_i)^2; \tag{5}$$

$F(a, b; c; z)$ is Gaussian hypergeometric function [16, Ch. 2, p. 56]; $\Gamma(\kappa)$ is the gamma function.

The fundamental solution given by (4) possesses the following potentially useful property:

$$q(\xi, x)|_{\xi_1=0} = q(\xi, x)|_{x_1=0} = 0. \tag{6}$$

Throughout this paper, it is assumed that the dimension of the space $m > 2$.

2 Green's formula

We consider the following identity:

$$x_1^{2\alpha}\left[uH_\alpha^m(v) - vH_\alpha^m(u)\right] = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left[x_1^{2\alpha}\left(v_{x_i}u - vu_{x_i}\right)\right]. \tag{7}$$

Integrating both sides of the identity (7) in a domain Ω located and bounded in the half-space $x_1 > 0$, and using the Gauss-Ostrogradsky formula, we obtain

$$\int_\Omega x_1^{2\alpha}\left[uH_\alpha^m(v) - vH_\alpha^m(u)\right]dx = \int_S (uB_{nx}^\alpha[v] - vB_{nx}^\alpha[u])dS, \tag{8}$$

where S is the boundary of Ω, n is the outer normal to the surface S and

$$B_{nx}^\alpha[\;] = x_1^{2\alpha}\sum_{i=1}^{m}\frac{\partial[\;]}{\partial x_i}cos(n, x_i) \tag{9}$$

is the conormal derivative with respect to x.

If u and v are solutions of the equation (3), then we find from the formula (8) that

$$\int_S (uB_{nx}^\alpha[v] - vB_{nx}^\alpha[u])dS = 0. \tag{10}$$

Assuming that $v = 1$ in (8) and replacing u by u^2, we obtain

$$\int_\Omega x_1^{2\alpha}\sum_{i=1}^{m}\left(\frac{\partial u}{\partial x_i}\right)^2 dx = \int_S uB_{nx}^\alpha[u]dS, \tag{11}$$

where $u(x)$ is the solution of equation (3)

The special case of (10) when $v = 1$ reduces to the following form:

$$\int_S B_{nx}^\alpha[u]dS = 0. \tag{12}$$

We note from (12) that the integral of the conormal derivative of the solution of the equation (3) along the boundary S of the domain Ω is equal to zero.

3 A double-layer potential $w(x)$

A surface Γ in the Euclidean space R_m, that satisfies the following three conditions is called the *Lyapunov surface* [17, Ch.18, p. 354]:

(i).There is a normal at any point of the surface Γ.

(ii).Let x and ξ be points of the surface Γ, n and ν are normals to the surface Γ at the points x and ξ, respectively, ϑ is an angle between these normals. There are positive constants a and κ, such as $\vartheta \leqq ar^{\kappa}$.

(iii).With respect to the surface Γ we shall assume that it approaches the hyperplane $x_1 = 0$ under the right angle.

Let Γ be a Lyapunov surface lying in the half-space $x_1 > 0$ and with the parameter equation $x_1 = x_1(t), ..., x_m = x_m(t)$, where $t := (t_1, ..., t_m) \in \Phi$, Φ is a domain of t. The boundary of the surface Γ on the hyperplane $x_1 = 0$ is denoted by γ.

Let Ω be a finite domain in R_m^+, bounded by the surface Γ and the hyperplane $x_1 = 0$. The boundary of the domain Ω on the hyperplane $x_1 = 0$ is denoted by Γ_1. It is obvious that γ is a boundary of the domain Γ_1: $\gamma = \partial\Gamma_1$.

We consider the following integral

$$w\left(x\right) = \int_{\Gamma} \mu\left(\xi\right) B_{n\xi}^{\alpha}[q\left(\xi; x\right)] d_{\xi}\Gamma, \tag{13}$$

where the density $\mu\left(x\right) \in C\left(\overline{\Gamma}\right)$, $q\left(\xi; x\right)$ is given in (4), n is the outer normal to the surface Γ and $B_{n\xi}^{\alpha}[\ ]$ is the conormal derivative with respect to ξ, defined in (9).

We call the integral (13) *a double-layer potential with density* $\mu\left(\xi\right)$. When $\mu\left(\xi\right) = 1$, we denote the double-layer potential (13) by $w_1\left(x\right)$.

We now investigate some properties of the double-layer potential $w_1\left(x\right)$.

Lemma 1 *[9] The following formula holds true*:

$$w_1(x) \equiv \int_{\Gamma} B_{n\xi}^{\alpha}[q\left(\xi; x\right)] d_{\xi}\Gamma = \begin{cases} i(x) - 1, & x \in \Omega, \\ i(x) - \dfrac{1}{2}, & x \in \Gamma, \\ i(x), & x \notin \Omega \cup \Gamma, \end{cases}$$

where

$$i(x) \equiv \int_{\Gamma_1} \left(\xi_1^{2\alpha} \frac{\partial q(x; \xi)}{\partial \xi_1} \right)\bigg|_{\xi_1 = 0} d_{\xi'}\Gamma_1 = (1 - 2\alpha) k x_1^{1-2\alpha} \int_{\Gamma_1} r_0^{2\alpha - m} d_{\xi'}\Gamma_1,$$

$$\xi' := (\xi_2, \xi_3, ..., \xi_m) \in \Gamma_1, \quad r_0^2 = x_1^2 + \sum_{i=2}^{m} \left(\xi_i - x_i\right)^2.$$

Here the domains Ω, Γ_1, γ *and the surface* Γ *are described as in this section and* $\bar{\Omega} := \Omega \cup \Gamma \cup \gamma \cup \Gamma_1$.

Theorem 1 *If* $x \in \Gamma$, *then the following inequality holds true*:

$$\left| B_{n\xi}^{\alpha}\left[q\left(\xi; x\right)\right] \right| \leqq \frac{C_1}{r_1^{2-2\alpha} r^{m-2}}, \tag{14}$$

where $m > 2$ and α is real parameter with $0 < 2\alpha < 1$ as in the equation (3), *and r and r_1 are as in (5), C_1 is a constant.*

Proof.

Using the differentiation formula for the Gaussian hypergeometric function, by virtue of (9), we obtain

$$\begin{aligned} B_{n\xi}^{\alpha}[q(\xi;x)] = & \frac{(m-2\alpha)k\xi_1^{1-2\alpha}x_1^{1-2\alpha}}{r_1^{2-2\alpha}r^{m-2}}\times \\ & \times F\left(1-\alpha-\frac{m}{2}, 1-\alpha; 2-2\alpha; 1-\frac{r^2}{r_1^2}\right) B_{n\xi}^{\alpha}\left[\ln\frac{1}{r}\right] \\ & + \frac{(1-2\alpha)k_2 x_1^{1-2\alpha}}{r_1^{2-2\alpha}r^{m-2}} F\left(1-\alpha-\frac{m}{2}, 1-\alpha; 1-2\alpha; 1-\frac{r^2}{r_1^2}\right)\cos(n,\xi_1). \end{aligned} \tag{15}$$

This immediately implies the estimate (14). Theorem 1 is thus proved.

Theorem 2 *If a surface Γ satisfies the conditions (i)–(iii), then the following inequality holds true*: $\int_\Gamma \left|B_{n\xi}^{\alpha}\left[q(\xi;x)\right]\right| d_\xi\Gamma \leqq C_2$, *where C_2 is a constant.*

Proof. Theorem 2 follows from conditions (*i*)–(*iii*) and formula (15).

Theorem 3 *The following limiting formulas hold true for a double-layer potential* (13):

$$w_i(t) = -\frac{1}{2}\mu(t) + \int_\Gamma \mu(s) K(s;t) d_s\Gamma, \tag{16}$$

$$w_e(t) = \frac{1}{2}\mu(t) + \int_\Gamma \mu(s) K(s;t) d_s\Gamma, \tag{17}$$

where

$$\mu(t) \in C\left(\overline{\Gamma}\right), s = (s_1, \cdots, s_m), t = (t_1, \cdots, t_m), K(s;t) = B_{ns}^{\alpha}\left[q(s;t)\right](s; t \in \Gamma),$$

$w_i(t)$ and $w_e(t)$ are limiting values of the double-layer potential (13) at the point $t \in \Gamma$ from the inside and the outside, respectively.

Proof. Theorem 3 follows from Lemma 1 and Theorem 2.

4 A simple-layer potential

We consider the following integral

$$v(x) = \int_\Gamma \rho(\xi) q(\xi;x) d_\xi\Gamma, \tag{18}$$

where the density $\rho(x) \in C\left(\overline{\Gamma}\right)$ and $q(\xi;x)$ is given in (4). We call the integral (18) *a simple-layer potential with density $\rho(\xi)$.*

The simple-layer potential (18) is defined throughout the half-space $x_1 > 0$ and is a continuous function when passing through the surface Γ. Obviously, a simple-layer potential

$v\,(x)$ is a regular solution of the equation (3) in any domain lying in the half-space $x_1 > 0$. It is easy to see that as the point x tends to infinity, a simple-layer potential $v(x)$ tends to zero. Indeed, we let the point x be on the hemisphere given by $x_1^2 + ... + x_m^2 = R^2, x_1 > 0$.

Then, by virtue of (4), we have

$$|v(x)| \leqq \int_\Gamma |\rho(t)||q(t;x)|d_t\Gamma \leqq C_3 R^{2-2\alpha-m} \ \ (R \geqq R_0),$$

where C_3 is a constant.

We take an arbitrary point $N(x_0)$ on the surface Γ and draw a normal at this point. Consider on this normal any point $M(x)$, not lying on the surface Γ, we find the conormal derivative of the simple-layer potential (18):

$$B^\alpha_{nx}\,[v(x)] = \int_\Gamma B^\alpha_{nx}\,[q\,(\xi;x)]\,d_t\Gamma. \tag{19}$$

The integral (19) exists also in the case when the point $M(x)$ coincides with the point N, which we mentioned above.

Theorem 4 *The following limiting formulas hold true for a simple-layer potential* (18):

$$B^\alpha_{nt}\,[v(t)]_i = \frac{1}{2}\rho\,(t) + \int_\Gamma \rho\,(s)\,K\,(s;t)\,d_s\Gamma, \tag{20}$$

$$B^\alpha_{nt}\,[v(t)]_e = -\frac{1}{2}\rho\,(t) + \int_\Gamma \rho\,(s)\,K_2\,(s;t)\,d_s\Gamma, \tag{21}$$

where $\rho \in C\left(\overline{\Gamma}\right), K\,(s;t) = B^\alpha_{nt}\,[q(s;t)]\ \ (s,t \in \Gamma)$, $B^\alpha_{nt}\,[v(t)]_i$ *and* $B^\alpha_{nt}\,[v(t)]_e$ *are limiting values of the normal derivative of simple-layer potential (18) at the point* $t \in \Gamma$ *from the inside and the outside, respectively.*

Making use of these formulas, the jump on the normal derivative of the simple-layer potential follows immediately:

$$B^\alpha_{nt}\,[v(t)]_i - B^\alpha_{nt}\,[v(t)]_e = \rho(t). \tag{22}$$

For future researches, it will be useful to note that, when the point x tends to infinity, the following inequality:

$$|B^\alpha_{nt}\,[v(t)]_i| \leqq C_4 R^{1-2\alpha-m}, \quad (R \geqq R_0)$$

is valid, where C_4 is a constant.

In exactly the same way as in the derivation of (11), it is not difficult to show that Green's formulas are applicable to the simple-layer potential (18) as follows:

$$\int_\Omega x_1^{2\alpha} \sum_{k=1}^m \left(\frac{\partial v}{\partial x_k}\right) dx = \int_S v(x) B^\alpha_{nx}\,[v(x)]_i\,dS, \tag{23}$$

$$\int_{\Omega'} x_1^{2\alpha} \sum_{k=1}^m \left(\frac{\partial v}{\partial x_k}\right) dx = -\int_S v(x) B^\alpha_{nx}\,[v(x)]_e\,dS. \tag{24}$$

Hereinafter, $\Omega' := R_m^+ \backslash \overline{\Omega}$ is a infinite domain.

5 Integral equations for denseness

Formulas (16), (17), (20) and (21) can be written as the following integral equations for densities:

$$\mu(s) - \lambda \int_{\Gamma} K(s;t)\, \mu(t)\, d_t\Gamma = f(s), \tag{25}$$

$$\rho(s) - \lambda \int_{\Gamma} K(t;s)\, \rho(t)\, d_t\Gamma = g(s), \tag{26}$$

where $\lambda = 2,\ f(s) = -2w_i(s),\ g(s) = -2B^{\alpha}_{ns}[v(s)]_e,\ \lambda = -2,\ f(s) = 2w_e(s),\ g(s) = 2B^{\alpha}_{ns}[v(s)]_i$.

Equations (25) and (26) are mutually conjugated and, by Theorem 1, Fredholm theory is applicable to them. We show that $\lambda = 2$ is not an eigenvalue of the kernel $K(s,t)$. This assertion is equivalent to the fact that the homogeneous integral equation

$$\rho(t) - 2\int_{\Gamma} K(s;t)\, \rho(s)\, d_s\Gamma = 0 \tag{27}$$

has no non-trivial solutions.

Let $\rho_0(t)$ be a continuous non-trivial solution of the equation (27). The simple-layer potential with density $\rho_0(t)$ gives us a function $v_0(x)$, which is a solution of the equation (3) in the domains Ω and Ω'. By virtue of the equation (27), the limiting values of the normal derivative of $B^{\alpha}_{ns}[v_0(s)]_e$ are zero. The formula (24) is applicable to the simple-layer potential $v_0(x)$, from which it follows that $v_0(x) = const$ in domain Ω'. At infinity, a simple layer potential is zero, and consequently $v_0(x) \equiv 0$ in Ω', and also on the surface Γ. Applying now (23), we find that $v_0(x) \equiv 0$ is valid also inside the region Ω. But then $B^{\alpha}_{ns}[v_0(s)]_i = 0$, and by virtue of formula (22) we obtain $\rho_0(t) \equiv 0$. Thus, clearly, the homogeneous equation (27) has only the trivial solution; consequently, $\lambda = 2$ is not an eigenvalue of the kernel $K(s;t)$.

Similarly, one can show that $\lambda = 2$ is not an eigenvalue of the kernel $K(s,t)$.

6 The uniqueness of the solution of the mixed problem

We apply the obtained results of potential theory to the solving the boundary value problem for the equation (3) in the domain Ω.

The mixed problem. Find a regular solution of the equation (3) in the domain Ω that is continuous in the closed domain $\overline{\Omega}$ and satisfies the following boundary conditions:

$$B^{\alpha}_{nx}[u]|_{\Gamma} = \varphi(x),\ \ x \in \Gamma,\ \ u(0,x') = \tau(x'),\ \ x' \in \overline{\Gamma}_1, \tag{28}$$

where $x' = (x_2, x_3, ..., x_m)$; $\varphi(x)$ and $\tau(x')$ are given continuous functions.

The uniqueness of the solution. We consider the identity (7). Integrating both sides of this identity along the domain Ω_ε and using the Gauss-Ostrogradsky formula, we obtain

$$\int_{\Omega_\varepsilon} x_1^{2\alpha}\left[uH^m_\alpha(v) - vH^m_\alpha(u)\right] dx = \int_{\partial\Omega_\varepsilon} (uB^{\alpha}_{nx}[v] - vB^{\alpha}_{nx}[u])dS.$$

Here Ω_ε is a sub-domain of Ω at distance ε from its boundaries Γ and Γ_1. One can easily check that the following equality holds:

$$\int_{\Omega_\varepsilon} x_1^{2\alpha} u H_\alpha^m(u) dx = \int_{\Omega_\varepsilon} x_1^{2\alpha} \sum_{i=1}^m \left(\frac{\partial u}{\partial x_i}\right)^2 dx - \int_{\Omega_\varepsilon} \sum_{i=1}^m \frac{\partial}{\partial x_i}\left(x_1^{2\alpha} u \frac{\partial u}{\partial x_i}\right) dx.$$

Application of the Gauss-Ostrogradsky formula to this equality after $\varepsilon \to 0$ yields

$$\int_\Omega x_1^{2\alpha} \sum_{i=1}^m \left(\frac{\partial u}{\partial x_i}\right)^2 dx = \int_{\Gamma_1} \tau(x')\nu(x') d_{x'}\Gamma_1 + \int_\Gamma u(x)\varphi(x) d_x\Gamma, \tag{29}$$

where $\nu(x') = \lim\limits_{x_1 \to 0} x_1^{2\alpha} \dfrac{\partial u}{\partial x_1}$, $x' \in \Gamma_1$.

If we consider the homogeneous mixed problem, then we find from (29) that

$$\int_\Omega x_1^{2\alpha} \sum_{i=1}^m \left(\frac{\partial u}{\partial x_i}\right)^2 dx = 0.$$

Hence, it follows that $u(x) = 0$ in $\overline{\Omega}$. Therefore, the following theorem holds true:

Theorem 5 *If the mixed problem has a regular solution, then it is unique.*

7 Green's function revisited

To solve this problem, we use the Green's function method. First, we construct the Green's function for solving the mixed problem for an equation in a domain bounded by an arbitrary surface and domain. In the end, we show that, thanks to the Green's function, the solution of the mixed problem in a special hemispherical domain takes a simpler form.

Definition 1 *We refer to $G(x;\xi)$ as Green's function of the mixed problem, if it satisfies the following conditions:*
Condition 1. *The function $G(x;\xi)$ is a regular solution of the equation (3) in the domain Ω, except at the point ξ, which is any fixed point of Ω.*
Condition 2. *The function $G(x;\xi)$ satisfies the boundary conditions given by*

$$B_{nx}^\alpha \left[G(x;\xi)\right]\big|_\Gamma = 0, \;\; x \in \Gamma; \lim_{x_1 \to 0} G(x;\xi) = 0, \;\; \tilde{x} \in \overline{\Gamma}_1, \tag{30}$$

Condition 3. *The function $G(x;\xi)$ can be represented as follows:*

$$G(x;\xi) = q(x;\xi) + v(x;\xi), \tag{31}$$

where $q(x;\xi)$ is a fundamental solution of the equation (3), defined in (4) and the function $v(x;\xi)$ is a regular solution of the equation (3) in the domain Ω.

The construction of the Green's function $G(x,\xi)$ reduces to finding its regular part $v(x;\xi)$ which, by virtue of (30), (31) and (6), must satisfy the following boundary conditions:

$$B_{nx}^{\alpha}\left[v\left(x;\xi\right)\right]\big|_{\Gamma} = -\,B_{nx}^{\alpha}\left[q\left(x;\xi\right)\right]\big|_{\Gamma}, \tag{32}$$

$$v(x;\xi)\big|_{x_1=0} = 0. \tag{33}$$

We look for the function $v(x;\xi)$ in the form of a double-layer potential given by

$$v\left(x;\xi\right) = \int_{\Gamma} \mu\left(\zeta;\xi\right) q\left(\zeta;x\right) d_{\zeta}\Gamma, \tag{34}$$

where $\zeta := (\zeta_1, ..., \zeta_m)$ are integration variables over the surface Γ.

By virtue of (6), a condition (33) is satisfied. Taking into account the equality (16) and the boundary condition (32), we obtain the integral equation for the density $\mu\left(s;\xi\right)$ as follows:

$$\mu\left(\eta;\xi\right) + 2\int_{\Gamma} K\left(\zeta;\eta\right)\mu\left(\zeta;\xi\right) d_{\zeta}\Gamma = -2B_{n\eta}^{\alpha}[q(\eta;\xi)], \tag{35}$$

where $\eta := (\eta_1, ..., \eta_m)$.

The right-hand side of (35) is a continuous function with respect to s (the point ξ lies inside Ω). By Theorem 1, Fredholm theory is applicable to the equation (35). In section 5 it was proved that $\lambda = -2$ is not an eigenvalue of the kernel and, consequently, the equation (35) is solvable and its continuous solution can be written in the following form:

$$\mu\left(\eta;\xi\right) = -2B_{n\eta}^{\alpha}[q(\eta;\xi)] + 4\int_{\Gamma} R\left(\zeta;\eta;-2\right) B_{n\zeta}^{\alpha}[q\left(\zeta;\xi\right)]d_{\zeta}\Gamma, \tag{36}$$

where $R(\zeta;\eta;\lambda)$ is the resolvent of kernel $K(\zeta;\eta)$, $\eta\in\Gamma$. Substituting (36) into (34), we obtain

$$v\left(x,\xi\right) = -2\int_{\Gamma} q(\zeta;\xi)B_{n\zeta}^{\alpha}\left[q\left(\zeta;x\right)\right] d_{\zeta}\Gamma + 4\int_{\Gamma}\int_{\Gamma} R\left(\zeta;\eta;-2\right) q\left(\eta;x\right) B_{n\zeta}^{\alpha}\left[q\left(\zeta;\xi\right)\right] d_{\zeta}\Gamma d_{\eta}\Gamma. \tag{37}$$

Let $\lambda(\eta;x)$ be a solution of the integral equation

$$\lambda(\eta;x) + 2\int_{\Gamma} K(\eta;\zeta)\lambda(\zeta;x)d_{\zeta}\Gamma = -2q(\eta;x). \tag{38}$$

Then

$$\lambda(\eta;x) = -2q(\eta;x) + 4\int_{\Gamma} R(\eta;\zeta;-2)q(\zeta;x)d_{\zeta}\Gamma. \tag{39}$$

Multiplying both sides of (39) by $B_{n\eta}^{\alpha}\left[q\left(\eta;\xi\right)\right]$, integrating by η over the surface Γ; then we get

$$\int_{\Gamma}\lambda(\eta;x)B_{n\eta}^{\alpha}\left[q\left(\eta;\xi\right)\right]d_{\eta}\Gamma = -2\int_{\Gamma} q(\eta;x)B_{n\eta}^{\alpha}\left[q\left(\eta;\xi\right)\right]d_{\eta}\Gamma$$
$$+4\int_{\Gamma}\int_{\Gamma} R(\eta;\zeta;-2)q(\zeta;x)B_{n\eta}^{\alpha}\left[q\left(\eta;\xi\right)\right]d_{\zeta}\Gamma d_{\eta}\Gamma,$$

or, by virtue of (37)

$$v(x;\xi) = \int_{\Gamma}\lambda(\zeta;x)B_{n\zeta}^{\alpha}\left[q\left(\eta;\xi\right)\right]d_{\zeta}\Gamma. \tag{40}$$

Lemma 2 *If points x and ξ are inside domain Ω, then Green's function $G\,(x;\xi)$ is symmetric about those points.*

From the lemma 2 and formula (31) it follows that its regular part $v(x;\xi)$ is symmetric with respect to the points x and ξ. Then the formula (40) can be rewritten as

$$v\,(x;\xi) = \int_{\Gamma} \lambda\,(\zeta;\xi)\, B^{\alpha}_{n\zeta}\,[q\,(\zeta;x)]\, d_{\zeta}\Gamma. \tag{41}$$

An equation (41) shows that $v(x;\xi)$, as a function of x, is the potential of a double layer with density $\lambda(\zeta;\xi)$. Then, according to the formula (25), we have

$$v_i(\eta;\xi) = -\frac{1}{2}\lambda(\eta;\xi) + \int_{\Gamma} K(\eta;\zeta)\lambda(\zeta;\xi)d_{\zeta}\Gamma,$$

whence by virtue of (38) and (31)

$$G(\eta;\xi) = -\lambda(\eta;\xi), \ \ \eta \in \Gamma. \tag{42}$$

Note that the Green's function $G_2(x;\xi)$ of the Dirichlet problem for the equation (3) in the work [9] is sought in the form

$$G_2(x;\xi) = q(x;\xi) + v_2(x;\xi),$$

where $q(x;\xi)$ is the fundamental solution, which is simultaneously present in the representation (31) of the Green's function of the mixed problem for the equation (3).

For a domain Ω_0 bounded by a hyperplane $x_1 = 0$ and a hemisphere given by

$$x_1^2 + x_2^2 + \cdots + x_m^2 = R^2,$$

Green's function of the Dirichlet problem has the following form:

$$G_{02}\,(x;\xi) = q\,(x;\xi) - \left(\frac{R}{\varrho}\right)^{m-2\alpha} \cdot q\left(x;\frac{R^2}{\varrho^2}\xi\right), \tag{43}$$

where

$$\xi_1^2 + \xi_2^2 + \cdots + \xi_m^2 = \varrho^2.$$

It is easy to show that its regular part $v_2(x;\xi)$ can be represented in the form

$$v_2(x;\xi) = -\int_{\Gamma} \lambda(\zeta;x)B^{\alpha}_{n\zeta}\,[v_2(\zeta;\xi)]\, d_{\zeta}\Gamma,$$

where $\lambda(\zeta;x)$ is a solution of the equation (38). Now subtracting from (31) the expression (43) and taking into account (40), (42) and (43), we get

$$G(x;\xi) = G_{02}(x;\xi) + H_2(x;\xi),$$

where

$$H_2(x;\xi) = \int_{\Gamma} \lambda(\zeta;x)B^{\alpha}_{n\zeta}\,[G_{02}(\zeta;\xi)]\, d_{\zeta}\Gamma. \tag{44}$$

8 Solving the mixed problem for equation (3).

Let $\xi := (\xi_1, ..., \xi_m)$ be a point inside the domain Ω. Consider the domain $\Omega_{\varepsilon,\delta} \subset \Omega$ bounded by the surface Γ_ε which is parallel to the surface Γ, and the domain $\Gamma_{1\varepsilon}$ lying on the hyperplane $x_1 = \delta > \varepsilon$. We choose ε and δ so small that the point x_0 is inside $\Omega_{\varepsilon,\delta}$. We cut out from the domain $\Omega_{\varepsilon,\delta}$ a ball of small radius ρ with center at the point x_0 and the remainder part of $\Omega_{\varepsilon,\delta}$ denote by $\Omega^{\rho}_{\varepsilon,\delta}$, in which the Green's function $G(x;\xi)$ is a regular solution of (3).

Let $u(x)$ be a regular solution of the equation (3) in the domain Ω that satisfies the boundary conditions (28). Applying the formula (10), we obtain

$$\begin{aligned}\int_{\Gamma_\varepsilon} \left(GB^{\alpha}_{nx}[u] - uB^{\alpha}_{nx}[G]\right) d_x\Gamma_\varepsilon + \int_{\Gamma_{1\delta}} \left(uB^{\alpha}_{nx}[G] - GB^{\alpha}_{nx}[u]\right)\Big|_{x=\delta} d_{x'}\Gamma_{1\delta} \\ = \int_{C_\rho} \left(GB^{\alpha}_{nx}[u] - uB^{\alpha}_{nx}[G]\right) d_x C_\rho.\end{aligned}$$

Passing to the limit as $\rho \to 0$ and then as $\varepsilon \to 0$ and $\delta \to 0$, we obtain

Main result.

Theorem 6 *Let the Lyapunov surface Γ approach Γ_1 at right angles. Then the function*

$$u(\xi) = \int_{\Gamma_1} \tau(\tilde{x})\, x_1^{2\alpha} \frac{\partial G(x;\xi)}{\partial x_1}\Big|_{x_1=0} d_{\tilde{x}}\Gamma_1 + \int_{\Gamma} \varphi(x) G(x;\xi) d_x\Gamma \tag{45}$$

is a solution to the mixed problem for the equation (3) in the domain Ω, where $\varphi(x) \in \Gamma$ and $\tau(\tilde{x}) \in \overline{\Gamma}_1$ are a given functions with conditions (28).

Proof. The proof of the theorem 6 is similar to the proof of the theorems in [9] and [10].

We remark that the solution (45) of the mixed problem is more convenient for further investigations. The resulting explicit integral representation (45) plays an important role in the study of problems for equation of the mixed type (that is, elliptic-hyperbolic or elliptic-parabolic types): it makes it easy to derive the basic functional relationship between the traces of the sought solution and of its derivative on the line of degeneration from the elliptic part of the mixed domain.

In the case of a hemisphere Ω_0 the solution (45) assumes a simpler form:

$$\begin{aligned} u(\xi) =& (1-2\alpha)k_2\xi_1^{1-2\alpha} \int_{\Gamma_1} \tau(x') \left\{ \left[\xi_1^2 + \sum_{i=2}^{m}(x_i - \xi_i)^2\right]^{\alpha-\frac{m}{2}} \right. \\ & \left. - \left[\sum_{i=2}^{m}\left(R - \frac{x_i\xi_i}{R}\right)^2 + \frac{1}{R^2}\sum_{i=2}^{m} x_i^2 \sum_{j=1,j\neq i}^{m} \xi_j^2 - (m-2)R^2\right]^{\alpha-\frac{m}{2}} \right\} d_{x'}\Gamma_1 \\ & + \int_{\Gamma_1} \tau(x') \left[H_2(x;\xi)\right]\Big|_{x_1=0} d_{x'}\Gamma_1 + \int_{\Gamma} \varphi(x) G(x;\xi) d_x\Gamma, \end{aligned}$$

where $H_2(x;\xi)$ is defined in (44).

9 Conclusion

In constructing the potential theory, fundamental solutions of the given elliptic equation are played an important role. In a recent work [15], the author managed to find all fundamental solutions of the multidimensional elliptic equation with several singular coefficients

$$\sum_{i=1}^{m} u_{x_i x_i} + \sum_{k=1}^{n} \frac{2\alpha_i}{x_k} u_{x_k} = 0$$

in explicit form, where m is a dimension of the elliptic equation, n is a number of the singular coefficients ($m \geq 2,\ 1 \leq n \leq m$); α_k are real numbers and $0 < 2\alpha_k < 1,\ k = \overline{1, n}$.

In addition, particular solutions of the more general equation

$$\sum_{j=1}^{k} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{k} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = \sum_{j=k+1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=k+1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j}, k = \overline{1, n-1}$$

are currently known [18].

It should be noted that the work [19] is also devoted to the hypergeometric expansions of solutions of the degenerating model parabolic equations of the third order

$$x^n y^m u_t - t^k y^m u_{xxx} - t^k x^n u_{yyy} = 0,\ m, n, k = const > 0. \tag{46}$$

The equation (46) has 9 linearly independent particular solutions which are expressed by, so called, a Kampé de Fériet's double hypergeometric series [20, p. 27]. In a recent work [21], expansions of the Kampé de Fériet's functions are established.

Despite the fact that fundamental solutions, particular solutions and expansion formulas are known even for a more general equation, the construction of potential theory for a singular elliptic equation was limited to the equations with one singular coefficient (3) and one paper [8] has been devoted when the number of singular coefficients exceeds one, but a dimension of the equation is equal to 2 (see Eq. (2) in Section 1).

For the future, it would be interesting to apply potential theory to the solution of boundary value problems for elliptic equations with two or more singular coefficients, for example, in the beginning for the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x} u_x + \frac{2\beta}{y} u_y = 0,\ 0 < 2\alpha, 2\beta < 1,\ x > 0, y > 0.$$

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