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# ON CONDITIONS FOR THE FINITENESS OF THE SPECTRUM OF A SECOND ORDER DIFFERENTIAL OPERATOR WITH INTEGRAL BOUNDARY CONDITIONS

In this paper we study the question of the finiteness of the spectrum of a second-order differential operator generated in the space $H = L_2(0,1)$ by integral boundary conditions. We have shown that the spectrum of such an operator is either infinite or empty. Previously, this result was known only in the case of two- or three-point boundary conditions. Next, we obtained a necessary and sufficient condition for the spectrum to be empty in terms of a system of two equations for the potential $q$ and the functions $\sigma_1$ and $\sigma_2$ that define the integral boundary conditions. If we assume that $\sigma_2$ belongs to the space $W_2^1[0,1]$, then the first equation is solvable with respect to $\sigma_1$ within some neighborhood of the zero $U$ of the space $H^3$. This allows us to resolve the indicated equation for $\sigma_1$ within a certain neighborhood of the zero $U$ of the space $H^3$. This scheme is not applicable to the second equation, but it is possible to identify a fairly wide class of functions $(q, \sigma_1\sigma_2) \in U$, on which this equation turns into an identity. The final part of the article is devoted to exploring the question: can the operator in question have an empty spectrum if the functions $\sigma_1, \sigma_2$ are not necessarily close to zero (in the space $H^2$)? We have constructed a class of functions $\sigma_1$ and $\sigma_2$ (in the form of polynomials with arbitrarily large norms) such that the spectrum of the corresponding operator is empty. The operating technique can be extended to the case when $H = L_2(\gamma)$, where $\gamma$ is a curve with a limited slope (that is, the absolute value of the slope of any chord of this curve does not exceed a certain number).

**Key words**: differential operator with integral boundary conditions, finiteness or infinity of the spectrum, transformation operators, Volterra boundary value problems.

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**Интегралдық шекаралық шартты екінші ретті дифференциалдық оператордың спектрінің шектілік шарттары туралы**

Жұмыста $H = L_2(0,1)$ кеңістігінде интегралдық шекаралық шарты арқылы құрылған екінші ретті дифференциалдық оператор спектрінің шектілігі туралы мәселе зерттеледі. Мұндай оператордың спектрі не шексіз, не бос екені көрсетілген. Бұрын бұл нәтиже екі немесе үш нүктелік шекаралық шарттар жағдайында ғана белгілі болды. Спектрдің бос болуының қажетті және жеткілікті шарты $q$ потенциалы және интегралдық шекаралық шарттарды анықтайтын $\sigma_1$ және $\sigma_2$ функциялары үшін екі теңдеу жүйесі тұрғысынан алынады. Егер $\sigma_2$ $W_2^1[0,1]$ кеңістігіне жатады деп болжасақ, онда бірінші теңдеу $H^3$ кеңістігінің $U$ нөлінің кейбір маңайында $\sigma_1$ қатысты шешіледі. Бұл схема екінші теңдеу үшін қолданылмайды, бірақ бұл теңдеу сәйкестікке айналатын $(q, \sigma_1\sigma_2) \in U$ функциялардың жеткілікті кең класын анықтауға болады. Мақаланың соңғы бөлімі келесі мәселені зерттеуге арналған: $\sigma_1, \sigma_2$ функциялары міндетті түрде нөлге жақын болмаса ($H^2$ кеңістігінде) қарастырылып отырған оператордың бос спектрі болуы мүмкін бе? Біз $\sigma_1$ және $\sigma_2$ функцияларының класын (еркін үлкен нормалары бар көпмүшелер түрінде) сәйкес оператордың спектрі бос болатындай етіп құрдық. Бұл жұмыстың әдісін $H = L_2(\gamma)$ болған жағдайға дейін кеңейтуге болады, мұнда шектелген көлбеуі бар қисық (яғни кез келген хорданың бұрыштық коэффициентінің модулі белгілі бір саннан аспайды).

**Түйін сөздер**: интегралдық шекаралық шарттары бар дифференциалдық оператор, спектрдің шектілігі немесе шексіздігі, түрлендіру операторлары, Вольтерралы шекаралық есептер.

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**Об условиях конечности спектра дифференциального оператора второго порядка с интегральными краевыми условиями**

В работе изучается вопрос о конечности спектра дифференциального оператора второго порядка, порожденного в пространстве $H = L_2(0,1)$ интегральными краевыми условиями. Показано, что спектр такого оператора либо бесконечен, либо пуст. Ранее этот результат был известен только в случае двух- или трехточечных краевых условий. Получено необходимое и достаточное условие пустоты спектра в терминах системы из двух уравнений для потенциала $q$ и функций $\sigma_1$ и $\sigma_2$, задающих интегральные краевые условия. Если предположить, что $\sigma_2$ принадлежит пространству $W_2^1[0,1]$, то первое уравнение разрешимо относительно $\sigma_1$ в пределах некоторой окрестности нуля $U$ пространства $H^3$. Ко второму уравнению такая схема неприменима, однако удается выделить достаточно широкий класс функций $(q, \sigma_1\sigma_2) \in U$, на котором это уравнение превращается в тождество. Заключительная часть статьи посвящена исследованию вопроса: может ли рассматриваемый оператор иметь пустой спектр, если функции $\sigma_1, \sigma_2$ не обязательно близки к нулю (в пространстве $H^2$)? Нами построен класс функций $\sigma_1$ и $\sigma_2$ (в виде многочленов со сколь угодно большими нормами), таких, что спектр соответствующего оператора пуст. Методика работы может быть распространена на случай, когда $H = L_2(\gamma)$, где $\gamma$ — кривая с ограниченным наклоном (то есть угловой коэффициент любой хорды по модулю не превосходит некоторого числа).
**Ключевые слова**: дифференциальный оператор с интегральными краевыми условиями, конечность или бесконечность спектра, операторы преобразования, вольтерровы краевые задачи.

## 1 Introduction

One of the main questions in the theory of inverse spectral problems is the description of spectral data necessary and sufficient (if possible) for the reconstruction of a particular spectral characteristic of the operator [1]. In this regard, not only quantitative (localization, asymptotics, etc.) information about the spectrum is important, but also qualitative information – its finiteness or infinity. In the proposed work, we study the question of the finiteness of the spectrum of the operator $L_V$, which is defined as follows. Let $L$ be an operator acting in $L_2(0,1)$ according to the rule

$$Ly = l(y) := -y'' + qy,$$
$$D(L) = D := \{y \in L_2(0,1) : \ y, y' \in AC([0,1]), l(y) \in L_2(0,1)\}.$$

Then $L_V$ is a restriction of $L$ defined by the conditions

$$V_j(y) := y^{(j-1)}(0) + \langle l(y), \sigma_j \rangle = 0, \quad j = 0, 1. \tag{1}$$

Here $q \in L^1(0,1)$ and

$$\langle f, g \rangle = \int_0^1 fg dz,$$

$\sigma_1, \sigma_2$ — some functions from $L_2(0,1)$.

Operators of type $L_V$ arise in the theory of turbulence [2] and in the theory of Markov processes [3, 4]. Various spectral properties of differential operators (of arbitrary order) with nonlocal boundary conditions of the form (1) were studied by M. Picone [5, 6], Ya. D. Tamarkin [7], A. M. Krall [8], A. A. Shkalikov [9], V. A. Ilyin and E. I. Moiseev [10, 11], B. E. Kanguzhin [12–14] and many others. A more detailed bibliography can be found in the reviews [15, 16].

As is known (see [16, Ch. III, § 1, Lemma 6] and [17, Theorem 2]), the integral conditions give a complete description of all restrictions of the operator $L$ that have a non-empty resolvent set. Moreover, the operator $L_V^{-1}$ is compact, $\sigma(L_V) = \{\lambda_k^2\}_{k=1}^N$ $(N \le \infty)$, where $\{\lambda_k\}$ are the zeros of the entire function

$$\Delta(\lambda) = \begin{vmatrix} V_1(c) & V_1(s) \\ V_2(c) & V_2(s) \end{vmatrix}, \tag{2}$$

$s, c$ are the solutions of the equation

$$l(y) = \lambda^2 y, \ \ z \in [0,1], \tag{3}$$

satisfying the conditions $s(0,\lambda) = c'(0,\lambda) = 0, \quad s'(0,\lambda) = c(0,\lambda) = 1$ (here and throughout below $\varphi'(z,\lambda)$ is the derivative with respect to $z$). Consequently, the spectrum of the operator $L_V$ consists of a finite or countable number of eigenvalues, each of which has a finite (algebraic) multiplicity.

Let us pose the question: under what conditions on the functions $\sigma_1, \sigma_2$ and $q$ is the spectrum of the operator $L_V$ finite?

The following statements are true.

**Theorem 1** *The spectrum of the operator $L_V$ is either infinite or empty.*

Since $L_V^{-1}$ is Volterra for $\sigma_1 = \sigma_2 = 0$, the spectrum of $L_V$ is empty. The question arises: can the spectrum of $L_V$ be empty for nonzero functions $\sigma_1 = \sigma_2$? If so, how rich is the set of such functions?

We have obtained a criterion for the emptiness of the spectrum of $L_V$, which allows us to distinguish a fairly wide class of functions $\sigma_1, \sigma_2$ for which the spectrum of $L_V$ is empty.

Let us start with the case $q = 0$. Introduce the function

$$A_0[\sigma_1, \sigma_2](x) = \int_x^1 [\sigma_1(t)\sigma_2(t-x) - \sigma_2(t)\sigma_1(t-x)]\, dt, \quad x \in [0,1]. \tag{4}$$

The function $A_0$ is continuous and $\|A_0[\sigma_1, \sigma_2]\|_{C[0,1]} \le 2\|\sigma_1\|\|\sigma_2\|$.

Here and throughout, $\|\cdot\|$ is the norm in the space $L_2(0,1)$.

**Theorem 2** *Let $q = 0$. Then the spectrum of the operator $L_V$ is empty if and only if the functions $\sigma_1, \sigma_2$ satisfy the equation*

$$A_0[\sigma_1, \sigma_2](x) + \int_x^1 [\sigma_1 + (t-x)\sigma_2]\, dt = 0, \quad x \in [0,1]. \tag{5}$$

Let $\delta > 0$ and

$$B_\delta = \left\{ f \in W_2^1[0,1] : \ \|f\|_{W_2^1(0,1)} := \left( \int_0^1 \left( |f|^2 + |f'|^2 \right) dt \right)^{1/2} < \delta, \ f(1) = 0 \right\}.$$

**Corollary 1** *There is a unique function*

$$F : \ B_{1/3} \longrightarrow L_2(0,1),$$

*such that*

(a) *$F$ is continuous on $B_{1/3}$,*

(b) *for any pair $(\sigma_1, \sigma_2) \in L_2(0,1) \times B_{1/3}$ the spectrum of the corresponding operator $L_V$ is empty if and only if $\sigma_1 = F(\sigma_2)$.*

Let us now consider the general case $q \neq 0$. Let $\mathcal{K}(\cdot,\cdot)$ be the kernel of the transformation operator for the solution $e(x,\lambda)$ of the equation (3) with initial conditions $e(0,\lambda) = 1, \ e'(0,\lambda) = i\lambda$ [18, Ch. 1, § 2]:

$$e(x,\lambda) = e^{i\lambda x} + \int_{-x}^{x} \mathcal{K}(x,t) e(t,\lambda) dt.$$

The solutions $s$ and $c$ of the equation (3) introduced above are represented in the form

$$s(x,\lambda) = \frac{\sin \lambda x}{\lambda} + \int_0^x \mathcal{K}_\infty(x,t) \frac{\sin \lambda t}{\lambda} dt, \tag{6}$$

$$c(x,\lambda) = \cos \lambda x + \int_0^x \mathcal{K}_0(x,t) \cos \lambda t dt, \tag{7}$$

$$\mathcal{K}_\infty(x,t) = \mathcal{K}(x,t) - \mathcal{K}(x,-t), \quad \mathcal{K}_0(x,t) = \mathcal{K}(x,t) + \mathcal{K}(x,-t).$$

Let $F_1 = K_+ + K_-, \ F_2 = K_+ - K_-$, where $K_\pm$ are operators acting in $L_2(0,1)$ according to the formulas

$$\left[ K_\pm f \right](x) = \int_x^1 \mathcal{K}(t, \pm x) f(t) dt. \tag{8}$$

Let's introduce the functions

$$B[f,g](x) = \frac{1}{2} \left( \int_x^1 \left[ g(t) f(t-x) - f(t) g(t-x) \right] dt + \int_0^x f(t) g(x-t) dt \right), \tag{9}$$

$$A_1[\sigma_1,\sigma_2,q](x) = B[(I+F_1)\sigma_1, (I+F_2)\sigma_2](x) - B[(I+F_1)\sigma_2, (I+F_2)\sigma_1](x),$$

$$A_2[\sigma_1,\sigma_2,q](x) = \int_x^1 \left[ (I+F_1)\sigma_1 + (t-x)(I+F_2)\sigma_2 \right] dt, \quad 0 \le x \le 1,$$

$$A_3[\sigma_1,\sigma_2,q](x) = \int_{x-1}^1 \left[ (K_-\sigma_1)(t)(1+K_+)\sigma_2(x-t) - \right.$$
$$\left. - (K_-\sigma_2)(t)(1+K_+)\sigma_1(x-t) \right] dt, \quad 1 \le x \le 2. \tag{10}$$

**Theorem 3** *The spectrum of the operator $L_V$ is empty if and only if the functions $q, \sigma_1, \sigma_2$ satisfy the system of equations on $[0,1]$*

$$A_1[\sigma_1,\sigma_2,q](x) - A_2[\sigma_1,\sigma_2,q](x) = 0, \quad 0 \le x \le 1, \tag{11}$$

$$A_3[\sigma_1,\sigma_2,q](x) = 0, \quad 1 \le x \le 2. \tag{12}$$

**Remark 1** *If the functions $\sigma_1, \sigma_2$ belong to the kernel of the operator $L$, then the conditions* (1) *take the form of classical (two-point) boundary conditions:*

$$a_{i0}y(0) + a_{i1}y'(0) + b_{i0}y(1) + b_{i1}y'(1) = 0, \quad i = 1, 2.$$

*In the paper [19] it is shown that in this situation the spectrum of the operator $L_V$ is empty if and only if*

*a)* $a_{11} = b_{11} = a_{20} = b_{20} = 0, \quad b_{10} = -\alpha a_{10}, \ b_{21} = \alpha a_{21},$ *where* $\alpha \neq 0, \infty, \pm 1;$
*b)* $q(x) = q(1-x), \quad x \in [0, 1/2].$

*If $\sigma_1, \sigma_2 \in W_2^1([0,1]\backslash\{1/2\})$ and $l(\sigma_k) = 0$ $(k = 1, 2)$, then the conditions* (1) *are reduced to three-point conditions:*

$$a_{i0}y(0) + a_{i1}y'(0) + b_{i0}y(1) + b_{i1}y'(1) + c_{i0}y(1/2) + c_{i1}y'(1/2) = 0, \quad i = 1, 2.$$

*It was shown in [20] that the spectrum of the operator $L_V$ is empty if*

*c)* $a_{11} = b_{11} = c_{11} = a_{20} = b_{20} = c_{20} = 0, \quad b_{10} = -k_1 a_{10},$
$c_{10} = k_2(1-k_1)a_{10}, \ b_{21} = k_1 a_{21}, c_{21} = -k_2(1-k_1)a_{21},$ *where* $k_1, k_2 \neq 0, \infty;$
*d)* $q(x) = q(1-x)$ *on* $[0, 1/2]$ *and* $q(x) = q(1/2 - x)$ *on* $[0, 1/4].$

Despite the cumbersome form of the system (11), (12), as in the case of $q = 0$, it is possible to locally resolve with respect to $\sigma_1$ within a certain neighborhood of zero $U$ of the space $L_2(0,1) \times W_2^1(0,1) \times L_2(0,1)$.

Let $\Pi_N = \{f \in L_2(0,1) : \ \|f\| \le N\}$ and $\Pi(M,\delta)$ $(M, \delta > 0)$ is the set of pairs $(\sigma_2, q) \in W_2^1(0,1) \times L_2(0,1)$ satisfying the conditions:

(i) $\|q\| \le M, \|\sigma_2\|_{W_2^1(0,1)} \le \delta, \ \sigma_2(1) = 0,$

(ii) There exists a constant $0 < a < 1$ (its own for each pair $(\sigma_2, q)$) such that $\mathrm{supp}(q) \subset [0,a], \ \mathrm{supp}(\sigma_2) \subset [a, 1]$.

**Corollary 2** *There exist $\Pi := \Pi_N \times \Pi_{M,\delta}$ and a unique function*

$$F: \ \Pi_{M,\delta} \longrightarrow \Pi_N,$$

*such that*

(a) *$F$ is continuous on $\Pi_{M,\delta}$,*

(b) *for any triple $(\sigma_1, \sigma_2, q) \in \Pi$ the spectrum of the corresponding operator $L_V$ is empty if and only if $\sigma_1 = F(\sigma_2, q)$.*

As noted above, for $\sigma_1 = \sigma_2 = 0$ the spectrum of the operator $L_V$ is empty. According to Corollaries 1 and 2 the spectrum of $L_V$ is empty for all $(\sigma_1, \sigma_2) \in L_2(0,1) \times W_2^1[0,1]$ sufficiently close to the point $(0,0)$ of the space $L_2(0,1) \times W_2^1[0,1]$. It does not follow that for other $(\sigma_1, \sigma_2)$ the spectrum of $L_V$ is not empty. We present one class of functions $(\sigma_1, \sigma_2)$ (in the form of polynomials) for which the spectrum of $L_V$ is empty. To reduce the number of the calculations, we restrict ourselves to the case $q = 0$.

**Theorem 4** *Let $q = 0, \sigma_2 = \mu(1-x),\ \mu \neq 0, 1,$ and*

$$\sigma_1(x) = -\frac{2k^2}{1-k^2}\int_0^1 sg(s)ds \cdot (1-x) - \int_x^1 g(s)ds, \quad k = \frac{\mu}{1-\mu}, \tag{13}$$

$$g(x) = k^2(1 - x + x^2/2) - k(1-x). \tag{14}$$

*Then the spectrum of the operator $L_V$ is empty.*

## 2 Proofs of Theorems 1, 2 and Corollary 1

### 2.1 Proof of Theorem 1

According to (1) and (2)

$$\Delta(\lambda) = \lambda^4\Delta_1(\lambda) + \lambda^2\left(\langle c, \sigma_1\rangle + \langle s, \sigma_2\rangle\right) + 1, \tag{15}$$

$$\Delta_1(\lambda) = \langle c, \sigma_1\rangle\langle s, \sigma_2\rangle - \langle s, \sigma_1\rangle\langle c, \sigma_2\rangle. \tag{16}$$

From relations (6) and (7) it is clear that $\Delta$ is an entire function of exponential type. Therefore, if the spectrum of the operator $L_V$ is finite, then

$$\Delta(\lambda) = e^{k\lambda}P(\lambda),$$

where $k = \text{const},\ P$ is a polynomial. Since the function $\Delta$ is even, then $k = 0$ and $P(\lambda) = p_0\lambda^{2m} + p_1\lambda^{2m-2} + \cdots + p_{m-2}\lambda^2 + p_m$ ($p_k = \text{const}$). According to equalities (15), (16) and (6), (7) we have

$$\Delta(\lambda)(1 + |\lambda|^3)^{-1} \in L_1(\mathbb{R}), \tag{17}$$

therefore, $m = 0$, so that

$$\Delta(\lambda) \equiv 1. \tag{18}$$

**Remark 2** *One of the key points in the proof of Theorem 1 is the estimate* (17) *(in fact, «half» of the estimate on $+\infty$ or $-\infty$ is sufficient). If in the definition of the operator $L_V$ the segment is replaced by a piecewise smooth curve $\gamma$, then the estimate* (17) *may be incorrect even in the case when* (1) *are the Dirichlet conditions [21, 22]. However, if $\gamma$ has a bounded slope (that is, the angular coefficients of all chords $\gamma$ do not exceed some number in absolute value), then there is a ray on which the estimate of type* (17) *will hold. Therefore, the assertion of Theorem 1 remains valid for the operator $L_V$ (with the conditions* (1)*) on any curve with a bounded slope.*

*In the works [23, 24], using a similar technique, the infinity of the spectrum of the Sturm–Liouville operator on the half-axis with a complex potential is proved.*

### 2.2 Proof of Theorem 2

Since for $q = 0$ $s(x,\lambda) = \sin\lambda x/\lambda$, $\quad c(x,\lambda) = \cos\lambda x$, then the equation (18) can be written as

$$\lambda^2\Phi_1(\lambda) + \lambda\Phi_2(\lambda) + \Phi_3(\lambda) = 0 \tag{19}$$

where

$$\Phi_1(\lambda) = \int_0^1 \sigma_1(x)\left(\int_0^1 \sigma_2(t)\sin\lambda(t-x)dt\right)dx,$$
$$\Phi_2(\lambda) = \int_0^1 \sigma_1(x)\cos\lambda x dx, \quad \Phi_3(\lambda) = \int_0^1 \sigma_2(x)\sin\lambda x dx.$$

Assuming

$$f_1(x) = \int_x^1 \sigma_1(t)dt, \quad f_2(x) = \int_x^1 (t-x)\sigma_2(t)dt,$$

we will have

$$\Phi_1(\lambda) = -\int_0^1 A_0[\sigma_1,\sigma_2](x)\sin\lambda x dx,$$
$$\Phi_2(\lambda) = f_1(0) - \lambda\int_0^1 f_1(x)\sin\lambda x dx, \quad \Phi_3(\lambda) = \lambda f_2(0) - \lambda^2\int_0^1 f_2(x)\sin\lambda x dx,$$

where the function $A_0[\sigma_1,\sigma_2]$ is defined by the formula (4). Then the equation (19) takes the form

$$\lambda\int_0^1 \{A_0[\sigma_1,\sigma_2](x) + f_1(x) + f_2(x)\}\sin\lambda x dx - f_1(0) - f_2(0) = 0.$$

It is easy to verify that this equation is equivalent to the equation (5).

### 2.3 Proof of the Corollary 1

Let $\sigma_2 \in B_{1/3}$. Then the left-hand side of the equation (5) is differentiable almost everywhere on $[0,1]$ and $|\sigma_2(0)| < 1/3$. Therefore, in the domain $L_2(0,1)\times B_{1/3}$, the equation (5) is equivalent to the equation

$$\sigma_1 + A\sigma_1 = g, \tag{20}$$

where

$$g(x) = -\int_x^1 \sigma_2(t)dt/\left(1+\sigma_2(0)\right),$$

$A$ is an operator acting in $L_2(0,1)$ according to the rule

$$[Af](x) = \frac{1}{1+\sigma_2(0)}\left[\int_x^1 \sigma_2'(t-x)f(t)dt + \int_0^{1-x}\sigma_2'(t+x)f(t)dt\right].$$

Since

$$\|Af\| \le \frac{2\|s_2'\|}{1-|\sigma_2(0)|}\|f\|$$

and $|\sigma_2(0)| \le \|\sigma_2'\| < 1/3$, then $\|A\| < 1$. Therefore, the pair $(\sigma_1, \sigma_2) \in L_2(0,1) \times B_{1/3}$ is a solution of the equation (20) if and only if $\sigma_1 = (I + A)^{-1}g$. For the function $F(\sigma_2) := (I + A)^{-1}g$, both statements of Corollary 1 are obviously true.

## 3 Proofs of Theorem 3 and Corollary 2

### 3.1 Proof of Theorem 3

Let $K_\infty$ and $K_0$ be the integral operators on the right-hand sides of equalities (6) and (7), respectively. According to (8), we have

$$\langle K_0 f, g\rangle = \langle f, F_1 g\rangle, \quad \langle K_\infty f, g\rangle = \langle f, F_2 g\rangle, \quad f, g \in L_2(0,1).$$

These relations allow the identity (18) to be written in the form

$$\lambda^2\Psi_1(\lambda) + \lambda\Psi_2(\lambda) + \Psi_3(\lambda) \equiv 0, \tag{21}$$

where

$$\Psi_1(\lambda) = \langle c_0, (I + F_1)\sigma_1\rangle\langle s_0, (I + F_2)\sigma_2\rangle - \langle s_0, (I + F_2)\sigma_1\rangle\langle c_0, (I + F_1)\sigma_2\rangle,$$
$$\Psi_2(\lambda) = \langle c_0, (I + F_1)\sigma_1\rangle, \quad \Psi_3(\lambda) = \langle s_0, (I + F_2)\sigma_2\rangle,$$
$$c_0 = \cos\lambda x, \quad s_0 = \sin\lambda x.$$

Direct calculations show that

$$\langle c_0, f\rangle\langle s_0, g\rangle(\lambda) = \int_0^2 \sin\lambda x C[f,g](x)dx,$$
$$C[f,g](x) = \begin{cases} B[f,g](x) & \text{on} \quad [0,1], \\ \dfrac{1}{2}\displaystyle\int_{x-1}^{1} f(t)g(x-t)dt & \text{on} \quad [1,2], \end{cases}$$

where the shape $B$ is defined by (9).

Next, integrating by parts, we have

$$\lambda\Psi_2(\lambda) + \Psi_3(\lambda) = -\lambda^2\int_0^1 \sin\lambda x A_2[\sigma_1, \sigma_2, q](x)dx + \lambda A_2[\sigma_1, \sigma_2, q](0).$$

Thus, the identity (21) takes the form

$$\lambda\int_0^2 \sin\lambda x D[\sigma_1, \sigma_2, q](x)dx + A_2[\sigma_1, \sigma_2, q](0) \equiv 0, \quad \lambda \in C,$$

where

$$D[\sigma_1, \sigma_2, q](x) = \begin{cases} A_1[\sigma_1, \sigma_2, q](x) - A_2[\sigma_1, \sigma_2, q](x) & \text{on} \quad [0,1], \\ A_3[\sigma_1, \sigma_2, q](x) & \text{by} \quad (1,2]. \end{cases}$$

The obtained identity for $\lambda = 0$ implies $A_2[\sigma_1, \sigma_2, q](0) = 0$, therefore it is equivalent to the system (11), (12). The theorem is proved.

### 3.2 Proof of the Corollary 2

We have

$$\left.\begin{aligned} A_1[\sigma_1,\sigma_2,q](x) &= A_0[[(I+K_+)\sigma_1,(I+K_+)\sigma_2](x)- \\ &\quad - A_0[[K_-\sigma_1,K_-\sigma_2](x) - C[(I+K_+)\sigma_1,K_-\sigma_2](x)+ \\ &\quad + C[K_-\sigma_1,(I+K_+)\sigma_2](x), \\ C[f,g](x) &= \int_0^x f(t)g(x-t)dt. \end{aligned}\right\} \tag{22}$$

From here and from the equality (4) it is clear that if $g \in W_2^1[0,1]$ and $g(1) = 0$, then the function $A_1[f,g]$ is differentiable almost everywhere on $[0,1]$ and

$$\left.\begin{aligned} \frac{d}{dx}A_0[[f,g](x) &= -g(0)f(x) - \int_x^1 f(t)g'(t-x)dt - \int_x^1 f(t-x)g'(t)dt, \\ \frac{d}{dx}C[[f,g](x) &= g(0)f(x) + \int_0^x f(t)g'(x-t)dt. \end{aligned}\right\} \tag{23}$$

If $q \in L_2(0,1)$, then the function $\mathcal{K}$ is absolutely continuous in both arguments [18, Chapter 1, § 2, Theorem 1.2.2] and

$$\iint_{[-1,1]\times[-1,1]} \left( \left|\frac{\partial \mathcal{K}(x,t)}{\partial x}\right|^2 + \left|\frac{\partial \mathcal{K}(x,t)}{\partial t}\right|^2 \right) dxdt < \infty. \tag{24}$$

Therefore, the function $A_1[\sigma_1,\sigma_2,q]$ is differentiable almost everywhere on $[0,1]$ and $\frac{d}{dx}A_1[\sigma_1,\sigma_2,q] \in L_2(0,1)$.

Let $q \in L_2(0,1)$ and let $\sigma_2$ satisfy the condition (i) with some $\delta > 0$. Further, let the functions $q,\sigma_1,\sigma_2$ satisfy the equation (11). According to what has been said, the left-hand side of this equation is differentiable almost everywhere on $[0,1]$. Differentiating it, we arrive at the equation

$$\sigma_1 + T[\sigma_2,q]\sigma_1 = R[\sigma_2,q],$$

where

$$R[\sigma_2,q](x) = -(I+F_1)^{-1}\int_x^1 (I+F_2)\sigma_2(t)dt,$$

$T[\sigma_2,q]$ is an operator acting in $L_2(0,1)$ according to the rule

$$T[\sigma_2,q]f = -(I+F_1)^{-1}\frac{d}{dx}A_1[f,\sigma_2,q].$$

If $(\sigma_2,q) \in \Pi_{M,\delta}$, then from the relations (22), (23) and (24), taking into account the limited invertibility of the operator $I+F_1$ (due to the Volterra property of $F_1$), we have

$$\|T[\sigma_2,q]f\| \le C_1(1+M)\delta\|f\|, \quad f \in L_2(0,1), \tag{25}$$

$$\|R[\sigma_2,q]\| \le C_2(1+M)\delta, \tag{26}$$

where $C_1, C_2$ are positive constants.

Let $M, \delta > 0$ be such that

$$C_1(1+M)\delta < 1. \tag{27}$$

Let

$$F[\sigma_2, q] = (I + T[\sigma_2, q])^{-1} R[\sigma_2, q], \quad (\sigma_2, q) \in \Pi_{M,\delta}. \tag{28}$$

According to estimates (25) and (26)

$$\sup_{\Pi_{M,\delta}} \|F[\sigma_2, q]\| \le N := (1 - C_1(1+M)\delta)^{-1} C_2 \delta.$$

If $q = 0$ a.e. on $(0, a)$, then $\mathcal{K}(x,t) = 0$ for all $|t| \le |x| \le a$ [18, Ch. 1, § 2]. Using (8), it is easy to show that $[K_- \sigma_2](x) \equiv 0$ on $[0,1]$ for any function $\sigma_2$ supported in $[0,a]$. By (28), if $\mathrm{supp}(\sigma_2) \in [0,a]$, then the support of $\sigma_1 = F[\sigma_2, q]$ also lies in $[0,a]$. From this, based on formula (10), we conclude that for any $(\sigma_2, q) \in \Pi_{M,\delta}$, the triple of functions $(\sigma_1 = F[\sigma_2, q], \sigma_2, q)$ satisfies equation (12). Thus, we have shown that if inequality (27) is satisfied, then in order for the triple $(\sigma_1, \sigma_2, q) \in \Pi_N \times \Pi_{M,\delta}$ to satisfy equation (11) it is necessary that

$$\sigma_1 = F[\sigma_2, q], \qquad (\sigma_2, q) \in \Pi_{M,\delta}. \tag{29}$$

Conversely, if (29) is true, then equation (11) is obtained by integrating over the interval $[x, 1]$ both parts of the equivalent (29) identity

$$\frac{d}{dx} A_1[\sigma_1, \sigma_2, q](x) - (I + F_1)(\sigma_1 + R[\sigma_2, q])(x) \equiv 0, \quad x \in [0,1].$$

## 4 Proof of Theorem 4

Let $\sigma_2(x) = \mu(1-x)$. Differentiating the left side of the equation (5) and taking into account the equalities $\sigma_2' = \mu$, $\sigma_2(0) = \mu$, we obtain

$$\sigma_1(x) + k\left(\int_x^1 \sigma_1(t)dt + \int_0^{1-x} \sigma_1(t)dt\right) = k(1-x)^2/2. \tag{30}$$

This equation is equivalent to the problem

$$\sigma_1'(x) - k(\sigma_1(x) + \sigma_1(1-x)) = -k(1-x), \tag{31}$$

$$\sigma_1(1) = 0. \tag{32}$$

From (30) we have

$$\sigma_1(1-x) + k\left(\int_{1-x}^1 \sigma_1(t)dt + \int_0^x \sigma_1(t)dt\right) = k/2. \tag{33}$$

Adding equations (30) and (33) term by term, we obtain

$$\sigma_1(x) + \sigma_1(1-x) = -2k \int_0^1 \sigma_1(t)dt + \frac{k}{2}\left(2 - 2x + x^2\right).$$

Therefore, the equation (31) will take the form

$$\sigma_1'(x) = -2k^2 c_0 + g(x), \quad c_0 = \int_0^1 \sigma_1(t)dt, \quad g(x) = k^2(1 - x + x^2/2) - k(1-x).$$

Hence, taking into account (32), we have

$$\sigma_1(x) = 2k^2 c_0(1-x) - \int_x^1 g(t)dt, \tag{34}$$

so that

$$c_0 = (1-k^2)^{-1} \int_0^1 x g(x)dx.$$

Substituting this expression into (34), we obtain (13).

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