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FINDING THE SOLUTION OF THE ELLIPTIC SYSTEM OF THE FIRST ORDER IN THE FORM OF A VECTOR-FUNCTION

The relationship between Douglis analytic functions and the analytic functions of the Laplace equation for elliptic equations with constant coefficients serves the same purpose. In this sense, some boundary value problems solved by Douglis using analytic functions remain relevant today. Douglis's Bitsadze-Samarsky problem on analytic functions was reconsidered by A.P. Soldatov. Meanwhile, for certain types of matrices and regions, V.G. Nikolaev proved the existence and uniqueness of the Schwarz problem solution. In any region G of the complex plane C, we considered a first order system $\dfrac{\partial \Phi}{\partial y} - J\dfrac{\partial \Phi}{\partial x} = F$, where the eigenvalues of the constant matrix $J \in C^{l\times l}$ lie in the upper half-plane, $Imv > 0$. In the case where $l = 1$, $J = J(z)$ is continuous and $ImJ > 0$, we obtain the Beltrami equation. In this paper, to find the solution of elliptic equation with constant coefficient, the corresponding J-analytic system was constructed by choosing the matrix J of size 2×2 and J has no real eigenvalues. In the postcomponential notation, the system consists of two differential equations depending on the variable z. Thus, the solution of this equation was found using the vector- function.

Key words: analytic function, Douglis analytic function, elliptic system, regular solution, vector-function.

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Бірінші ретті эллиптикалық жүйенің шешімін вектор-функция түрінде табу

Аналитикалық функциялар мен Лаплас теңдеуінің арасында қатыс бар. Сол сияқты Дуглис бойынша аналитикалық функциялар мен тұрақты коэффициентті элиптикалық теңдеулер арасында да қатыс бар болғандықтан қазіргі уақытта Дуглис бойынша аналитикалық функцияларға қойылған әртүрлі шекаралық есептер өзекті болып отыр. А.П. Солдатов Дуглис бойынша аналитикалық функцияларға қойылған Бицадзе-Самарский есебін қарастырды. В.Г. Николаев матрицалар мен облыстардың арнайы типтері үшін қойылған Шварц есебінің шешімінің жалғыздығын алды. Біз C комплекс жазықтықтың кез келген G облысында $z = x + iy$ айнымалыға байланысты $\dfrac{\partial \Phi}{\partial y} - J\dfrac{\partial \Phi}{\partial x} = F$ бірінші ретті жүйе қарастырамыз, мұнда J 2×2 тұрақты өлшемді тұрақты матрицаның меншікті мәндері жоғары жарты жазықтықта орналасқан, $Imv > 0$. $J = J(z)$ үзіліссіз және $ImJ > 0$ жағдайда Бельтрами теңдеуін аламыз. Әр компоненті арқылы жазсақ, жүйе z айнымалыға байланысты екі бірінші ретті дифференциалдық теңдеулерден құралады. Бұл жүйені тепе-теңдікке айналдыратын екі вектор-функция берілген теңдеудің шешімі болып табылады.

Түйін сөздер: аналитикалық функция, Дуглис бойынша аналитикалық функция, эллиптикалық жүйе, регулярлық шешім, вектор-функция.

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Нахождение решения эллиптической системы первого порядка в виде вектор-функции

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Существует связь между аналитическими функциями и уравнением Лапласа. Точно так же, поскольку существует связь между аналитическими функциями по Дуглису и эллиптическими уравнениями с постоянными коэффициентами, в настоящее время актуальны различные граничные задачи, применяемые к аналитическим функциям по Дуглису. А.П. Солдатов рассмотрел задачу Бицадзе-Самарского применительно к аналитическим функциям по Дуглису. В.Г. Николаевым получена единственность решения задачи Шварца для особых типов матрицы и областей.

В данной работе рассматривается эллиптическая система первого порядка $\frac{\partial \Phi}{\partial y} - J\frac{\partial \Phi}{\partial x} = F$, заданная постоянными матричными коэффициентами, зависящими от переменной $z = x + iy$ в области G на комплексной плоскости C. Матрица имеет размер 2×2 и собственные значения постоянной матрицы J лежат в верхней полуплоскости, $Imv > 0$. В случае, когда $J = J(z)$ непрерывен и $ImJ > 0$, мы получаем уравнение Бельтрами. Если записать через каждую компоненту, система состоит из двух дифференциальных уравнений первого порядка, зависящих от переменной z. Таким образом, решение данного уравнения было найдено в виде вектор-функции

Ключевые слова: аналитическая функция, аналитическая функция по Дуглису, эллиптическая система, регулярное решение, вектор-функция.

1 Introduction

Analytic functions are known to provide the solutions of elliptic equations, and the Laplace equation's $\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ solution is known to define the real part of an analytic function. I.N. Vequa [1] created a unified approach to the study of this concept. By considering the solutions of elliptic systems of two equations with two unknowns reliant on two variables, L. Bers [2] and I.N. Vequa expanded the concept of analyticity. The general solution and derivatives of an elliptic system were further investigated by A.B. Bitsadze [3] using an analytic vector function. These solutions are called generalised analytic functions. By defining the class and algebra of functions satisfying the main part of elliptic systems consisting of $2r$ equations with $2r$ unknowns relying on two variables, A. Douglis [4] established the groundwork for J-analytic functions. These functions have been referred to as hyperanalytic (r is a positive integer).

This system can be represented by a single "hypercomplex" equation in Douglis algebra; he presented a complete study of the theory of hyperanalytic functions in [4]. These topics can also be considered in Wendland's monograph [5]. Later B. Boyarsky [6], D. Horvats [7], R. Hilbert [8], A.P. Soldatov [9] and others improved it. They developed an analogue of the theory of analytic functions. These functions are also called Douglas analytic functions. A.P. Soldatov demonstrated how analyzing the elliptic system with the Douglis analytic function considerably simplifies the analysis of A.B. Bitsadze. The relationship between the analytic functions of the Laplace equation and elliptic equations with constant coefficients is the same as that of the Douglis analytic functions. Several boundary value issues applied to Douglis analytic functions are still relevant today since complex analysis has been generalized in the plane and has many real-world applications. The Bitsadze-Samarsky problem for J-analytic functions was considered by A.P. Soldatov. The Schwarz problem's uniqueness for particular types of matrices and regions was discovered by V.G. Nikolaev [10].

$$\partial_{\overline{z}}\Phi + A(z)\Phi + B(z)\overline{\Phi} = 0$$

coefficients $A(z)$, $B(z)$ of the system $z \in G$ belong to the class of functions $L_p(G)$, $p > 2$ in the G-bounded region, p-integration satisfies the "weak" condition. If the coefficients $A(z)$, $B(z)$ tend to infinity at an isolated point in the region G, then the order of the singularity must be less than one. The case when the point singularities of the coefficients are not less than unity was first discovered by I.N. Vekua considered quasi-summarizable by introducing the class of functions. More precisely, he considered a class of coefficients such that φA and $\psi B \in L_p(G)$, $p > 2$ are analytic functions of $\varphi(z)$, $\psi(z)$, $z \in G$. Solution of the system of coefficients of the quasiset

$$\Phi(z) = U(z) exp \omega(z)$$

in the form Here $U(z)$ is an analytic function $z \in G$,

$$\omega(z) = \frac{1}{\pi\varphi(z)} \iint\limits_G \frac{\varphi(\zeta)A(\zeta)}{\zeta - z} d\xi d\eta + \frac{1}{\pi\psi(z)} \iint\limits_G \frac{\psi(\zeta)B(\zeta)}{\zeta - z} \frac{\overline{\Phi(\zeta)}}{\Phi(\zeta)} d\xi d\eta$$

This formula establishes a connection between the set of solutions of the system by quasi-summarizable coefficients and analytic functions of the complex variable. If $B(z) = 0$, then $\omega(z)$ and $\Phi(z)$ are independent of each other, and the efficiency of the system solution increases. If $B(z) \neq 0$, the quasi-summarizability condition is too general to solve the system. Therefore, researchers work with a specific object.

In this paper, the solution of elliptic system with constant coefficient is reduced to the condition $B(z) = 0$ by searching in the form of vector function.

2 Materials and Methods

Consider the following equation in any domain G of the complex plane C:

$$\partial_{\bar{z}}\Phi - \beta J \partial_z \Phi + \frac{1}{2} A_J \Phi + \frac{1}{2} B_J \overline{\Phi} = 0 \tag{1}$$

Where we aim to find a regular analytic solution belonging to the class

$$C_\alpha(G), \quad \alpha = \frac{p-2}{p} \tag{2}$$

Here, $0 \leq \beta < 1$ is the elliptic condition, $J \in C^{l \times l}$ is a complex matrix, and A, $B \in L_p(G)$, $p > 2$.

In the case $l = 1$, $J(z) = \frac{z}{\bar{z}}$, $A = \frac{a(\varphi)}{\bar{z}}$, $B = \frac{b(\varphi)}{\bar{z}}$, $a(\varphi)$, $b(\varphi) \in C[0, 2\pi]$ the equation becomes:

$$\partial_{\bar{z}}\Phi - \beta \frac{z}{\bar{z}} \partial_z \Phi + \frac{1}{2} \frac{a(\varphi)}{\bar{z}} \Phi + \frac{1}{2} \frac{b(\varphi)}{\bar{z}} \overline{\Phi} = 0 \tag{3}$$

The solution to the Beltrami equation in the class $C(G) \cap W_p^1(G)$, $1 < p < 2$ was found in [11]. Here, $G = \left\{ z = re^{i\phi} \, : \, 0 \leq r < R, \; 0 \leq \phi \leq 2\pi \right\}$, $R > 0$.

This work considers the case $l = 2$. Let J be a nilpotent complex matrix of size (2×2). For each complex number $z = x \cdot 1 + y \cdot i$, using the identity matrix I, $z_J = x \cdot 1 + y \cdot J = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ we obtain the matrices:

$$A_J = \begin{pmatrix} ReA & ImA \\ 0 & ReA \end{pmatrix}, \quad B_J = \begin{pmatrix} ReB & ImB \\ 0 & ReB \end{pmatrix}$$

The fundamental elements of analytic functions J have been extended in works [4-9]. We seek the solution in the vector-function form:

$$\Phi = \begin{pmatrix} Re\Phi \\ Im\Phi \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$$

The corresponding conjugate function is: $\overline{\Phi} = \begin{pmatrix} \Phi_1 \\ -\Phi_2 \end{pmatrix}$. Applying the Wirtinger formula to the vector-function $\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ we get:

$$\partial_{\overline{z}}\Phi = \begin{pmatrix} \dfrac{\partial\Phi_1}{\partial x} + i\dfrac{\partial\Phi_1}{\partial y} \\ \dfrac{\partial\Phi_2}{\partial x} + i\dfrac{\partial\Phi_2}{\partial y} \end{pmatrix}, \qquad \partial_z\Phi = \begin{pmatrix} \dfrac{\partial\Phi_1}{\partial x} - i\dfrac{\partial\Phi_1}{\partial y} \\ \dfrac{\partial\Phi_2}{\partial x} - i\dfrac{\partial\Phi_2}{\partial y} \end{pmatrix}$$

Rewriting equation (1) in matrix form, we obtain:

$$\begin{cases} \partial_{\overline{z}}\Phi_1 - \beta\partial_z\Phi_2 + Re(A+B)\Phi_1 + Im(A-B)\Phi_2 = 0, \\ \partial_{\overline{z}}\Phi_2 - Re(B-A)\Phi_2 = 0 \end{cases} \tag{4}$$

We obtain a system of differential equations depending on the variable z. The solution to the second homogeneous equation in this system is:

$$\Phi_2(z) = U_1(z) \cdot exp\left(\frac{1}{\pi}\iint\limits_G \frac{Re(A-B)(\xi)}{\xi - z} d\xi d\eta\right) \tag{5}$$

where $U_1(z) \in U_0(G)$, $U_0(G)$ – is the set of analytic functions.

Using the solution (5), we introduce the notation:

$$F(z) =: \beta\partial_z\Phi_2 - Im(A-B)(z) \cdot \Phi_2(z)$$

Thus, the first equation of system (4) can be written as:

$$\partial_{\overline{z}}\Phi_1 + Re(A+B)\Phi_1 = F \tag{6}$$

The general solution of this inhomogeneous equation is equal to the sum of a specific inhomogeneous equation solution and the general solution of the related homogeneous equation. The homogeneous equation's general solution:

$$\partial_{\overline{z}}\Phi_1 + Re(A+B)\Phi_1 = 0$$

The general solution of the equation will be in the following form:

$$\Phi_1(z) = U_2(z) \cdot exp\left(\frac{1}{\pi}\iint\limits_G \frac{Re(A+B)(\xi)}{\xi - z} d\xi d\eta\right) \tag{7}$$

where $U_2(z) \in U_0(G)$.

To find a particular solution of the inhomogeneous equation (6), we first show that the equation (6) and:

$$\partial_{\overline{z}}(exp(\omega(z))\Phi_1) = F(z)exp(\omega(z)) \tag{8}$$

Let us show that the equation is mutually pairwise. Here

$$\omega(z) = \frac{-1}{\pi}\iint\limits_G \frac{Re(A+B)(\xi)}{\xi - z}d\xi d\eta + \Phi_0(z),$$

where $\Phi_0(z) \in U_0(G)$. In equation (7), following the property of the derivative,

$$\Phi_1 exp(\omega)\partial_{\overline{z}}\omega + exp(\omega)\partial_{\overline{z}}\Phi_1 = F\,exp(\omega)$$

and considering that $\partial_{\overline{z}}\Phi_0(z) = 0$ and $\partial_{\overline{z}}\omega = \partial_{\overline{z}}\left(\frac{-1}{\pi}\iint\limits_G \frac{Re(A+B)(\xi)}{\xi - z}d\xi d\eta\right) = Re(A + B)(z)$, we get equation (6). The solution to equation (8) is as follows:

$$\Phi_1(z)exp(\omega(z)) = U_0(z) - \frac{1}{\pi}\iint\limits_G \frac{F(\xi)exp(\omega(\xi))}{\xi - z}d\xi d\eta$$

From this equation,

$$\Phi_1(z) = U_3(z)exp(-\omega(z)) - \frac{1}{\pi}\iint\limits_G \frac{F(\xi)exp(\omega(\xi))}{(\xi - z)exp(\omega(z))}d\xi d\eta$$

where $U_3(z) \in U_0(G)$. From the form of the integrals on the right-hand side of equations (5) and (7), it follows that the solution belongs to the class $C_\alpha(G)$, $\alpha = \frac{p-2}{p}$.

Thus, the following theorem is valid.

Theorem. *The functions $\Phi_1(z)\Phi_2(z)$ found using formulas (7) and (5) are solutions of the elliptic system (1) in the class (2) given in any domain G of the complex plane $\mathbb{C}$.*

3 Conclusion

The fundamental solution of the generalized Beltrami system

$$\frac{\partial\Phi}{\partial y} - J\frac{\partial\Phi}{\partial x} = F$$

is of the matrix-function form

$$E(z) = \frac{1}{2\pi i}z_J^{-1}$$

In other words, for a continuous differentiable function $F(z)$,

$$(TF)(z) = \frac{1}{2\pi i}\int\limits_C (t-z)_J^{-1}F(t)dt_1dt_2$$

the integral is a classical solution to the generalized Beltrami system.

In this article, using theoretical functional methods, the solution of the Beltrami system in vector-function form was found.

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