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NUMERICAL MODELING OF OPTIONS IN DIFFUSION (B, S) STOCK MARKETS

This article examines the calculation of the option price $V(t, x)$, the stock price $x(t)$, and the optimal stopping (execution) time τ; $(\equiv t)$ over both finite and infinite time horizons. It then delves into determining a fair value for American-style options, leveraging the optimal stopping time within the framework of diffusion processes in stock markets, represented by (B, S). Additionally, the article explores the pricing of European-style options, starting with the buyer's perspective and then transitioning to the seller's viewpoint. The problems are solved either analytically, when the optimal stopping time is pre-determined, or numerically using methods like the sweep method and finite element techniques.These methods are applied by reducing the problem to Stefan's problem, where $Y^*(t, x)$ represents the rational option value, τ_T^* indicates the rational execution time, and $x^*(t)$ corresponds to the rational stock price.
Key words: option prices, stock prices, equity diffusion markets, options of American and European types, Stefan's problem, numerical modeling.

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Диффузиондық (B, S) акциялар нарығындағы опциондарды сандық модельдеу

Бұл мақалада опцион бағасын $V(t, x)$, акция бағасын $x(t)$ және тиімді тоқтату (орындау) уақытын τ $(\equiv t)$ ақырлы және шексіз уақыт аралықтарында есептеудің ерекшеліктері талқыланады. Кейінірек диффузиялық (B, S) – нарықтарында тиімді тоқтату мезетін ескере отырып, Американдық типтегі опциондардың әділ бағасын анықтау мәселесі қарастырылады. Одан әрі, Еуропалық типтегі опциондардың бағасын дұрыс есептеу мәселесі зерттеледі. Алдымен, опцион сатып алушының көзқарасынан қарастырылып, оның опционы талданады, содан соң сатушының опционы қарастырылады. Барлық есептер, егер тиімді тоқтату уақыты алдын ала белгілі болса, дәл шешілуі мүмкін немесе сандық әдістер арқылы — қуалау және ақырлы элементтер әдісін пайдаланып, Стефан есебіне келтіріліп, $Y^*(t, x)$ – опционның рационалды құны, τ_T^* – тиімді орындау уақыты және $x^*(t)$ – акция бағасының рационалды мәні бойынша шешілуі мүмкін.
Түйін сөздер: опцион бағасы, акция бағасы, шындық диффузиялық нарық, Американдық және Еуропалық типті опциондар, Стефан есебі, сандық модельдеу.

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Численное моделирование опционов на диффузионных (B, S)-рынках акций

В данной статье рассматриваются особенности вычисления стоимости опциона $V(t,\,x)$, цены акции $x(t)$ и оптимального момента остановки (или исполнения) $\tau\ \ (\equiv t)$ как для конечных, так и для бесконечных временных интервалов. Далее изучается задача определения справедливой стоимости опционов американского типа на основе оптимального момента остановки, в контексте диффузионных рынков акций $(B,\,S)$. Затем обсуждается проблема определения рациональной цены опционов Европейского типа. Вначале рассматривается ситуация с точки зрения покупателя опциона, после чего анализируется ситуация с точки зрения продавца. Все поставленные задачи решаются точно, если заранее найден оптимальный момент остановки, либо численно — с использованием методов прогонки и конечных элементов, путем преобразования их в задачу Стефана, где $Y^*(t,\,x)$ представляет собой рациональную цену опциона, τ_T^* – оптимальный момент исполнения, а $x^*(t)$ – рациональную цену акции.
Ключевые слова: цена опциона, цена акции, справедливый диффузионный рынок, опционы Американского и Европейского типов, задача Стефана, численное моделирование.

1 Introduction

Building on the work in [1], this paper investigates various aspects of calculating the option price $V(t,x)$, the stock price $x(t)$, and the optimal stopping (execution) time τ $(\equiv t)$ over finite and infinite time intervals. In then explores the determination of a fair price for American-style options, utilizing the optimal stopping time within diffusion-based $(B,\,S)$-stock market models. The discussion proceeds to address the pricing of European-style options, starting with an analysis on the buyer's perspective, particularly the call option, followed by a focus on the put option. The problems are solved either exactly when the optimal stopping time is predetermined or numerically by reformulating them into the Stefan problem. In mathematical physics, the Stefan problem arises in the study of physical processes associated with the phase transformation of matter and consists in finding a function $u = u(t,x)$ that describes the temperature regime of the phases and the separation boundary $x = x(t),\ t \geq 0$ of these phases.

In the case of standard buyer and seller options, a two-phase situation also takes place – when searching for optimal stopping rules, we can restrict ourselves to considering only two simply connected phases: the area of continuation of observations C^T and the area D^T.

All problems can be solved analytically if the optimal stopping time is known beforehand or numerically if it is not.

The results of numerical modeling of the Stefan's problem by the sweep method and the finite element method for standard call and ask options are presented. As well as a comparative analysis of the numerical results by the sweep method and the finite element method (FEA).

Statement of the problem from the book [1]. Standard American-type buyer and seller options and optimal process stopping are considered in the works [2], [3], [4], [5], [6], [7]. Numerical methods for solving the Stefan problem and other numerical methods for solving stochastic (diffusion) partial differential equations are considered in the works [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18].

2 Numerical solution of the Stefan's problem for the call option

In the area $C^T = \Big\{\big(t,\, x\big):\; x < x^*(t),\; t \in \big[0,\, T\big)\Big\}$ consider the equation

$$-\frac{\partial Y^*(t,\, x)}{\partial t} + \beta Y^*(t,\, x) = \mathrm{L}Y^*(t,\, x), \tag{1}$$

where $\beta = \lambda + r,\;\; \mathrm{L}Y^*(t,\, x) = rx\dfrac{\partial Y^*(t,\, x)}{\partial x} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 Y^*(t,\, x)}{\partial x^2}$ and in the area $D^T \cup \Big\{\big(T,\, x\big):\; x \in E\Big\}$ consider

$$Y^*(t,\, x) = g(x) \tag{2}$$

at the boundary $x^* = x^*(t),\; 0 \le t < T$, the section of "two phases the Dirichlet condition is fulfilled:

$$Y^*\big(t,\, x^*(t)\big) = g\big(x^*(t)\big); \tag{3}$$

and the Neumann condition:

$$\left.\frac{\partial Y^*(t,\, x)}{\partial x}\right|_{x \uparrow x^*(t)} = \left.\frac{dg(x)}{dx}\right|_{x \downarrow x^*(t)}. \tag{4}$$

Let us discretize the phase domain $C^T = \Big\{(t,\, x):\;\; x < x^*(t),\;\; t \in \big[0,\, T\big)\Big\}$ with t respect to the step τ, $t_n = n\tau,\; n = 0,\, 1,\, 2,\, \ldots,\, N,\; \tau = \frac{T}{N}$, with x respect to the step h, $x_i = ih,\; i = 0,\, 1,\, \ldots$. We also discretize the area $D^T \cup \Big\{\big(T,\, x\big):\; x \in E\Big\}$. Omit index * above $Y^*(t,\, x)$). We approximate (1) by an implicit scheme, and for the discrete domain C^T_{ni} we obtain the difference equation

$$\alpha_i Y^{n+1}_{i+1} + \beta_i Y^{n+1}_i + \gamma_i Y^{n+1}_{i-1} = -\frac{1}{\tau} Y^n_i, \tag{5}$$

where $\alpha_i = -\Big(\frac{rx_i}{h} + \frac{\sigma^2 x_i^2}{2h^2}\Big)$, $\beta_i = \Big(\frac{rx_i}{h} + \frac{\sigma^2 x_i^2}{h^2} + \beta - \frac{1}{\tau}\Big)$, $\gamma_i = -\frac{\sigma^2 x_i^2}{2h^2}$. In the discrete domain D^T_{ni}, we write the Dirichlet condition (3):

$$Y^n_i = g_i \;\text{ or }\; Y^n\big(x^*_i\big) = g(x^*_i\big) \tag{6}$$

and the Neumann condition (4):

$$\left.\frac{Y^n_{i+1} - Y^n_i}{h}\right|_{x_i = x^*_i + 0} = \left.\frac{g_{i+1} - g_i}{h}\right|_{x_i = x^*_i - 0}. \tag{7}$$

If the condition $\beta > \frac{1}{\tau}$, is satisfied, equation (5) can be solved, for instance, using the sweep method. To solve (5) we ensure that at the boundary $\left(x^*\right)_i^n$, the two-phase conditions (6) and (7) are met. At each step, we verify that the "front" $\left(x^*\right)_i^n$ is defined at a grid point. If not, we can adjust the step sizes τ and h.

Next, issues of numerical modeling of the Stefan's problem are considered. [8], [9], [10], [17], [18].

3 Numerical modeling by the method of double-sweep method the Stefan's problem for standard call options

As a first example for a call option, we consider an American call option with financial variables $K = 10$, $\sigma = 0.6$, $r = 0.25$, $\delta = 0.2$, $x_0 = 10$ and $T = 1$. For these data, proposes a value of $Y^*\big(0,\ K\big) = 2.18728$, which corresponds to the variant shown in following Figure 1

The selected values for the test data $\big(K = 10,\ \sigma = 0.6,\ r = 0.25,\ \delta = 0,\ T = 1\big)$ are shown in Table 1 and visualized in Figure 2.

Table 1: Values $Y^*(0, x)$ of the American call option at K=10, σ=0.6, r=0.25, δ=0, T=1.

m (grid dimension), $Y^*_{\max}$	Sweep method	Finite element method (FEM)
100	2.181171	2.186701
200	2.186031	2.187181
400	2.186941	2.187251
800	2.187191	2.187271
1600	2.187261	2.187281

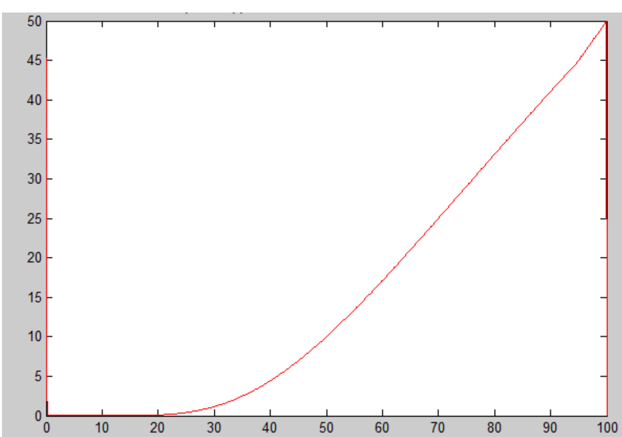

Figure 1: American call option pricing for $K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $T = 5/12$ (grid dimension 1600×1600).

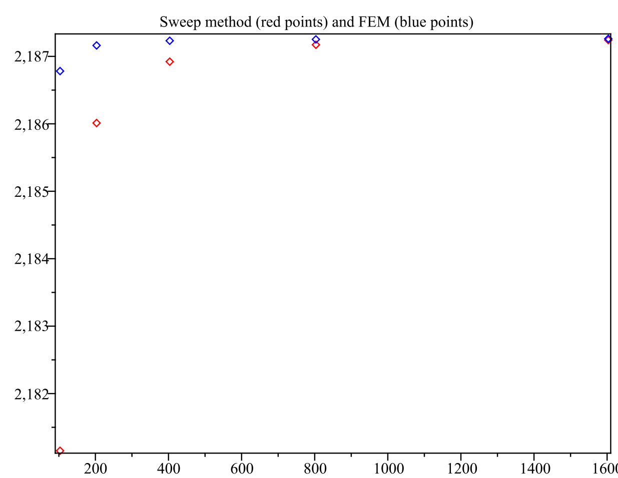

Figure 2: Convergence of the price an American call option $Y^*(0,\ x)$ $\big(K = 10,\ \sigma = 0.6,\ r = 0.25,\ \delta = 0.2,\ T = 1\big)$ with grid dimension.

As a second example for call option, we consider an American call option with financial variables $K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $x_0 = 50$ and $T = 5/12$. For these data proposes a value of $Y^*\big(0,\ K\big) = 21.28638$, which corresponds to the variant shown in Figure 3.

The selected values for the test data $\big(K = 50,\ \sigma = 0.4,\ r = 0.1,\ \delta = 0,\ T = 5/12\big)$ are shown in Table 2 and visualized in Figure 4.

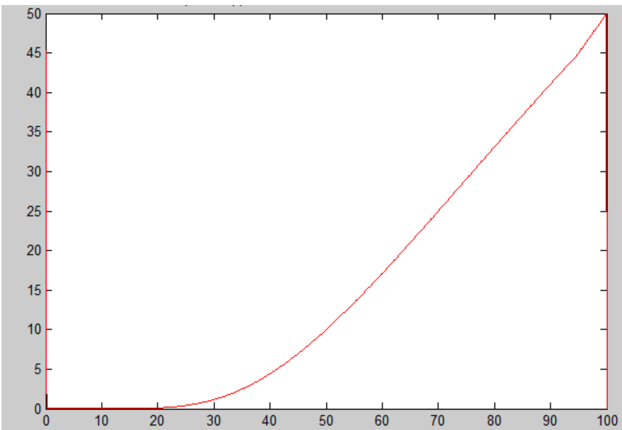

Figure 3: American call option pricing for $K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $T = 5/12$ (grid dimension 1600×1600).

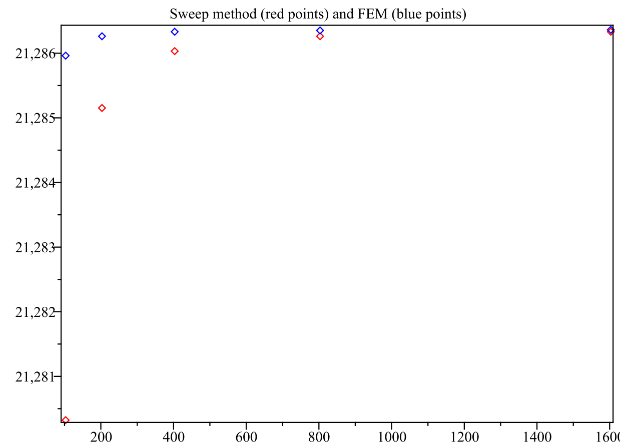

Figure 4: Convergence of the price of an American call option $Y^*(0,\, x)$ $\big(K = 50,\ \sigma = 0.4,\ r = 0.1,\ \delta = 0,\ T = 5/12\big)$ with increasing grid dimension.

Table 2: Values $Y^*(0, x)$ of the American call option at K=50, σ=0.4, r=0.1, δ=0, T=5/12.

m (grid dimension), $Y^*_{\max}$	Sweep method	Finite element method (FEM),
100	21.280341	21.285981
200	21.285171	21.286281
400	21.286051	21.286351
800	21.286281	21.286371
1600	21.286351	21.286381

4 Numerical modeling by the method of running the Stefan's problem for standard put options

As the first example for a put option, we consider an American put option with financial variables $K = 10$, $\sigma = 0.6$, $r = 0.25$, $\delta = 0.2$, $x_0 = 10$, $T = 1$, which is shown in Figure 5.

This curve, shown in Figure 6, defines the option's early exercise strategy.

As a second example for a put option, we consider an American put option with financial variables K=50, σ=0.4, r=0.1, δ=0, x_0=50 and T=5/12, which is shown in Figure 7 We also calculate the point $x^*(0)$ for early exercise of the put option. Numerical results are given in Table 3 and illustrated in Figure 8.

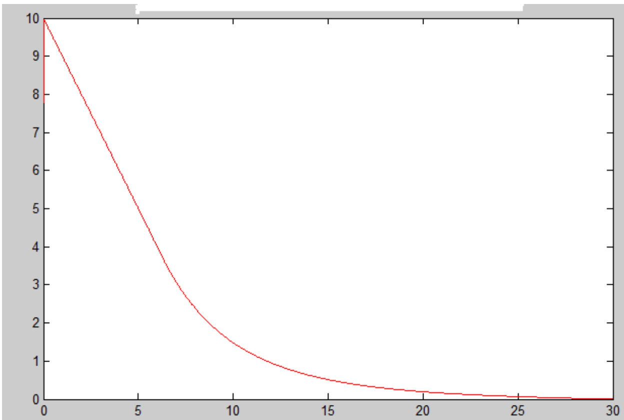

Figure 5: American put option pricing function $Y^*(0, x)$ for $K = 10$, $\sigma = 0.6$, $r = 0.25$, $\delta = 0.2$, $x_0 = 10$, $T = 1$ (grid dimension 1600×1600).

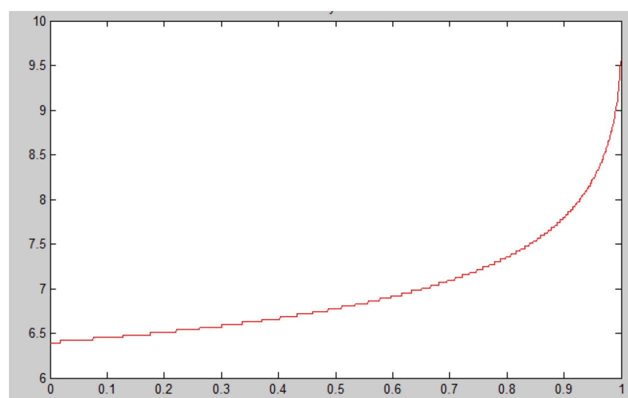

Figure 6: Time structure of the early exercise boundary $x^*(t)$ of a put option $K = 10$, $\sigma = 0.6$, $r = 0.25$, $\delta = 0$, $T = 1$.

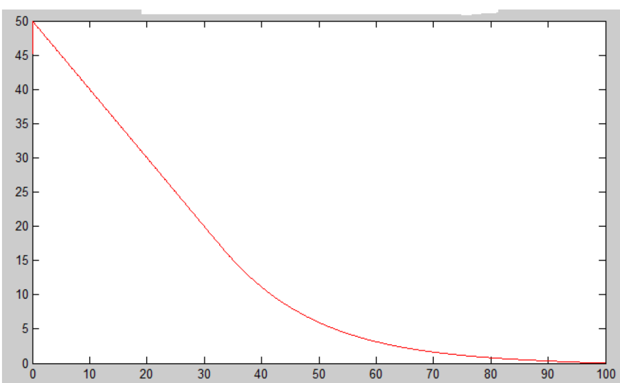

Figure 7: The pricing function $Y^*(0, x)$ of an American put option at $K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $T = 5/12$ (grid dimension 1600×1600).

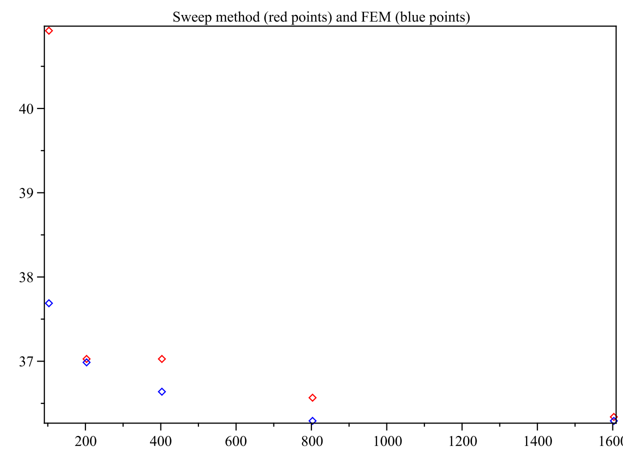

Figure 8: Convergence of the price of an American put option $\big(K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $T = 5/12\big)$ with increasing grid dimension.

Table 3: The boundary $x^*(0)$ of the early exercise of the K=50, σ=0.4, r=0.1, δ=0, T=5/12.

m (grid dimension), $Y^*_{\max}$	Sweep method	Finite element method (FEM)
100	40.93651	37.70211
200	37.04091	37.00011
400	37.04091	36.65201
800	36.58081	36.30571
1600	36.35291	36.30571

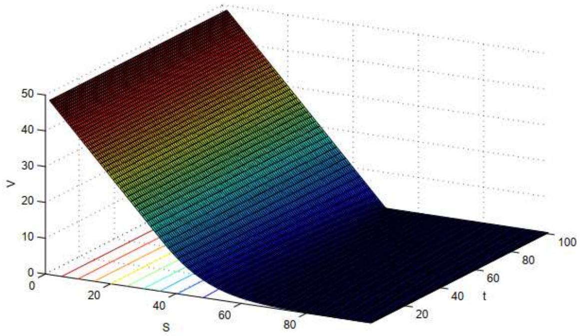

Figure 9: Structure $Y^*(t,\, x)$ of an American put option.

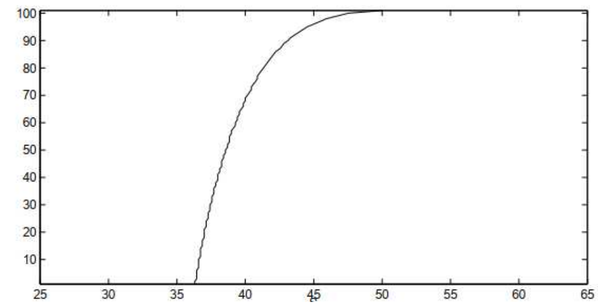

Figure 10: Structure of the free margin $x^*(t)$ of an American put option.

5 Conclusion

The advantage of finite difference methods (the sweep method) compared to the finite element method (FEA), in particular, the solution by the sweep method provides knowledge about the development of the option value function for each time step, i.e. the entire term structure of an American put option can easily visualize. See Figure 9. The accuracy of the sweep method is higher than FEA, see Figures 2, 4 and 8. At present $t = 0$, which is the "leading edge" of the surface, the shape of the cost function $Y^*(0,\, x)$ can be clearly seen, as shown in Figure 9. By selecting the $Y^*(t,\, x)$ functions for each t during the life of the option, one can obtain a complete term structure, and by the maturity date T approaches the non-smooth payoff function $Y^*\big(K - x\big)^+$. Of the entire surface of options, of particular interest is the development of the high contact point over time, namely $x^*(t)$, which is shown in Figure 10. This curve can be obtained by projecting the upper contact point at each time step onto the $x^*(t) - t$-plane, and it determines the option's early exercise strategy.

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