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COMPUTATION OF FUZZY EIGENVALUES AND EIGENVECTORS OF GENERALIZED FUZZY EIGENVALUE PROBLEMS USING NONLINEAR PROGRAMMING APPROACH

Computation of the fuzzy eigenvalues and the associated fuzzy eigenvectors of a fuzzy matrix is often a more challenging task compared to the classical eigenvalue problem for crisp matrices. Most of the methods developed so far focus specifically on the standard fuzzy eigenvalue problem and cannot be directly applied to more general cases. In this paper, we consider the generalized fuzzy eigenvalue problem ($\mathbb{GFEP}$) of the form $AV = \lambda BV$ in a fuzzy environment, where A and B are fuzzy matrices and the parameter λ represents the fuzzy eigenvalue. To obtain the fuzzy eigenpair $(\tilde{\lambda}, \tilde{V})$ of $\mathbb{GFEP}$, we propose a method that converts the problem into a system of linear and nonlinear equations and then applies interval-based calculations using a few inequalities. Additionally, we provide a numerical example to illustrate the application of the method and to verify its efficiency, demonstrating that the proposed approach can effectively compute fuzzy eigenvalues and eigenvectors in generalized settings.

Key words: fuzziness, fuzzy eigenvalues, fuzzy eigenvectors, generalized eigenvalue problem, fuzzy triangular matrix.

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Сызықты емес бағдарламалау әдісін пайдалана отырып, жалпыланған белгісіз меншікті мәндер мен меншікті векторларды есептеу

Белгісіз матрицаның белгісіз меншікті мәндері мен сәйкес үздік векторларын есептеу дәлірек (классикалық) матрицалар үшін классикалық меншікті мәндер есептеу мәселесімен салыстырғанда күрделірек болып табылады. Қазіргі уақытқа дейін әзірленген көптеген әдістер негізінен стандартты белгісіз меншікті мәндер есептеу тапсырмасына бағытталған және оларды кеңейтілген жағдайларға тікелей қолдану мүмкін емес. Осы жұмыста жалпыланған белгісіз меншікті мәндер есептеу тапсырмасы ($\mathbb{GFEP}$) қарастырылады: $AV = \lambda BV$, мұндағы A және B — белгісіз матрицалар, ал λ — белгісіз меншікті мәнді білдіреді. $\mathbb{GFEP}$ тапсырмасының белгісіз меншікті мәндер мен векторлар жұбын $(\tilde{\lambda}, \tilde{V})$ алу үшін, тапсырманы сызықты және сызықты емес теңдеулер жүйесіне келтіретін әдіс ұсынылады, содан кейін бірнеше теңсіздіктерді пайдаланып интервалдық есептеулер жүргізіледі. Сонымен қатар, әдістің қолданылуын көрсету және тиімділігін тексеру үшін сандық мысал берілген, бұл ұсынылған тәсілдің жалпыланған тапсырмаларда белгісіз меншікті мәндер мен меншікті векторларды тиімді есептеуге мүмкіндік беретінін көрсетеді.

Түйін сөздер: белгісіздік, белгісіз меншікті мәндер, белгісіз меншікті векторлар, жалпыланған меншікті мәндерге арналған есеп, белгісіз үшбұрышты матрица

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Вычисление нечетких собственных значений и собственных векторов обобщенных нечетких задач собственных значений с использованием метода нелинейного программирования

Вычисление нечетких собственных значений и соответствующих нечетких собственных векторов нечеткой матрицы часто является более сложной задачей по сравнению с классической задачей собственных значений для точных (классических) матриц. Большинство методов, разработанных до настоящего времени, ориентированы в основном на стандартную задачу нечетких собственных значений и не могут быть напрямую применены к более общим случаям. В этой работе рассматривается обобщенная задача нечетких собственных значений ($\mathbb{GFEP}$) вида $AV = \lambda BV$ в нечеткой среде, где A и B – нечеткие матрицы, а параметр λ обозначает нечеткое собственное значение. Для получения нечеткой пары собственных значений и векторов $(\tilde{\lambda}, \tilde{V})$ задачи $\mathbb{GFEP}$ предлагается метод, который сводит задачу к системе линейных и нелинейных уравнений, после чего выполняются расчеты на интервалах с использованием нескольких неравенств. Кроме того, приводится численный пример, демонстрирующий применение метода и проверяющий его эффективность, показывая, что предложенный подход позволяет эффективно вычислять нечеткие собственные значения и собственные векторы в обобщенных задачах.

Ключевые слова: нечеткость, нечеткие собственные значения, нечеткие собственные векторы, обобщенная задача на собственные значения, нечеткая треугольная матрица.

1 Introduction

The concept of eigenvalues and eigenvectors plays a pivotal role in various mathematical disciplines and finds wide variety of applications in fields such as physics, engineering, and computer science [12]. However, when dealing with systems that involve uncertainty, imprecision, or vagueness, the classical eigenvalue problem framework has a failure to provide comprehensive analysis. To address this limitation, the concept of fuzzy eigenvalue problem has emerged as a powerful tool for handling fuzziness and uncertainty in eigenvalue-based analyses [6]. The fuzzy eigenvalue problem extends the classical eigenvalue problem by incorporating fuzzy set theory, which allows for the representation of uncertain or imprecise information in a more flexible manner. In a fuzzy eigenvalue problem, the eigenvalues and eigenvectors are expressed as fuzzy sets or fuzzy numbers, enabling the capture of various degrees of membership or possibility. It is important to note that solving the fuzzy eigenvalue problem can be computationally challenging due to the increased complexity introduced by the fuzzy or uncertain nature of the data. Therefore, specialized algorithms and numerical techniques are often required to obtain meaningful solutions. There are different approaches to solve the fuzzy eigenvalue problem, such as interval analysis [20], fuzzy-affine approach [19], etc., depending on the specific context and requirements of the problem. The goal is to obtain a fuzzy solution that characterizes the uncertainty in the eigenvalues and eigenvectors based on the uncertainty in the input matrix.The system of nonlinear equations $\tilde{A}\tilde{V} = \tilde{\lambda}\tilde{V}$ where $\tilde{A} = (\tilde{a_{ij}})_{n \times n}$, $\tilde{V}^t = (\tilde{v_1}, \tilde{v_2}, ..., \tilde{v_n})_{1 \times n}$ such that $\tilde{a_{ij}}$, $\tilde{v_j}$, $\tilde{\lambda}$ are fuzzy numbers, for all $i, j = 1, \ldots, n$ is called a dual fully fuzzy linear system ($\mathbb{DFFLS}$). The common approach for solving a $\mathbb{DFFLS}$ is the $\alpha - cut$ theory and interval calculations. It was Buckley [6] who introduced $\alpha - cut$ approach for finding eigenvalues of fuzzy matrix, but this has a limitation due to the fact that only fuzzy positive matrix was considered. Buckley et al. [8] introduced another approach for calculating fuzzy eigenvalues. Their method utilized artificial neural networks for this purpose. However, similar to the previous methods [6], this approach also exhibited certain restrictions and deficiencies. Also, for more references, see [7]. Using the same $\alpha - cut$ approach Chiao [10] studied the generalized fuzzy eigenvalues of the form $Av = \lambda Bv$ under some restrictions. Dehghan et al. [11] presented several methods to solve square and non-square fully fuzzy systems, including the Cramer's rule, Gaussian elimination,

LU decomposition (Doolittle algorithm), and linear programming (LP). However, the non-negative solutions doesn't exist in their approaches. A useful approach with parametric functions and variables provided by the system's fuzzy coefficients was proposed by Vroman et al. [24]. Additionally, they demonstrated that their algorithm is superior to Buckley's approach. Theodorou et al. [22] explored the fuzzy eigenvalues of the corresponding fuzzy analysis. To transform the fuzzy system into an classical one, they used a two-step process. Later, Yiannis et al. [26] developed a practical method to use Corresponding Analysis with fuzzy data to complete this theoretical result. Tian [23] proposed an extension of crisp eigenvectors of a crisp matrix to fuzzy eigenvalues. His work focused on studying the structure of fuzzy eigenspaces and the relationships between crisp eigenspaces in a crisp matrix. Based on this analysis, Tian then examined a system of fuzzy linear equations and provided a solution for it. Xia & Friswell [25] proposed a perturbation approach which provides the precise maximum and minimum of the structural eigenvalue's first order deviation to set tighter boundaries on the eigenvalues of a structural vibration problem with unknown fuzzy parameters.In Allahviranloo et al. [5], a technique to solve fully fuzzy linear system ($\mathbb{FFLS}$) was proposed. To do this, they solved a 1-cut of an $\mathbb{FFLS}$, assigned some unknown symmetric spreads to each row of the solution, and then calculated the symmetric spreads by solving a 2n-linear equation system.. In Allahviranloo et al. [3], the authors presented a practical method for solving $\mathbb{DFFLS}$. They approached the problem by considering the 1-cut of the system and then devised a method to allocate unknown symmetric expansions for each cut of the system. By using interval algebra, they formulated a system of linear equations to calculate fuzzy eigenvalues and fuzzy eigenvectors. Few literature on computations of fuzzy eigenvalues and fuzzy eigenvectors are available in [1], [2],, [9], [20], [21], [15], and the references therein. In 2016, Mahato [17] extended the interval filtering method used by Hladik et al. [13] to propose interval and fuzzy filtering techniques to reach tighter limitations of $\mathbb{GFEP}$. Rout and Chakraverty [19] proposed an effective approach to address the fully fuzzy non-linear eigenvalue problem of a damped spring mass structural system. Their suggested method was based on a fuzzy affine technique. They found that this approach yielded better enclosures compared to standard fuzzy arithmetic. Recently, Khosravi et al. [16] provided a new approach for finding fuzzy triangular matrix's eigenvalues and fuzzy eigenvectors. First they obtained the eigenvalues and eigenvectors of the matrix by solving $1 - cut$ of the system.Then based on this solutions the system is converted to nonlinear programming problem to obtain the support of the triangular fuzzy eigenvalues and eigenvectors. Till now, researchers have developed various methods for solving $\mathbb{DFFLS}$. But, for solving $\mathbb{GFEP}$, no specific method are available in the existing literature. Existing literature of $\mathbb{GFEP}$ ensures the exploration of even more diverse and intricate fuzzy systems. It has various application in diverse scientific domain such as decision making under uncertainty, fuzzy pattern recognition [14], correspondence analysis [22], and damped spring-mass structural system [19]. Inspired by the works of Khosravi et al. [16], in the current paper, we propose a method for finding fuzzy eigenvalues and fuzzy eigenvectors of $\mathbb{GFEP}$ using a nonlinear programming approach.

The rest of the paper is organized as follows: Section 2 contains some basic results, which will be used in the whole work. Section 3, contains main result to obtain the eigenpair of the generalized fuzzy eigenvalue system. In section 4, numerical examples are presented to check the efficiency of the method and finally in Section 5 a conclusion is drawn on the whole work.

2 Preliminaries

The following definitions and basic results will be used throughout the research works. $\mathbb{R}$ represents the set of real numbers and $\otimes$ stands for usual Kronecker product.

Definition 1 *[16] An interval number [n] is defined as the set of real numbers such that $[n] = [\underline{n}, \overline{n}] = \{x \in \mathbb{R} : \underline{n} \leq x \leq \overline{n}\}$.*

Definition 2 *[16] Let $[m] = [\underline{m}, \overline{m}]$ and $[n] = [\underline{n}, \overline{n}]$ be two interval numbers, then the basic arithmetic operations of interval numbers is defined as below*

$$[\underline{m}, \overline{m}] + [\underline{n}, \overline{n}] = [\underline{m} + \underline{n}, \overline{m} + \overline{n}], \quad [\underline{m}, \overline{m}] - [\underline{n}, \overline{n}] = [\underline{m} - \overline{n}, \overline{m} - \underline{n}],$$

$$[\underline{m}, \overline{m}] \times [\underline{n}, \overline{n}] = [\underline{p}, \overline{p}] = \begin{cases} \underline{p} = min\{\underline{m}.\underline{n}, \overline{m}.\underline{n}, \underline{m}.\overline{n}, \overline{m}.\overline{n}\} \\ \overline{p} = max\{\underline{m}.\underline{n}, \overline{m}.\underline{n}, \underline{m}.\overline{n}, \overline{m}.\overline{n}\}, \end{cases}$$

$$k.[\underline{m}, \overline{m}] = \begin{cases} [k\underline{m}, k\overline{m}] & k \geq 0 \\ [k\overline{m}, k\underline{m}] & k < 0 \ where \ k \in \mathbb{R}. \end{cases} \tag{1}$$

Definition 3 *[3] The width of an interval number $[v] = [\underline{v}, \overline{v}]$ is defined as $\mathbb{W}([v]) = \overline{v} - \underline{v}$.*

Definition 4 *[18] A fuzzy number $\tilde{A}$ is a fuzzy set on the real line $\mathbb{R}$, must satisfy the following conditions.*

1. *$\mu_{\tilde{A}}(x_0)$ is piece-wise continuous.*

2. *There exists at least one $x_0 \in R$ with $\mu_{\tilde{A}}(x_0) = 1$.*

3. *$\tilde{A}$ must be normal and convex.*

Definition 5 *Triangular Fuzzy Number (TFN): [18] A TFN $\tilde{A}$ is a fuzzy subset of real line $\mathbb{R}$, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions.*

1. *$\mu_{\tilde{A}}$ is continuous mapping from R to the closed interval $[0, 1]$.*

2. *$\mu_{\tilde{A}}(x) = 0$, where $-\infty < x \leq a_1$.*

3. *$\mu_{\tilde{A}}(x)$ is strictly increasing with constant rate on $a_1 \leq x \leq b_1$.*

4. *$\mu_{\tilde{A}}(x) = 1$, where $x = b_1$.*

5. *$\mu_{\tilde{A}}(x)$ is strictly decreasing with constant rate on $b_1 \leq x \leq c_1$.*

6. *$\mu_{\tilde{A}}(x) = 0$, where $c_1 \leq x \leq \infty$.*

Definition 6 *[16] $\alpha - cut$ of a fuzzy number $\tilde{m}$ is defined as $[\tilde{m}]^{\alpha} = \{x \in \mathbb{R} : \tilde{m}(x) \geq \alpha\}$ where $0 \leq \alpha \leq 1$ and the support and core of $\tilde{m}$ are defined by the sets $\mathbb{S}(\tilde{m}) = \{x \in \mathbb{R} : \tilde{m}(x) > 0\}$ and $\mathbb{C}(\tilde{m}) = \{x \in \tilde{R} : \tilde{m}(x) = 1\}$.*
    Clearly, for TFN $\tilde{m} = (m_1, m_2, m_3)$ we have $\mathbb{S}(\tilde{m}) = [m_1, m_3]$ and $\mathbb{C}(\tilde{m}) = m_2$.

Fow two fuzzy numbers $\tilde{m} = [\underline{m}(\alpha), \overline{m}(\alpha)]$ and $\tilde{n} = [\underline{n}(\alpha), \overline{n}(\alpha)]$, and $k \in \mathbb{R}$, the $\alpha - cuts$ of $\tilde{m} + \tilde{n}$ and $k.\tilde{m}$ are defined as

$$
\begin{cases}
[\tilde{m} + \tilde{n}]^\alpha = [\tilde{m}]^\alpha + [\tilde{n}]^\alpha = [\underline{m}(\alpha) + \underline{n}(\alpha), \overline{m}(\alpha) + \overline{n}(\alpha)], \\
[\tilde{m} - \tilde{n}]^\alpha = [\tilde{m}]^\alpha - [\tilde{n}]^\alpha = [\underline{m}(\alpha) - \overline{n}(\alpha), \overline{m}(\alpha) - \underline{n}(\alpha)], \\
[k.\tilde{m}]^\alpha = k.[\tilde{m}^\alpha] = \begin{cases} [k\underline{m}(\alpha), k\overline{m}(\alpha)], & k \geq 0, \\ [k\overline{m}(\alpha), k\underline{m}(\alpha)], & k < 0. \end{cases}
\end{cases}
\tag{2}
$$

Definition 7 *[16] For two arbitrary* TFN*s* $\tilde{M} = (m_1, m_2, m_3)$ *and* $\tilde{N} = (n_1, n_2, n_3)$, *addition, subtraction, and scalar multiplication are defined as follows:*

$$\tilde{M} + \tilde{N} = (m_1, m_2, m_3) + (n_1, n_2, n_3) = (m_1 + n_1, m_2 + n_2, m_3 + n_3)$$
$$\tilde{M} - \tilde{N} = (m_1, m_2, m_3) - (n_1, n_2, n_3) = (m_1 - n_3, m_2 - n_2, m_3 - n_1)$$
$$k.\tilde{M} = k.(m_1, m_2, m_3) = \begin{cases} (k.m_1, k.m_2, k.m_3), & k \geq 0, \\ (k.m_3, k.m_2, k.m_1), & k < 0. \end{cases}$$

Definition 8 *[16] A vector* $\tilde{V} = (\tilde{v_1}, \tilde{v_2}, ..., \tilde{v_n})^t$, $\tilde{v_i} \in$ TFN, $1 \leq i \leq n$, *is called a* TFN *vector. We denote it by* $\tilde{X} \in$ TFN$_t^n$.

Definition 9 *[19] A matrix* $[\tilde{N}] = (\tilde{n}_{pq})$ *of order* $m \times n$ *can be referred as fuzzy matrix , if each element* $(\tilde{n}_{pq})$ *for* $p = 1, 2, ..., m$ *and* $q = 1, 2, ..., n$ *is a fuzzy number.*

Definition 10 *[16] The* TFN $\tilde{\lambda} \neq 0$ *is called the generalized fuzzy eigenvalue of the fuzzy matrices* $\tilde{A}$ *and* $\tilde{B}$, *if there is a fuzzy eigenvector* $\tilde{V} \neq 0$ *such that* $\tilde{A}\tilde{V} = \tilde{\lambda}\tilde{B}\tilde{V}$.

It can be written in $n \times n$ *nonlinear system of equations as follows:*

$$
\begin{cases}
\tilde{a}_{11}\tilde{v}_1 + \tilde{a}_{12}\tilde{v}_2 + ... + \tilde{a}_{1n}\tilde{v}_n = \tilde{\lambda}(\tilde{b}_{11}\tilde{v}_1 + \tilde{b}_{12}\tilde{v}_2 + ... + \tilde{b}_{1n}\tilde{v}_n) \\
\tilde{a}_{21}\tilde{v}_1 + \tilde{a}_{22}\tilde{v}_2 + ... + \tilde{a}_{2n}\tilde{v}_n = \tilde{\lambda}(\tilde{b}_{21}\tilde{v}_1 + \tilde{b}_{22}\tilde{v}_2 + ... + \tilde{b}_{2n}\tilde{v}_n) \\
. \\
. \\
. \\
\tilde{a}_{n1}\tilde{v}_1 + \tilde{a}_{n2}\tilde{v}_2 + ... + \tilde{a}_{nn}\tilde{v}_n = \tilde{\lambda}(\tilde{b}_{n1}\tilde{v}_1 + \tilde{b}_{n2}\tilde{v}_2 + ... + \tilde{b}_{nn}\tilde{v}_n)
\end{cases}
,
\tag{3}
$$

where $\tilde{a}_{pq}$, $\tilde{\lambda}$, $\tilde{b}_{pq}$ *and* $\tilde{v}_p$, $1 \leq p, q \leq n$, *are fuzzy numbers.*

Definition 11 *[16] The* $n \times n$ *nonlinear system*

$$
\begin{cases}
[\tilde{a}_{11}]^\alpha[\tilde{v}_1]^\alpha + [\tilde{a}_{12}]^\alpha[\tilde{v}_2]^\alpha + ... + [\tilde{a}_{1n}]^\alpha[\tilde{v}_n]^\alpha = [\tilde{\lambda}]^\alpha([\tilde{b}_{11}]^\alpha[\tilde{v}_1]^\alpha + [\tilde{b}_{12}]^\alpha[\tilde{v}_2]^\alpha + ... + [\tilde{b}_{1n}]^\alpha[\tilde{v}_n]^\alpha) \\
[\tilde{a}_{21}]^\alpha[\tilde{v}_1]^\alpha + [\tilde{a}_{22}]^\alpha[\tilde{v}_2]^\alpha + ... + [\tilde{a}_{2n}]^\alpha[\tilde{v}_n]^\alpha = [\tilde{\lambda}]^\alpha([\tilde{b}_{21}]^\alpha[\tilde{v}_1]^\alpha + [\tilde{b}_{22}]^\alpha[\tilde{v}_2]^\alpha + ... + [\tilde{b}_{2n}]^\alpha[\tilde{v}_n]^\alpha) \\
. \\
. \\
. \\
[\tilde{a}_{n1}]^\alpha[\tilde{v}_1]^\alpha + [\tilde{a}_{n2}]^\alpha[\tilde{v}_2]^\alpha + ... + [\tilde{a}_{nn}]^\alpha[\tilde{v}_n]^\alpha = [\tilde{\lambda}]^\alpha([\tilde{b}_{n1}]^\alpha[\tilde{v}_1]^\alpha + [\tilde{b}_{n2}]^\alpha[\tilde{v}_2]^\alpha + ... + [\tilde{b}_{nn}]^\alpha[\tilde{v}_n]^\alpha)
\end{cases}
\tag{4}
$$

is called the $\alpha - cut$ system of the generalized fuzzy nonlinear system where $[\tilde{a}_{pq}]^{\alpha}$, $[\tilde{\lambda}]^{\alpha}$, $[\tilde{b}_{pq}]^{\alpha}$ and $[\tilde{v}_p]^{\alpha}$, $1 \leq p,q \leq n$ and $\alpha \in [0,1]$, are the $\alpha - cut$ of the fuzzy numbers $\tilde{a}_{pq}$, $\tilde{\lambda}$, $\tilde{b}_{pq}$ and $\tilde{v}_p$, respectively. Its matrix form is written as $[\tilde{A}]^{\alpha}[\tilde{V}]^{\alpha} = [\tilde{\lambda}]^{\alpha}[\tilde{B}]^{\alpha}[\tilde{V}]^{\alpha}$.

3 Main result

In this section, we extend the results of section 3.1 of [16] to obtain the eigenpair $(\tilde{\lambda}, \tilde{V})$ of the $\mathbb{GFEP}$ $\tilde{A}\tilde{V} = \tilde{\lambda}\tilde{B}\tilde{V}$ such that , $\underline{a}_{ij} \geq 0$ or $\overline{a}_{ij} \leq 0$ and $\underline{b}_{ij} \geq 0$, $1 \leq i,j \leq n$.

Now, to obtain a suitable solution , we have to solve the fuzzy interval system

$$[\tilde{A}]_1 \otimes [\tilde{V}]_1 = [\tilde{\lambda}]_1 \otimes [\tilde{B}]_1 \otimes [\tilde{V}]_1, \tag{5}$$

$$[\tilde{A}]_0 \otimes [\tilde{V}]_0 = [\tilde{\lambda}]_0 \otimes [\tilde{B}]_0 \otimes [\tilde{V}]_0. \tag{6}$$

First, by solving the system 5, we obtain the eigenvalues $\lambda_i^{'s}$ and corresponding eigenvectors $V_i^{'s}, 1 \leq i \leq n$ for the crisp system $[\tilde{A}]_1, [\tilde{B}]_1$. After obtaining the eigenpairs for the crisp system 5, next we determine the left and right width for the eigenpairs $(\lambda_i, V_i), 1 \leq i \leq n$.

Consider the eigenpair $(\lambda_m, V_m), m \in \{1, 2, ..., n\}$

Now, we put the following restrictions to obtain the left and right boundaries of the eigenpair (λ_m, V_m)

1. We consider the fuzzy eigenvalues to be positive.

2. We consider the entries of matrix $\tilde{B}$ to be positive.

Then using the results of Section 3.1 of [16] we consider the following two cases:
Case I: $0 \notin \mathbb{S}(\tilde{V}_j), 1 \leq j \leq n$. Consider the following partition.
$N^+ = \{j | \underline{V_j} \geq 0 \& V_j \neq 0\}$
$N^- = \{j | \overline{V_j} \leq 0\}$
$A_i^- = \{j | \overline{a_{ij}} \leq 0 \ \& \ \underline{a_{ij}} \neq 0\}$
$A_i^+ = \{j | \underline{a_{ij}} \geq 0\}, i = 1, 2, ..., n.$
$B_i^+ = \{j | \underline{b_{ij}} \geq 0\}, i = 1, 2, ..., n$
We consider only positive fuzzy eigenvalues. So we have,
$\lambda_k, \underline{\lambda}_k, \overline{\lambda}_k > 0$
Therefore, we can rewrite equation 6 based on the signs of eigenpairs(λ_m, V_m) from equation 5 as follows:

$$\sum_{A_i^+ N^+} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] + \sum_{A_i^+ N^-} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] + \sum_{A_i^- N^+} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] + \sum_{A_i^- N^-} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}]$$

$$= [\underline{\lambda_k}, \overline{\lambda^k}] \otimes \{ \sum_{B_i^+ N^+} [\underline{b_{ij}}, \overline{b_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] + \sum_{B_i^+ N^-} [\underline{b_{ij}}, \overline{b_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] \}, \ 1 \leq i \leq n. \tag{7}$$

where,

$$\sum_{A_i^+ N^+} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] = \sum_{A_i^+ N^+} [\underline{a_{ij}v_j}, \overline{a_{ij}v_j}],$$

$$\sum_{A_i^+ N^-} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] = \sum_{A_i^+ N^-} [\overline{a_{ij}}\underline{v_j}, \underline{a_{ij}}\overline{v_j}],$$

$$\sum_{A_i^- N^+} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] = \sum_{A_i^- N^+} [\underline{a_{ij}}\overline{v_j}, \overline{a_{ij}}\underline{v_j}],$$

$$\sum_{A_i^- N^-} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] = \sum_{A_i^- N^-} [\overline{a_{ij}}\overline{v_j}, \underline{a_{ij}}\underline{v_j}]. \tag{8}$$

and the right hand side equation 7 can be expressed as:

$$[\underline{\lambda_k}, \overline{\lambda^k}] \otimes \{ \sum_{B_i^+ N^+} [\underline{b_{ij}}, \overline{b_{ij}}] \otimes [\underline{v_j}, \overline{v_j}]\} = \sum_{B_i^+ N^+} [\underline{\lambda_k b_{ij} v_j}, \overline{\lambda_k b_{ij} v_j}],$$

$$[\underline{\lambda_k}, \overline{\lambda^k}] \otimes \{ \sum_{B_i^+ N^-} [\underline{b_{ij}}, \overline{b_{ij}}] \otimes [\underline{v_j}, \overline{v_j}]\} = \sum_{B_i^+ N^-} [\overline{\lambda_k b_{ij}}\underline{v_j}, \underline{\lambda_k b_{ij}}\overline{v_j}]. \tag{9}$$

Case II: $0 \notin \mathbb{S}(\tilde{\lambda}_k)$ then $\underline{\lambda_k},\ \lambda_k,\ \overline{\lambda_k} > 0$
$P^+ = \{j | \overline{V_j} \geq 0\ \& V_j \neq 0\}$
$P^- = \{j | \underline{V_j} \leq 0\}$
Therefore, we can rewrite equation 6 based on the signs of eigenpairs(λ_m, V_m) from equation 5 as follows:

$$\sum_{A_i^+ P^+} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] + \sum_{A_i^+ P^-} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] + \sum_{A_i^- P^+} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] + \sum_{A_i^- P^-} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}]$$

$$= [\underline{\lambda_k}, \overline{\lambda^k}] \otimes \{ \sum_{B_i^+ P^+} [\underline{b_{ij}}, \overline{b_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] + \sum_{B_i^+ P^-} [\underline{b_{ij}}, \overline{b_{ij}}] \otimes [\underline{v_j}, \overline{v_j}]\},\ 1 \leq i \leq n. \tag{10}$$

Where,

$$\sum_{A_i^+ P^+} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] = \sum_{A_i^+ P^+} [\underline{a_{ij}}v_{j1} + \overline{a_{ij}}v_{j2}, \overline{a_{ij}v_j}],$$

$$\sum_{A_i^+ P^-} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] = \sum_{A_i^+ P^-} [\overline{a_{ij}}\underline{v_j}, \overline{a_{ij}}v_{j1} + \underline{a_{ij}}v_{j2}],$$

$$\sum_{A_i^- P^+} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] = \sum_{A_i^- P^+} [\underline{a_{ij}}\overline{v_j}, \underline{a_{ij}}v_{j2} + \overline{a_{ij}}v_{j1}],$$

$$\sum_{A_i^- P^-} [\underline{a_{ij}}, \overline{a_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] = \sum_{A_i^- P^-} [\underline{a_{ij}} v_{j1} + \overline{a_{ij}} v_{j2}, \underline{a_{ij}} v_{j1}]. \tag{11}$$

$$v_{j1} \geq 0, v_{j2} \leq 0, 1 \leq j \leq n.$$

and the right hand side equation 10 can be expressed as:

$$[\underline{\lambda_k}, \overline{\lambda^k}] \otimes \{ \sum_{B_i^+ P^+} [\underline{b_{ij}}, \overline{b_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] \} = \sum_{B_i^+ P^+} [\underline{\lambda_k}(\underline{b_{ij}} v_{j1} + \overline{b_{ij}} v_{j2}), \overline{\lambda_k b_{ij} v_j}],$$

$$[\underline{\lambda_k}, \overline{\lambda^k}] \otimes \{ \sum_{B_i^+ P^-} [\underline{b_{ij}}, \overline{b_{ij}}] \otimes [\underline{v_j}, \overline{v_j}] \} = \sum_{B_i^+ P^-} [\overline{\lambda_k b_{ij}} \underline{v_j}, \underline{\lambda_k}(\overline{b_{ij}} v_{j1} + \underline{b_{ij}} v_{j2})]. \tag{12}$$

Where $v_{j1} + v_{j2} = \underline{v_j}$ or $v_{j1} + v_{j2} = \overline{v_j}$ Now,by solving **Case I** and **Case II**, we can find the values of $\underline{\lambda_k}, \overline{\lambda_k}$ and $\underline{v_j}, \overline{v_j}, 1 \leq j \leq n$, which is a suitable solution for the system 6.

Another important point is to control the width of the fuzzy eigenvalue $\tilde{\lambda}$ and fuzzy eigenvector $\tilde{v}_j, (j = 1, ..., n)$ in the $\alpha - cuts$.So, we use the following nonlinear programming models:

Model 1: $Max \ \overline{\lambda_k} - \underline{\lambda_k}$, where the objective functions are **Case I** or **Case II**.

Model 2: $Max \ \sum_{j=1}^{n} \overline{v_j} - \underline{v_j}$, where the objective functions are **Case I** or **Case II**.

The above two models can be easily solved by using **LINGO 20** software.

4 Numerical examples

Example 1 *Consider two fuzzy matrices $\tilde{A}$ and $\tilde{B}$,assuming that the TFN $\tilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda})$ is a fuzzy eigenvalue and the fuzzy triangular vector $\tilde{V} = (\tilde{v}_1, \tilde{v}_2)$ is a fuzzy eigenvector of the system, we have:*

$$\tilde{A}\tilde{V} = \tilde{\lambda}\tilde{B}\tilde{V}$$

$$\implies \begin{bmatrix} [-4 \ -3 \ -2] & [1 \ 2 \ 3] \\ [1 \ 3 \ 5] & [-5 \ -3 \ -1] \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} [2 \ 4 \ 6] & [1 \ 2 \ 3] \\ [4 \ 5 \ 6] & [0.7 \ 2 \ 4] \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$$

$$\implies \begin{bmatrix} [-4 \ -3 \ -2] & [1 \ 2 \ 3] \\ [1 \ 3 \ 5] & [-5 \ -3 \ -1] \end{bmatrix} \begin{bmatrix} [v_{11} \ v_1 \ v_{12}] \\ [v_{21} \ v_2 \ v_{22}] \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} [2 \ 4 \ 6] & [1 \ 2 \ 3] \\ [4 \ 5 \ 6] & [0.7 \ 2 \ 4] \end{bmatrix} \begin{bmatrix} [v_{11} \ v_1 \ v_{12}] \\ [v_{21} \ v_2 \ v_{22}] \end{bmatrix}$$

where $\tilde{v}_1, \tilde{v}_2$ are TFN. Now, $1 - cut$ of the above system is

$$\begin{bmatrix} -3 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} 4 & 2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

*Using **MATLAB** the crisp solutions are obtained as:*

$$\lambda = \begin{bmatrix} -0.0878 & 0 \\ 0 & 17.0878 \end{bmatrix} ; V = \begin{bmatrix} -0.8213 & 0.4509 \\ -1.0000 & -1.0000 \end{bmatrix}.$$

Now, considering only the positive crisp eigenvalue $\lambda_1 = 17.0878 \implies \tilde{\lambda}_1 = [\underline{\lambda_1}, 17.0878, \overline{\lambda_2}]$ and

$$v = \begin{bmatrix} 0.4509 \\ -1.0000 \end{bmatrix} \implies \tilde{V} = \begin{bmatrix} [v_{11}, 0.4509, v_{12}] \\ [v_{21}, -1.0000, v_{22}] \end{bmatrix}, \text{ we simplify the } 0 - cut \text{ of the above}$$

system as follow:

$$\begin{bmatrix} [-4, -2] & [1, 3] \\ [1, 5] & [-3, -1] \end{bmatrix} \begin{bmatrix} [v_{11}, v_{12}] \\ [v_{21}, v_{22}] \end{bmatrix} = [\lambda_1, \lambda_2]. \begin{bmatrix} [2, 6] & [1, 3] \\ [4, 6] & [0.7, 4] \end{bmatrix} \begin{bmatrix} [v_{11}, v_{12}] \\ [v_{21}, v_{22}] \end{bmatrix}$$

$$\implies \begin{bmatrix} [-4, -2] * [v_{11}, v_{12}] + [1, 3] * [v_{21}, v_{22}] \\ [1, 5] * [v_{11}, v_{12}] + [-3, -1] * [v_{21}, v_{22}] \end{bmatrix} = [\lambda_1, \lambda_2]. \begin{bmatrix} [2, 6] * [v_{11}, v_{12}] + [1, 3] * [v_{21}, v_{22}] \\ [4, 6] * [v_{11}, v_{12}] + [0.7, 4] * [v_{21}, v_{22}] \end{bmatrix}$$

$$\implies \begin{bmatrix} [-4v_{12} + 3v_{21}, -2v_{11} + v_{22}] \\ [v_{11} - v_{22}, 5v_{12} - 3v_{21}] \end{bmatrix} = [\lambda_1, \lambda_2]. \begin{bmatrix} [2v_{11} + 3v_{21}, 6v_{12} + v_{22}] \\ [4v_{11} + 4v_{21}, 6v_{12} + 0.7v_{22}] \end{bmatrix}$$

$$\implies \begin{bmatrix} [-4v_{12} + 3v_{21}, -2v_{11} + v_{22}] \\ [v_{11} - v_{22}, 5v_{12} - 3v_{21}] \end{bmatrix} = \begin{bmatrix} [\lambda_2 * (2v_{11} + 3v_{21}), \lambda_1 * (6v_{12} + v_{22})] \\ [\lambda_2 * (4v_{11} + 4v_{21}), \lambda_1 * (6v_{12} + 0.7v_{22})] \end{bmatrix}.$$

Now, **Model 1** *and* **Model 2** *can be formulated as*

Model 1: *MAX z*=$\lambda_2 - \lambda_1$ *s.t.*
$$\begin{cases} -4v_{12} + 3v_{11} = \lambda_2 * (2v_{11} + 3v_{21}) \\ -2v_{11} + v_{22} = \lambda_1 * (6v_{12} + v_{22}) \\ v_{11} - v_{22} = \lambda_2 * (4v_{11} + 4v_{21}) \\ 5v_{12} - 3v_{21} = \lambda_1 * (6v_{12} + 0.7v_{22}) \\ v_{11} \leq 0.4509, v_{12} \geq 0.4509 \\ v_{21} \leq -1.0000, v_{22} \geq -1.0000 \\ \lambda_1 \leq 17.0878, \lambda_1 > 0, \lambda_2 \geq 17.0878 \end{cases},$$

where $\tilde{\lambda}_1 = [\lambda_1, \lambda, \lambda_2]$ *and* $\tilde{V} = \begin{bmatrix} [v_{11}, v_1, v_{12}] \\ [v_{21}, v_2, v_{22}] \end{bmatrix}.$

Model 2: *MAX z*=$\lambda_2 - \lambda_1$ *s.t.*
$$\begin{cases} -4v_{12} + 3v_{21} = \lambda_2 * (2v_{111} + 6v_{112} + 3v_{21}) \\ -4v_{112} - 2v_{111} + 3v_{221} + v_{222} = \lambda_1 * (6v_{12} + 3v_{221} + v_{222}) \\ v_{111} + 5v_{112} - 3v_{221} - v_{222} = \lambda_2 * (4v_{111} + 6v_{112} + 4v_{21}) \\ 5v_{12} - 3v_{21} = \lambda_1 * (6v_{12} + 4v_{221} + 0.7v_{222}) \\ v_{11} = v_{111} + v_{112} \leq 0.4509, v_{12} \geq 0.4509 \\ v_{21} \leq -1.0000, v_{22} = v_{221} + v_{222} \geq -1.0000 \\ v_{111} \geq 0, v_{111} \leq 20 * s_1 \\ v_{112} \leq 0, v_{112} \geq -20 + 20 * s_1 \\ v_{221} \geq 0, v_{221} \leq 20 * s_2 \\ v_{222} \leq 0, v_{222} \geq -20 + 20 * s_2 \\ \lambda_1 \leq 17.0878, \lambda_2 \geq 17.0878, \lambda_1 > 0. \end{cases},$$

where $\tilde{\lambda}_1 = [\lambda_1, \lambda, \lambda_2]$ *and* $\tilde{V} = \begin{bmatrix} [v_{111} + v_{112}, v_1, v_{12}] \\ [v_{21}, v_2, v_{221} + v_{222}] \end{bmatrix}.$

Then, using **LINGO 20** *software , we will see that* **Model 1** *is infeasible and* **Model 2** *has the following solution:*

$\lambda_2 = 25.0303, \lambda_1 = 0.4297$

$v_{12} = 12.9569$, $v_{111} = 0.4047, v_{112} = 0.0000$

$v_{21} = -1.0000$, $v_{221} = 20.0000$, $v_{222} = 0.0000.$

So, the required fuzzy eigenvalue and fuzzy eigenvector of $\mathbb{GFEP}$ is obtained as:
$\tilde{\lambda} = (0.4297, 17.0878, 25.0303)$
$\tilde{v}_1 = (0.4047, 0.4509, 12.9569)$
$\tilde{v}_2 = (-1.0000, -1.0000, 20.0000)$
The membership function of the generalized fuzzy eigenvalue and eigenvectors are shown in Figure 1 and Figure 2.
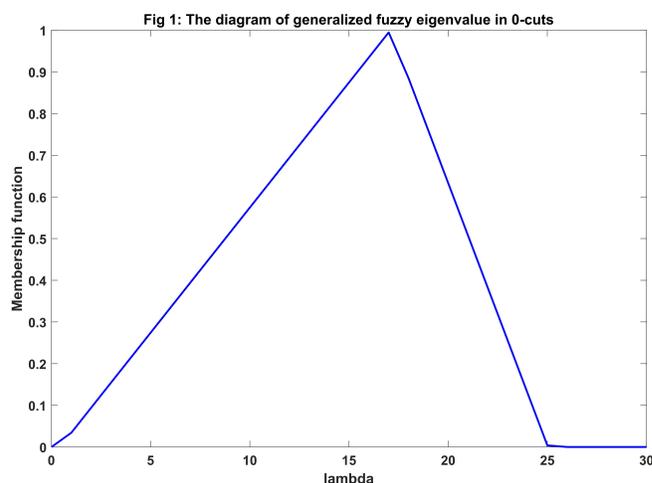

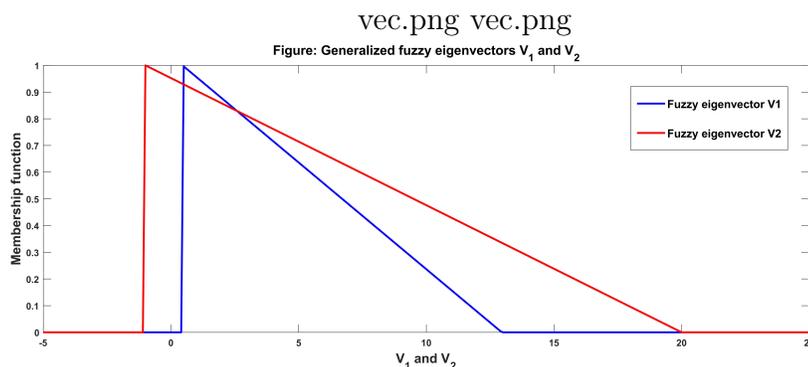**Figure** 1: The diagram of generalized fuzzy eigenvalue $\tilde{\lambda}$

Figure 2: The diagram of generalized fuzzy eigenvectors $\tilde{v_1}$ and $\tilde{v_2}$

5 Conclusion

We described a unified framework of $\mathbb{GFEP}$. We computed fuzzy eigenvalue pair of $\mathbb{GFEP}$ using nonlinear programming approach. In the suggested method, there are few limitations for the choice of sign of the eigenvector $\tilde{V}$. We calculated the fuzzy eigenvalues and eigenvectors in 0-cuts of the system using the suggested approach. Finally, numerical examples are demonstrated to check the feasibility of the method, and it ensures that the approaches are efficient and workable to compute the fuzzy eigenpair for the $\mathbb{GFEP}$. Study on k-parameter eigenvlaue problems may be considered as a new avenue of future research in this area.

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