УДК 517.95

D. Suragan [1,2]
[1]*Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[2]*Institute Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
e-mail: suragan@list.ru

A comparison theorem for eigenvalues of the Newton potential

Dedicated to Prof Mukhtarbay Otelbaev on the occasion of his seventieth birthday

There is a wealth of interesting results comparing between Dirichlet and Neumann eigenvalues. In this paper we compare the eigenvalues of the Newton potential with the Dirichlet eigenvalues and the Neumann eigenvalues in a bounded domain in R^d. First we show that the spectral problem of the Newton potential is equivalent to a spectral problem of a non-local boundary value problem of the Laplace operator then it is proved that the nth eigenvalue of the Laplacian with the non-local boundary condition is between the nth eigenvalue of Neumann Laplacian and the nth eigenvalue of Dirichlet Laplacian in a bounded domain of any dimensional euclidian space.
Key words: spectral problems, Dirichlet problem, eigenvalues of Laplacian, Neumann problem, Newton potential.

Д. Сұраған

Ньютон потенциалының меншікті мәндерін салыстыру теоремасы

Бұл мақалада Ньютон потенциалының меншікті мәндерін салыстыру теоремасы дәлелденді. Евклид кеністігіндегі ашық шенелген обылыста Ньютон потенциалының n-ші меншікті мәні Лаплас операторына қойылған Нейман есебінің n-ші меншікті мәнінен үлкен екендігі және сол Ньютон потенциалының n-ші меншікті мәні Лаплас операторына қойылған Дирихле есебінің n-ші меншікті мәнінен кіші емес екендігі дәлелденді.

Түйін сөздер: спектральдық есеп, Дирихле есебі, Лапласиан меншікті мәндері, Нейман есебі, ньютон потенциалы.

Д. Сураган

Теорема сравнения для собственных значений Ньютоного потенциала

В этой работе доказано одна теоремы сравнения собственных значений Ньютоного потенциала в ограниченной области евклидова пространства R^D с собственными значениями задачи Дирихле для уравнения Лапласа и задачи Неймана для уравнения Лапласа в той же ограниченной области евклидова пространства. Доказано, что n-ное собственное значение Ньютонова потенциала в ограниченной области евклидова пространства R^D больше, чем n-ное собственное значение задачи Неймана для уравнения Лапласа в той же ограниченной области евклидова пространства. А также доказано,

что n-ное собственное значение Ньютонова потенциала в ограниченной области евклидова пространства R^D не больше чем n-ное собственное значение задачи Дирихле для уравнения Лапласа в той же ограниченной евклидова пространства.

Ключевые слова:спектральная задача, задача Дирихле, собственные значения Лапласиана, задача Неймана, потенциал ньютона.

Introduction. There are vast inequalities comparing the eigenvalues $\lambda_1^D < \lambda_2^D \leq \lambda_3^D \leq ...$ of the Dirichlet Laplacian and those of the Neumann Laplacian denoted by $\lambda_1^N < \lambda_2^N \leq \lambda_3^N \leq ...$,(enumerate their eigenvalues in increasing order) each time repeated according to multiplicity. The Mini-Max Theorem asserts that $\lambda_n^N \leq \lambda_n^D, n \in N$. In 1952-1954, Polya [14] and Szego [12] proved that there exists a $\gamma > 0$ independent of Ω such that $\lambda_2^N \leq \gamma\lambda_1^D$

Shortly after, in 1955, Payne [13] showed that

$$\lambda_{n+d}^N \leq \lambda_n^D, n \in N, \tag{1}$$

whenever Ω is a convex, planar domain with C^2-boundary. Developing an idea used in [13], Levine and Weinberger'86 [9] proved inequality (1) for arbitrary bounded convex domains in R^d without any regularity assumption. They also showed that (1) remains true if convexity is replaced by more general conditions on the mean curvature of the boundary (which is assumed to be C^{2-d}). However, without any geometric condition, in dimension 2, it may happen that $\lambda_3^N > \lambda_1^D$ for $\Omega \subset R^2$, (see [1]). It was only in 1991 that Friedlander [5] proved the inequality $\lambda_{n+1}^N \leq \lambda_n^D, n \in N$ for arbitrary domains in R^2 of class C^1 without any restriction on the geometry. However, his assumption on the C^1-regularity of the boundary is crucial for his arguments (which are actually given for C^∞ -domains, referring to a general approximation result of C^1-domains by C^∞ -domains with convergence of the corresponding eigenvalues in [2]). In view of the preceding diverse results involving geometric and regularity assumptions one may wonder whether the C^1 -assumption is optimal in Friedlander's, even though some hypothesis on Ω is needed to guarantee that the Neumann Laplacian has compact resolvent. Mazzeo'91 proved that the analogue of Friedlander's result is valid for all compact domains in a symmetric space of non-compact type [11]. A more recent result is taken by Filonov [3] that $d \geq 2$, a domain $\Omega \subset R^d$ is such that the embedding $H^1(\Omega) \subset L_2(\Omega)$ is compact, and the measure of Ω is finite, $|\Omega| < \infty$, then $\lambda_{n+1}^N < \lambda_n^D, n \in N$, in 2010 Frank and Laptev proved that the analogue of Filonov's result is valid for the sub-Laplacian for any domain in the Heisenberg group [4]. Gesztesy and Mitrea [6] extended Friedlander's inequalities between Neumann and Dirichlet Laplacian eigenvalues to those between one type of nonlocal Robin and Dirichlet Laplacian eigenvalues associated with bounded Lipschitz domains, following an approach introduced by Filonov for this type of problems.

Let consider the spectral problem on eigenvalues of the Newton potential in a bounded domain $\Omega \subset R^d, d > 1$ with a boundary $\partial\Omega \in C^{2,\alpha}, \alpha \in (0,1)$

$$u(x) = \lambda \int_\Omega \varepsilon_d(x-y)u(y)dy, \tag{2}$$

where

$$\varepsilon_d(x-y) = \begin{cases} -\frac{1}{2\pi}\ln|x-y|, d=2, \\ \frac{1}{(d-2)\sigma_d}|x-y|^{2-d}, d \geq 3, \end{cases}$$

is a fundamental solution of the Laplace equation i.e., $-\Delta\varepsilon_d(x-y)=\delta(x-y)$ in R^d, δ is the delta-function, $|x-y|=\left[\sum\limits_{k=1}^{d}(x_k-y_k)^2\right]^{1/2}$ is the distance between two points $x=(x_1,...,x_d)$ and $y=(y_1,...,y_d)$ in d-dimensional Euclidean space R^d, $\sigma_d=\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the area of the unit sphere in R^d and Γ is the Gamma function.

In work [7] we explicitly computed the eigenvalues of the Newton potential (2) in the 2-disk and the 3-disk.

We denote eigenvalues of the Newton potential (2) by $\lambda_n^{NP}, n\in N$ and enumerate their eigenvalues in increasing order (with multiplicity taken into account). By using eigenvalue counting functions and some important lemmas for the Newton potential, now we shall compare the eigenvalues of the Newton potential with the Dirichlet eigenvalues and the Neumann eigenvalues in a bounded domain Ω in R^d with a boundary $\partial\Omega\in C^{2,\alpha}, \alpha\in(0,1)$. We obtain the following main result.

Теорема 1 *In a bounded domain $\Omega\subset R^d$, $d\geq 2$ with a boundary $\partial\Omega\in C^{2,\alpha}, \alpha\in(0,1)$ we have*

$$\lambda_n^N<\lambda_n^{NP}\leq\lambda_n^D, n\in N, \tag{3}$$

where λ_n^D, λ_n^N and λ_n^{NP} are the eigenvalues of the Dirichlet problem, the Neumann problem and the Newton potential, respectively.

In the preliminaries section 2, there are proved some important lemmas and we will use these in the proof section 3. In the section 2 we also consider the one-dimensional case in $(0,1)$.

1. Preliminaries

Лемма 1 *For any function $f\in L_2(\Omega)$, $suppf\subset\Omega$ the Newton potential*

$$u=\int\limits_{\Omega}\varepsilon_d(x-y)f(y)dy, \tag{4}$$

satisfies the boundary condition

$$-u(x)+2\int\limits_{\partial\Omega}\frac{\partial\varepsilon_d(x-y)}{\partial n_y}u(y)dS_y-2\int\limits_{\partial\Omega}\varepsilon_d(x-y)\frac{\partial u(y)}{\partial n_y}dS_y=0, x\in\partial\Omega. \tag{5}$$

Conversely, if a function $u\in H^2(\Omega)$ satisfies

$$-\Delta u=f, x\in\Omega \tag{6}$$

and the boundary condition (5), then it determines the Newton potential (4), where $\Omega\in R^d, d>1$ is a bounded domain with boundary $\partial\Omega\in C^{2,\alpha}, \alpha\in(0,1)$ and $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative on the boundary.

First, we assume that $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$.

A direct calculation shows that, for any $x \in \Omega$, we have

$$u(x) = \int\limits_{\Omega} \varepsilon_d(x-y) f(y) dy = -\int\limits_{\Omega} \varepsilon_d(x-y) \Delta_y u(y) dy =$$

$$\int\limits_{\partial\Omega} \Big(-\varepsilon_d(x-y) \frac{\partial u(y)}{\partial n_y} + \frac{\partial \varepsilon_d(x-y)}{\partial n_y} u(y) \Big) dS_y - \int\limits_{\Omega} \Delta_y \varepsilon_d(x-y) u(y) dy =$$

$$u(x) + \int\limits_{\partial\Omega} \Big(\frac{\partial \varepsilon_d(x-y)}{\partial n_y} u(y) - \varepsilon_d(x-y) \frac{\partial u(y)}{\partial n_y} \Big) dS_y,$$

where $\frac{\partial}{\partial n_y} = n_1 \frac{\partial}{\partial y_1} + ... + n_n \frac{\partial}{\partial y_n}$ is the outer normal derivative and $n_1, ..., n_n$ are the components of the unit normal.

This implies

$$\int\limits_{\partial\Omega} \Big(\frac{\partial \varepsilon_d(x-y)}{\partial n_y} u(y) - \varepsilon_d(x-y) \frac{\partial u(y)}{\partial n_y} \Big) dS_y = 0, \;\; x \in \Omega. \tag{7}$$

Applying properties of the double-layer potential and single-layer potential to (7) with $x \to \partial\Omega$, we obtain

$$-\frac{u(x)}{2} + \int\limits_{\partial\Omega} \Big(\frac{\partial \varepsilon_d(x-y)}{\partial n_y} u(y) - \varepsilon_d(x-y) \frac{\partial u(y)}{\partial n_y} \Big) dS_y = 0, \; x \in \partial\Omega. \tag{8}$$

i.e. (8) is a boundary condition for the Newton potential (4). Next, it is easy to show by passing to the limit that relation (8) remains valid for all $u \in H^2(\Omega)$. Thus, the Newton potential (4) satisfies boundary condition (5).

Conversely, if a function $u_1 \in H^2(\Omega)$ satisfies the equation $-\Delta u_1 = f$ and boundary condition (5), then it coincides with the Newton potential (4). Indeed, if this is not so, then the function $v = u - u_1 \in H^2(\Omega)$, where u is the Newton potential (4), satisfies the homogeneous equation $\Delta v = 0$ and the boundary condition

$$-\frac{v(x)}{2} + \int\limits_{\partial\Omega} \Big(\frac{\partial \varepsilon_d(x-y)}{\partial n_y} v(y) - \varepsilon_d(x-y) \frac{\partial v(y)}{\partial n_y} \Big) dS_y = 0, \; x \in \partial\Omega. \tag{9}$$

As above, applying the Green formula to $v \in H^2(\Omega)$, we see that $\int\limits_{\Omega} \varepsilon_d(x-y) \Delta_y v(y) dy = -v(x) + \int\limits_{\partial\Omega} \Big(\frac{\partial \varepsilon_d(x-y)}{\partial n_y} v(y) - \varepsilon_d(x-y) \frac{\partial v(y)}{\partial n_y} \Big) dS_y = 0, \;\; \forall x \in \Omega.$

Passing to the limit as $x \to \partial\Omega$, we obtain

$$-v(x) - \frac{v(x)}{2} + \int\limits_{\partial\Omega} \Big(\frac{\partial \varepsilon_d(x-y)}{\partial n_y} v(y) - \varepsilon_d(x-y) \frac{\partial v(y)}{\partial n_y} \Big) dS_y = 0$$

Here from (9) we get

$$v(x)|_{x\in\partial\Omega}=0. \tag{10}$$

By virtue of the uniqueness of a solution to the Dirichlet problem for the Laplace equation, we have $v(x)=u(x)-u_1(x)=0$ for any $x\in\Omega$, i.e. $u_1=u$, u_1 coincides with the Newton potential. This completes the proof of Lemma 1.

It follows from Lemma 1, the spectral problem on eigenvalues of the Newton potential (2) is equivalent to the boundary value spectral problem

$$-\Delta u=\lambda u, x\in\Omega \tag{11}$$

$$-u(x)+2\int\limits_{\partial\Omega}\frac{\partial\varepsilon_d(x-y)}{\partial n_y}u(y)dS_y-2\int\limits_{\partial\Omega}\varepsilon_d(x-y)\frac{\partial u(y)}{\partial n_y}dS_y=0, x\in\partial\Omega. \tag{12}$$

Remark 1. The operator (11)-(12) is self-adjoint as its inverse (2) is self-adjoint operator. Hereafter, we denote the self-adjoint operator (11)-(12) by $-\Delta_\Omega^{NP}$.

Remark 2. It is easy to check the nonlocal boundary value problem (11), (12) is not equivalent to another regular boundary value problem for Laplacian such as Neumann, Robin and so on. For example if the nonlocal boundary value problem (11), (12) is equivalent a Robin-type boundary value problem then at least the following inequalities are known [see, for example, 2, also cf. 6] $\lambda_n^N\leq\lambda_n\leq\lambda_n^D, n\in N$ but we consider completely different case.

Лемма 2 *Let $d>2$. If u is the Newton potential (4) then*

$$\int\limits_{\partial\Omega}\frac{\partial u}{\partial n}\overline{u}dS\leq 0. \tag{13}$$

From (4) we have

$$\Delta u(x)=0, x\in R^d\setminus\Omega, \tag{14}$$

$$u(x)=O\left(\frac{1}{|x|^{\frac{d-1}{2}}}\right), |x|\to\infty, d>2. \tag{15}$$

We can then compute by using (14) and the Green's first formula

$$0=\int\limits_{R^d\setminus\Omega}\Delta u\cdot\overline{u}dx=-\int\limits_{R^d\setminus\Omega}|\nabla u|^2dx+\int\limits_{\partial\Omega}\frac{\partial u}{\partial n^-}\overline{u}dS+\lim_{r\to\infty}\int\limits_{\partial\Omega_r}\frac{\partial u}{\partial n}\overline{u}dS,$$

where $\partial\Omega_r=\{x\in R^d,|x|=r\}$ is a d-sphere and $\frac{\partial}{\partial n^-}$ denotes the inner normal derivative on the boundary $\partial\Omega$ i.e., $\frac{\partial}{\partial n^-}=-\frac{\partial}{\partial n}$.

From (15), we have

$$\lim_{r\to\infty}\int\limits_{\Omega_r}\frac{\partial u}{\partial n}\overline{u}dS=0.$$

It follows

$$\int\limits_{\partial\Omega} \frac{\partial u}{\partial n}\overline{u}dS = -\int\limits_{R^d\setminus\Omega} |\nabla u|^2 dx \leq 0.$$

Remark 3. In the case $d = 2$ in a disk by using explicit formula of eigenvalues [see 7] we see that (3) is valid then we prove Theorem 1 with help of the conform mapping.

Лемма 3 *We have that*

$$H^1_{NP}(\Omega) \cap ker(-\Delta^N_\Omega - \mu) = \{0\}, \mu > 0, \tag{16}$$

where $H^1_{NP}(\Omega)$ is a set of functions $u \in H^1(\Omega)$ which satisfy the boundary condition (5) of the Newton potential, $-\Delta^N_\Omega$ is the Neumann operator.

if $u \in H^1_{NP}(\Omega) \cap ker(-\Delta^N_\Omega - \mu)$ then it follows from Lemma 1

$$-u(x) + 2\int\limits_{\partial\Omega} \frac{\partial\varepsilon_d(x-y)}{\partial n_y}u(y)dS_y = 0, x \in \partial\Omega. \tag{17}$$

Hence (17) is the second-kind Fredholm equation, it has the unique solution [see 10]

$$u(x) = 0, x \in \partial\Omega. \tag{18}$$

As $H^1_0(\Omega) \cap ker(-\Delta^N_\Omega - \mu) = \{0\}$, [see 3 or 8] $\mu > 0$, we get $u = 0$.

Лемма 4 *[see 10] We have*

$$\frac{\partial u(x)}{\partial n_x} = \frac{\partial U}{\partial n_x}, x \in \partial\Omega \in C^{2,\alpha}, \alpha \in (0,1) \tag{19}$$

where $U := 2\int\limits_{\partial\Omega} \frac{\partial\varepsilon_d(x-y)}{\partial n_y}u(y)dS_y - 2\int\limits_{\partial\Omega} \varepsilon_d(x-y)\frac{\partial u(y)}{\partial n_y}dS_y$ and u is the Newton potential (2). And (19) is solvable according to

$$\frac{\partial u(x)}{\partial n_x} = V(x,u), x \in \partial\Omega,$$

function V is defined on x and u on $\partial\Omega$.

d=1

Consider the spectral problem for the one-dimensional Newton potential ($d = 1$)

$$u(x) = \lambda\int\limits_0^1 -\frac{1}{2}|x-t|u(t)dt. \tag{20}$$

And we have

$$-\frac{d^2u}{dx^2} = \lambda u.$$

Integrating by part, we obtain

$$u(x) = \lambda \int_0^1 -\frac{1}{2}|x-t|u(t)dt = \int_0^1 \frac{1}{2}|x-t|u''(t)dt$$

$$= \frac{1}{2}\left[\int_0^x (x-t)u''(t)dt - \int_x^1 (x-t)u''(t)dt\right]$$

$$= u(x) - x\frac{u'(0)+u'(1)}{2} - \frac{-u'(1)+u(0)+u(1)}{2}.$$

Thus,

$$x(u'(0)+u'(1)) + (-u'(1)+u(0)+u(1)) = 0, \forall x \in (0,1).$$

Therefore, the boundary conditions for the one-dimensional Newton potential are $u'(0) + u'(1) = 0, -u'(1) + u(0) + u(1) = 0$.

So the spectral problem for the one-dimensional Newton potential is equivalent to the following boundary value spectral problem

$$-\frac{d^2u}{dx^2} = \lambda u, x \in (0,1) \tag{21}$$

with boundary conditions

$$u'(0) + u'(1) = 0, -u'(1) + u(0) + u(1) = 0. \tag{22}$$

Solving the boundary value spectral problem (21), (22) we find two series of eigenvalues

$$\lambda_{1k}^{NP} = ((2k-1)\pi)^2 \text{ and } \lambda_{2k}^{NP} = 4z_k^2 \text{ where } cotz_k = -z_k, k \in N$$

We enumerate these eigenvalues in increasing order and denote by $\lambda_n^{NP}, n \in N$.

Теорема 2 *If $d = 1$, we have*

$$\lambda_n^N < \lambda_n^{NP} \leq \lambda_n^D, n \in N, \tag{23}$$

clearly

$$\lambda_{n+1}^N = \lambda_n^{NP} = \lambda_n^D, n = 2l-1, l \in N,$$
$$\lambda_n^{NP} < \lambda_{n+1}^N = \lambda_n^D, n = 2l, l \in N.$$

Corollary 1. Theorem 1 is also valid for the one-dimensional case.

Short proof of Theorem 2. a) Dirichlet boundary conditions:

$$-\nu'' = \lambda^D \nu, x \in (0,1), \tag{24}$$

with $\nu(0) = 0 = \nu(1)$. (24) has eigenvalues

$$\lambda_n^D = (n\pi)^2, n \in N,$$

b) Neumann boundary conditions:

$$-w'' = \lambda^N w, x \in (0,1), \tag{25}$$

with $w'(0) = 0 = w'(1)$. (25) has eigenvalues

$$\lambda_n^N = ((n-1)\pi)^2, n \in N.$$

c) Newton potential boundary conditions:

$$-u'' = \lambda^{NP} u, x \in (0,1), \tag{26}$$

with $u'(0) + u'(1) = 0, -u'(1) + u(0) + u(1) = 0$. (26) has eigenvalues

$$\lambda_1^{NP} = \min_{k \in N}(((2k-1)\pi)^2, 4z_k^2),$$

$$\lambda_n^{NP} = \min_{\{(((2k-1)\pi)^2, 4z_k^2)\} \setminus \{\lambda_i^{NP}\}_1^{n-1}}(((2k-1)\pi)^2, 4z_k^2), cotz_k = -z_k, n > 1.$$

Fourier analysis shows that the eigenfunctions form a basis. Furthermore, all eigenvalues λ_n^D, λ_n^N and λ_n^{NP} are positive and $\lambda_n^D, \lambda_n^N, \lambda_n^{NP} \to \infty$ and the whole spectrum is discrete. From a), b) and c) it is easy to check (23).

2. Proof of Theorem 1.

Now we are in the position to prove Theorem 1 when $n > 2$ (see Remark 3 and Theorem 2). We introduce the following counting function for the self-adjoint boundary value problem (11)-(12) i.e., for the Newton potential (2)

$$N_{NP}(\lambda) = N(\lambda, -\Delta_\Omega^{NP})$$

$$= \max(\dim L : L \subset H_{NP}^1(\Omega), \int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \overline{u} dS \leq \lambda \int_\Omega |u|^2 dx, u \in L) \tag{27}$$

and we also introduce the counting function for the Neumann Laplacian

$$N_N(\lambda) = N(\lambda, -\Delta_\Omega^N) = \max(\dim L : L \subset H^1(\Omega), \int_\Omega |\nabla u|^2 dx \leq \lambda \int_\Omega |u|^2 dx, u \in L) \tag{28}$$

Let Q be the subspace of $H_{NP}^1(\Omega)$ (see Lemma 3) such that

$$\dim Q = N_{NP}(\mu), \int_\Omega |\nabla u|^2 dx \leq \mu \int_\Omega |u|^2 dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} \overline{u} dS$$

for $u \in Q, \mu > 0$. Consider the following direct sum

$$P = Q \dot{+} ker(-\Delta_\Omega^N - \mu).$$

Clearly $P \subset H^1(\Omega)$. Next, we take a vector $u + \nu \in P$, where $u \in Q$ and $\nu \in ker(-\Delta_\Omega^N - \mu)$.

We can then compute

$$\int\limits_{\Omega} |\nabla(u+\nu)|^2 dx = \int\limits_{\Omega} (|\nabla u|^2 + |\nabla \nu|^2) dx + 2Re(\int\limits_{\Omega} \nabla \nu \overline{\nabla u} dx) = I_1 + I_2.$$

According to Lemma 2

$$I_1 = \int\limits_{\Omega} (|\nabla u|^2 + |\nabla \nu|^2) dx \leq \mu \int\limits_{\Omega} (|u|^2 + |\nu|^2) dx + \int\limits_{\partial\Omega} \frac{\partial u}{\partial n} \overline{u} dS \leq \mu \int\limits_{\Omega} (|u|^2 + |\nu|^2) dx.$$

Hence,

$$I_2 = 2Re(\int\limits_{\Omega} \nabla \nu \overline{\nabla u} dx) = 2Re(\int\limits_{\Omega} (-\Delta \nu) \overline{u} dx + \int\limits_{\partial\Omega} \frac{\partial \nu}{\partial n} \overline{u} dS) = 2\mu Re(\int\limits_{\Omega} \nu \overline{u} dx).$$

Thus, altogether,

$$\int\limits_{\Omega} |\nabla(u+\nu)|^2 dx \leq \mu \int\limits_{\Omega} |u+\nu|^2 dx.$$

It means

$$N_N(\mu) \geq \dim P$$

Since Q is a subspace of $H^1_{NP}(\Omega)$, Lemma 3 asserts that Q and $ker(-\Delta^N_\Omega - \mu)$ are disjoint, thus

$$N_N(\mu) \geq \dim P = N_{NP}(\mu) + \dim ker(-\Delta^N_\Omega - \mu).$$

If we set $\mu = \lambda^N_n$ then we see that

$$N_N(\lambda^N_n) \geq N_{NP}(\lambda^N_n) + 1,$$

what means $\lambda^N_n < \lambda^{NP}_n$.

According to Lemma 4, we can also introduce the following counting function for the self-adjoint boundary value problem (11)-(12) (see [6], Lemma 1)

$$N_{NP}(\lambda) = N(\lambda, -\Delta^{NP}_\Omega) =$$

$$\max(\dim L : L \subset H^1(\Omega), \int\limits_{\Omega} |\nabla u|^2 dx - \int\limits_{\partial\Omega} V(x,u)\overline{u(x)} dS_x \leq \lambda \int\limits_{\Omega} |u|^2 dx, u \in L) \qquad (29)$$

and we introduce the counting function for the Dirichlet Laplacian

$$N_D(\lambda) = N(\lambda, -\Delta^D_\Omega) = \max(\dim L : L \subset H^1_0(\Omega), \int\limits_{\Omega} |\nabla u|^2 dx \leq \lambda \int\limits_{\Omega} |u|^2 dx, u \in L) \quad (30)$$

From (29) and (30) we can see that $\lambda^{NP}_n \leq \lambda^D_n$.

Theorem 1 is completely proved.

Conclusion. We proved the eigenvalues of the Newton potential with the Dirichlet eigenvalues and the Neumann eigenvalues in a bounded domain in R^d. First we show that the spectral

problem of the Newton potential is equivalent to a spectral problem of a non-local boundary value problem of the Laplace operator then it is proved that the nth eigenvalue of the Laplacian with the non-local boundary condition is between the nth eigenvalue of Neumann Laplacian and the nth eigenvalue of Dirichlet Laplacian in a bounded domain of any dimensional euclidian space.

Acknowledgements. The author is grateful to Ari Laptev for helpful discussions and correspondence.

References

[1] *Aviles P.* Symmetry theorems related to Pompeiu's problem // Amer. J. Math.- 1986. - Vol. 108 . - P. 1023–1036.

[2] *Courant R., Hilbert D.* Methods of Mathematical Physics. - New York: Interscience., 1953. - 239 p.

[3] *Filonov N.* On an inequality between Dirichlet and Neumann eigenvalues for the Laplace Operator // St. Petersburg Math. J. - 2005. - Vol. 16, №2. - P. 413–416.

[4] *Frank R.L., Laptev A.* Inequalities between Dirichlet and Neumann eigenvalues on the Heisenberg group // Internat. Math. Res. Notices - 2010. - Vol. 15. - P. 2889–2902.

[5] *Friedlander L.* Some inequalities between Dirichlet and Neumann eigenvalues // Arch. Rational Mech. Anal. - 1991. - Vol. 116. - P. 150–160.

[6] *Gesztesy F., Mitrea M.* Nonlocal Robin Laplacians and some remarks on a paper by Filonov on eigenvalue inequalities // J. Diff. Eq. - 2009. - Vol. 247. - P. 2871–2896.

[7] *Kal'menov T.Sh., Suragan D.* To Spectral Problems for the Volume Potential // Doklady Mathematics. -2008. - Vol. 80, №2. - P. 646–649.

[8] *Laptev A.* Spectral Inequalities, Lectures.- London: - Imperial College London, 2010. - 48 p.

[9] *Levine H. A.* Weinberger H.F. Inequalities between Dirichlet and Neumann eigenvalues // Arch. Rational Mech. Anal. - 1986. - Vol. 94. - P. 193–208.

[10] *Mazzeo R.* Remarks on a paper of Friedlander concerning inequalities between Dirichlet and Neumann eigenvalues // Internat. Math. Res. Notices - 1991. - Vol. 4. - P. 41–48.

[11] *Szego R.* Inequalities for certain eigenvalues of a membrane of given area // J. Rational Mech. Anal. - 1954. - Vol. 3. - P. 343–356.

[12] *Payne L. E.* Inequalities for certain eigenvalues of a membrane of given area // J. Rational Mech. Anal. - 1955. - Vol. 4. - P. 517–529.

Поступила в редакцию 13 ноября 2012 года