1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

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SOLVABILITY OF INVERSE PROBLEM OF A PSEUDOPARABOLIC EQUATION WITH FRACTIONAL CAPUTO DERIVATIVE

Inverse problem on recovering the coefficient of the right-hand side for a pseudoparabolic equation with a Caputo fractional derivative is studied. Overdetermination condition of the inverse problem is given in integral form. Existence and uniqueness theorems are proved for regular solutions (i.e., having all Sobolev generalized derivatives entering the equation) for a pseudoparabolic equation with the Caputo fractional derivative. Also, we propose an algorithm for numerical solution of the considered inverse problem. Numerical experiments are carried out for a one-dimensional problem, illustrating the obtained theoretical results. Inverse problems with fractional derivatives belong to the class of problems that are associated with determining unknown parameters or functions in mathematical models described by equations with fractional derivatives. Such problems arise in various applications where models with fractional derivatives are used, for example, in mechanics, heat conductivity, biology, finance and other areas.
Key words: inverse problem, pseudoparabolic equation, Caputo fractional derivative, Galerkin approximations, regular solution, numerical solution, numerical experiments.

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Бөлшек туындылы псевдопараболалық теңдеу үшін кері есептің шешімділігі

Капуто бөлшек туындылы псевдопараболалық теңдеудің оң жақтағы коэффициентін қалпына келтіру кері есебі зерттелген. Кері есепте қосымша шарты интегралдық түрде берілген. Капуто бөлшек туындылы псевдопараболалық теңдеу үшін кері есептің регулярлы шешімнің (яғни, теңдеуге қатысатын барлық жалпыланған Соболев туындылары үшін) бар және жалғыздығы туралы теоремалар дәлелденген. Қарастырылып отырған кері есептің сандық шешу алгоритмі де ұсынылған. Алынған теориялық нәтижелерді суреттейтін бір өлшемді кері есеп үшін сандық эксперименттер жүргізілді. Бөлшек туындылы кері есептер бөлшек туындылы теңдеулер арқылы сипатталған математикалық модельдердегі белгісіз параметрлерді немесе функцияларды анықтаумен байланысты есептер класына жатады. Мұндай мәселелер механика, жылуөткізгіштік, биология, қаржы және басқа да салалар бөлшек туынды модельдерді пайдаланатын әртүрлі қолданбалы облыстарда пайда болады.
Түйін сөздер: кері есеп, псевдопараболалық теңдеу, Капуто бөлшек туындысы, Галеркин жуықтаулары, регулярлық шешім, сандық шешім, сандық тәжірибелер.

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Разрешимость обратной задачи псевдопараболического уравнения с дробной производной Капуто

Исследована обратная задача по восстановлению коэффициента правой части для псевдопараболического уравнения с дробной производной Капуто. Условие переопределения обратной задачи приведено в интегральной форме. Доказаны теоремы существования и единственности регулярных решений (т.е. имеющих входящие в уравнение все обобщенные производные Соболева) для псевдопараболического уравнения с дробной производной Капуто. Также предложен алгоритм численного решения рассматриваемой обратной задачи. Проведены численные эксперименты для одномерной задачи, иллюстрирующие полученные теоретические результаты. Обратные задачи с дробными производными относятся к классу задач, которые связаны с определением неизвестных параметров или функций в математических моделях, описываемых уравнениями с дробными производными. Такие задачи возникают в различных приложениях, где используются модели с дробными производными, например, в механике, теплопроводности, биологии, финансах и других областях.
Ключевые слова: обратная задача, псевдопараболическое уравнение, дробная производная Капуто, приближения Галеркина, регулярное решение, численное решение, численные эксперименты.

1 Introduction

In this paper, we consider inverse problems for a pseudo-parabolic equation with an integral overdetermination condition. Local solvability of the inverse problem for a pseudo-parabolic equation with fractional Caputo derivative is proved. In addition, numerical solutions to the inverse problem are obtained in this paper. In a particular case, the results make it possible to solve the problem of determining the intensity of liquid between cracks and pores in the model equation of filtration in a fractured medium.

In this paper, a fractional differential mathematical model of the geofiltration process based on concepts of the Caputo derivative is considered. The use of this derivative in the fractional differential dynamics of filtration processes in porous media makes it possible to study filtration models of greater generality. We mainly consider theoretical results on inverse problems for pseudo-parabolic equations with time-fractional Caputo derivatives of the $\alpha \in (0, 1]$ order.

The study of pseudo-parabolic equations begins with the work of S.L. Sobolev [1]. This work aroused the greatest interest in the study of nonclassical equations. The term Sobolev-type equations was introduced by E.R. Showalter [2, 3].

The Sobolev-type equations, referred to as pseudo-parabolic equations, arise in the description of filtration processes, heat and mass transfer processes, wave processes, hydrodynamics, and in other areas. A large number of works [1-10] are devoted to the study solvability of initial-boundary value problems for Sobolev-type pseudo-parabolic equations.

Theory and numerical analysis of inverse problems is an actively developing area of modern mathematics. Unlike the classical boundary value problems of mathematical physics, in inverse problems, along with an unknown function that satisfies the considered equation, the coefficient (coefficients) of the equation, or coefficient that determines the right side of the equation, is found. For the correctness of these problems, as a rule, along with the boundary value and initial conditions, some other conditions, called the overdetermination conditions, are also specified.

The study of inverse problems for pseudo-parabolic equations began in the 1980s, and, today, is a relevant and popular area of science. Inverse problems with final time observation have been well studied, and many theoretical studies have been published for classical partial differential equations. As a monograph, reference should be made to the works [11-17].

In the paper of A.Sh. Lyubanova [18], coefficient inverse problems have been studied for a pseudo-parabolic equation of the form:

$$(u + L_1 u)_t + L_2 u = f$$

with differential operators L_1 and L_2. Equations of this type arise in modeling heat transfer processes, filtration processes in fractured media, in mathematical models of two-phase fluid filtration in porous media, with capillary effects. The leading coefficients of the operators L_1 and L_2 describe a physical property of the medium (permeability, compressibility, thermal or electrical conductivity, etc.).

In [19], the existence and uniqueness of regular solutions to the linear inverse problem of the Sobolev-type equation are proved.

In [20], the inverse problem for a pseudo-parabolic equation with p-Laplacian is considered. The existence of a weak solution is proved, and the asymptotic behavior of solutions is shown as $t \to \infty$. Sufficient conditions for the "blow up"of a solution in a finite time are obtained, as well as sufficient conditions for the disappearance of a solution in a finite time are obtained.

Inverse problems for the Sobolev-type equation were studied in many works of scientists, such as A.I.Kozhanov, A.Sh.Lyubanova, S.N.Antontsev, H.Khompysh, M.Shahrouzi, J.Ferreira, E.Pişkin and others [18-24].

Recently, there has been a growing interest in inverse problems with fractional derivatives. Usually, in these works, the fractional time derivative is considered, we note some of them [25-29] and many others.

In [28], the coefficient inverse problem for a nonlinear diffusion equation with a time-fractional derivative is considered:

$$\frac{\partial^\beta u}{\partial t^\beta} = \nabla \cdot (a(u)\nabla u) + f(x,t), \quad (x,t) \in \Omega_T,$$

$$u(x,0) = 0, \quad x \in \bar{\Omega},$$

$$u(x,t) = 0, \quad (x,t) \in \Gamma_1 \times [0,T], \quad \Gamma_1 \subset \partial\Omega,$$

$$\begin{array}{c} a(u)\frac{\partial u}{\partial n} = \varphi(x,t), \quad (x,t) \in \Gamma_2 \times [0,T], \quad \Gamma_2 \subset \partial\Omega, \\ u(x,t) = g(x,t), \quad (x,t) \in \Gamma_2 \times [0,T], \quad \Gamma_2 \subset \partial\Omega, \end{array}$$

where $\Omega_T = \Omega \times (0,T)$, the domain $\Omega \subset R^n$ $(n \geq 1)$ is assumed to be bounded simply connected with a piecewise smooth boundary Γ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \Gamma$, $meas(\Gamma_i) \neq 0$, $i = 1, 2$.

Firstly, the uniqueness of the direct problem is proved. Then continuous dependence of the solution of the corresponding direct problem on the coefficient is proved. After that, the existence of a quasi-solution of the inverse problem in the corresponding class of admissible coefficients is proved.

In [29], the inverse problem is studied for pseudo-parabolic equations with the final overdetermination condition:

$$D_t^\alpha[u(t) + Lu(t)] + Mu(t) = f(t) \ \ in \ \ H,$$

$$u(0) = \varphi \in H,$$
$$u(T) = \psi \in H,$$

where H is a separable Hilbert space, L and M are self-adjoint operators in H. The existence and uniqueness of a solution to this inverse problem in the Hilbert space is proved.

It is necessary to note that the works [30-45] are also dedicated to solvability and numerical methods for differential equations of fractional order. In [30-32], [38], [39], [43], [44], using the fixed-point theorem, the solvability of local and nonlocal boundary value problems for fractional Caputo differential equations and some fractional differential equations.

Inverse problems for quasi-linear and non-linear equations with fractional deriva-tives are not fully investigated, and we hope that this work partially fills it.

2 Problem statement.

In a cylinder $Q_T = \{(x,t) : \ x \in \Omega, \ \Omega \subset R^n, \ 0 < t < T\}$, we consider inverse problem for a pseudo parabolic equation with a non-local overdetermination condition. Find a pair of functions $\{u(x,t), \ f(t)\}$, which satisfy:

$$D_{0,t}^\alpha(u - \chi\Delta u) - a\Delta u + c(x,t)u = b(x,t)|u|^{p-2}u + |\nabla u|^q + f(t)h(x,t), \tag{1}$$

the initial condition

$$u(x,0) = u_0(x), \tag{2}$$

the boundary value condition

$$u|_S = 0, \tag{3}$$

and integral overdetermination condition:

$$\int_\Omega u(x,t)(\omega(x) - \chi\Delta\omega)dx = \varphi(t). \tag{4}$$

Here $\Omega \subset R^n$, $n \geq 1$, is a bounded domain, boundary $\partial\Omega$ is smooth enough, $u_0(x)$, $c(x,t)$, $b(x,t)$, $h(x,t)$ and $\omega(x)$ are known functions, χ, a, p and q are positive constants, $D_{0,t}^\alpha$ is a fractional Caputo derivative of the order $0 < \alpha < 1$:

$$D_{0,t}^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_s(x,s)}{(t-s)^\alpha}ds, & 0 < \alpha < 1, \\ u_t(x,t), & \alpha = 1. \end{cases}$$

The functions $u_0(x),\ c(x,t),\ b(x,t),\ h(x),\ \omega(x)$ satisfy the following conditions:

$$\begin{array}{l} 0 < c_1' \le c(x,t) \le c_2' < \infty,\ \ \forall (x,t) \in Q_T, \\ 0 \le b(x,t) \le b_1 < \infty,\ \ \forall (x,t) \in Q_T, \\ h(x,t) \in L_\infty \left(0,T;\ \in L_2(\Omega) \cap L_{\frac{p}{p-2}}(\Omega) \cap L_{\frac{p}{p-1}}(\Omega)\right),\ \ p>2, \\ h_1(t) \equiv \int_\Omega h(x,t)\omega(x)dx \neq 0,\ \ \forall t \in [0,T], \\ h_1(t) \in L_\infty(0,T), \\ \omega \in \overset{0}{W}{}_2^1(\Omega) \cap L_{\frac{p}{p-1}}(\Omega) \cap L_p(\Omega),\ p>2, \\ \varphi(t),\ D^\alpha_{0,t}\varphi(t) \in L_\infty(0,T),\ \ u_0 \in \overset{0}{W}{}_2^1(\Omega) \cap W_2^2(\Omega). \end{array} \tag{5}$$

3 Main Part

In this inverse problem for constants p and q, we consider various cases:
1^{st} case, when $0 < q \le 1,\ 2 < p < \frac{2N}{N-2},\ N \ge 3.$
2^{nd} case, when $1 \le q \le 1 + \frac{2}{N},\ 2 < p < \frac{2N}{N-2},\ N \ge 3.$
By $V_2^\alpha(Q_T),\ 0<\alpha<1$, we denote a space with the norm

$$\begin{array}{l} \|u\|^2_{V_2^\alpha(Q_T)} = \|u\|^2_{L_\infty(0,T;\ L_2(\Omega))} + \left\|D^\alpha_{0,t}u\right\|^2_{L_\infty(0,T;\ L_2(\Omega))} + \\ + \|u\|^2_{L_\infty(0,T;\ \overset{0}{W}{}_2^1(\Omega))} + \|u\|^2_{L_\infty(0,T;\ W_2^2(\Omega))} + \\ + \left\|D^\alpha_{0,t}u\right\|^2_{L_\infty(0,T;\ \overset{0}{W}{}_2^1(\Omega))} + \left\|D^\alpha_{0,t}u\right\|^2_{L_\infty(0,T;\ W_2^2(\Omega))}. \end{array}$$

Lemma 1 *The problem* (1) - (4) *is equivalent to the following problem for a nonlinear pseudoparabolic equation containing a nonlinear nonlocal operator of the function* $u(x,t)$

$$\begin{array}{l} D^\alpha_{0,t}(u - \chi\Delta u) - a\Delta u + c(x,t)u = b(x,t)|u|^{p-2}u \\ +|\nabla u|^q + F(t,u)h(x,t),\ \ x \in \Omega,\ \ \ t>0, \end{array} \tag{6}$$

$$u(x,0) = u_0(x),\ \ x \in \Omega,\ \ \ \ u|_S = 0. \tag{7}$$

Here

$$\begin{array}{l} F(t,u) = \frac{1}{h_1(t)}\left(D^\alpha_{0,t}\varphi(t) + a\int_\Omega \nabla u \nabla \omega dx + \int_\Omega c(x,t)u\omega dx \right. \\ \left. - \int_\Omega b(x,t)|u|^{p-2}u\omega dx - \int_\Omega |\nabla u|^q \omega dx\right). \end{array} \tag{8}$$

Proof 1 *Indeed, from* (1) *it follows that*

$$\begin{array}{l} \int_\Omega D^\alpha_{0,t}(u - \chi\Delta u)\omega dx - a\int_\Omega \Delta u\omega dx + \int_\Omega c(x,t)u\omega dx \\ = \int_\Omega b(x,t)|u|^{p-2}u\omega dx + \int_\Omega |\nabla u|^q\omega dx + \int_\Omega f(t)h(x,t)\omega dx, \end{array} \tag{9}$$

then, if the conditions (4) *and* (5) *hold, we get*

$$\begin{array}{l} F(t,u) = \frac{1}{h_1(t)}\left(D^\alpha_{0,t}\varphi(t) + a\int_\Omega \nabla u \nabla \omega dx + \int_\Omega c(x,t)u\omega dx \right. \\ \left. - \int_\Omega b(x,t)|u|^{p-2}u\omega dx - \int_\Omega |\nabla u|^q \omega dx\right). \end{array} \tag{10}$$

Therefore, (10) *holds.*

Now we consider the problem (6) - (7)*. If* (8) *holds, then it obviously implies the equality* (10)*. Thus,*

$$F(t,u)=\frac{1}{h_1(t)}\left(D^{\alpha}_{0,t}\varphi(t)+a\int_\Omega\nabla u\nabla\omega dx+\int_\Omega c(x,t)u\omega dx\right.$$
$$\left.-\int_\Omega b(x,t)|u|^{p-2}u\omega dx-\int_\Omega|\nabla u|^q\omega dx\right)$$
$$=\frac{1}{h_1(t)}\left(D^{\alpha}_{0,t}\varphi(t)-a\int_\Omega\Delta u\omega dx+\int_\Omega c(x,t)u\omega dx\right.$$
$$\left.-\int_\Omega b(x,t)|u|^{p-2}u\omega dx-\int_\Omega|\nabla u|^q\omega dx\right).$$

Due to (9)*, we obtain*

$$F(t,u)=\frac{1}{h_1(t)}\left(D^{\alpha}_{0,t}\varphi(t)+a\int_\Omega\nabla u\nabla\omega dx+\int_\Omega c(x,t)u\omega dx\right.$$
$$\left.-\int_\Omega b(x,t)|u|^{p-2}u\omega dx-\int_\Omega|\nabla u|^q\omega dx\right)$$
$$=\frac{1}{h_1(t)}\left(D^{\alpha}_{0,t}\varphi(t)+\int_\Omega c(x,t)u\omega dx-\int_\Omega D^{\alpha}_{0,t}(u-\chi\Delta u)\omega dx\right.$$
$$+\int_\Omega b(x,t)|u|^{p-2}u\omega dx+\int_\Omega|\nabla u|^q\omega dx-\int_\Omega c(x,t)u\omega dx$$
$$\left.+\int_\Omega f(t)h(x,t)\omega dx-\int_\Omega b(x,t)|u|^{p-2}u\omega dx-\int_\Omega|\nabla u|^q\omega dx\right).$$

$$D^{\alpha}_{0,t}\varphi(t)-\int_\Omega D^{\alpha}_{0,t}(u-\chi\Delta u)\omega dx=0.$$

Thus, $D^{\alpha}_{0,t}\left(\varphi(t)-\int_\Omega(u-\chi\Delta u)\omega dx\right)=0$*. Denote* $v(t)=\varphi(t)-\int_\Omega(u-\chi\Delta u)\omega(x)dx$*. Then the function* $v(t)$ *can be found as a solution of the Cauchy problem:* $D^{\alpha}_{0,t}v(t)=0,\ \ v(0)=0$*.* ($v(0)=0$ *follows from the condition* (5)*). Unique solution of the problem is the function* $v(t)=0$*, consequently,* $\int_\Omega(u-\chi\Delta u)\omega(x)dx=\varphi(t)$*. Lemma is proved.*

Definition 1 *A weak generalized solution of the problem* (6) - (7) *is a function* $u(x,t)$ *from the space* $V^{\alpha}_2(Q_T),\ 0<\alpha<1$ *that satisfies the following integral identity:*

$$\int_\Omega\left(D^{\alpha}_{0,t}(u-\chi\varDelta u)-a\varDelta u+c(x,t)u\right)wdx$$

$$=\int_\Omega b(x,t)|u|^{p-2}uwdx+\int_\Omega|\nabla u|^q wdx+\int_\Omega F(u,t)hwdx,$$
$$\text{almost everywhere } t\in[0,T], \tag{11}$$

$$u(x,0)=u_0(x)\in\overset{0}{W}{}^1_2(\Omega)\cap W^2_2(\varOmega),$$

for all $w(x,t)\in L_2(0,T;\overset{0}{W}{}^1_2(\Omega))$*.*

3.1 Existence of a local solution. Galerkin approximations.

Theorem 1 *Suppose that the condition* (5)*, as well as cases 1 and 2, holds. Then on the interval* $(0,T)$*,* $T<T_0$*, there exists a weak generalized solution* $u(x,t)\in V^{\alpha}_2(Q_T)$ *to the problem* (6) - (7)*.*

Proof 2 *In the space* $\overset{0}{W}{}^1_2(\Omega)\cap W^2_2(\Omega)$*, we choose some system of functions* $\{\Psi_j(x)\}$ *forming a basis in this space. For example,*

$$\begin{cases}\Delta\Psi_j(x)+\lambda_j\Psi_j(x)=0,\\ \Psi_j(x)|_{\partial\Omega}=0.\end{cases}$$

We will look for an approximate solution of the problem (6) - (7) *in the form*

$$u_m(x,t) = \sum_{k=1}^{m} v_{mk}(t)\Psi_k(x), \tag{12}$$

where the coefficients $v_{mk}(t)$ *are searched from the conditions:*

$$\begin{array}{l} \int_\Omega \left(D^\alpha_{0,t} v_{mk}(t)(\Psi_k(x) - \chi\Delta\Psi_k(x)) - a v_{mk}(t)\Delta\Psi_k(x) \right. \\ \left. + c(x,t) v_{mk}(t)\Psi_k(x) \right) \Psi_j(x)dx = \int_\Omega b(x,t)|u_m|^{p-2}u_m\Psi_j(x)dx \\ + \int_\Omega |\nabla u_m|^q \Psi_j(x)dx + \int_\Omega F(u_m,t)h\Psi_j(x)dx, \quad j = 1, ...m, \end{array} \tag{13}$$

$$u_{m0} = u_m(0) = \sum_{k=1}^{m} v_{mk}(0)\Psi_k = \sum_{k=1}^{m} \alpha_k \Psi_k, \tag{14}$$

moreover,

$$u_{m0} \to u_0 \quad \textit{strongly in } \overset{0}{W}{}^1_2(\Omega) \cap W^2_2(\Omega) \textit{ as } m \to \infty. \tag{15}$$

Denote

$$\vec{v}_m \equiv \{v_{1m}(t), ..., v_{mm}(t)\}^T, \vec{\alpha} \equiv \{\alpha_1, ..., \alpha_m\}^T, a_{kj} = \int_\Omega [\Psi_k\Psi_j + \chi\nabla\Psi_k\nabla\Psi_j]\, dx,$$

$$\begin{array}{l} C_{kj} = a\int_\Omega \nabla\Psi_k\nabla\Psi_j dx + \int_\Omega c(x,t)\Psi_k\Psi_j dx, b_{kj} \\ = \int_\Omega b\,(x,t)\,|u_m|^{p-2}\Psi_k\Psi_j dx + \int_\Omega |\nabla u_m|^q \Psi_j dx + \int_\Omega F(u_m,t)h\Psi_j dx, \end{array}$$

$$A_m \equiv \{a_{jk}\}, \vec{C}_m \equiv \{C_{jk}\}, \vec{G}_m(\vec{v}_m) \equiv \{b_{jk}(\vec{v}_m)\}\,\vec{v}_m.$$

Then the system of equations (13) *and the condition* (14) *takes the matrix form*

$$\begin{array}{l} A_m D^\alpha_{0,t}\vec{v}_m + \vec{C}_m\vec{v}_m \equiv \vec{G}_m(\vec{v}_m), \\ \vec{v}_m(0) = \vec{\alpha}. \end{array} \tag{16}$$

The matrix A_m *is invertible. In fact, the quadratic form*

$$\sum_{k,j=1}^{m} a_{kj}\xi_k\xi_j = \int_\Omega |w|^2 dx + \chi\int_\Omega |\nabla w|^2 dx, \quad w = \sum_{l=1}^{m} \xi_l\Psi_l,$$

is equal to zero if and only if $w = 0$. *Considering the positivity of the matrix* A_m*, the problem* (16) *can be reduced to the following form:*

$$D^\alpha_{0,t}\vec{v}_m + A_m^{-1}\vec{C}_m\vec{v}_m \equiv A_m^{-1}\vec{G}_m(\vec{v}_m), \quad \vec{v}_m(0) = \vec{\alpha}. \tag{17}$$

According to Cauchy's theorem, the problem (17) *has at least one solution* $\vec{v}_m$ *in some time interval* $t \in (0, T_m)$, $T_m > 0$. *At the next step, we obtain the a priori estimates which prove that the Cauchy problem* (17) *has the local solution in the interval* $[0, T]$.

Lemma 2 *([41]) For any absolutely continuous function $v(t)$ on $[0,T]$, the following inequality holds:*

$$v(t)D_{0,t}^{\alpha}v(t) \geq \frac{1}{2}D_{0,t}^{\alpha}v^2(t), \quad 0<\alpha<1.$$

Lemma 3 *([44]) If $u \in \overset{0}{W}{}_2^1(\Omega)$, we have the inequality:*

$$\|u\|_{\sigma,\Omega}^2 \leq C^2 \|\nabla u\|_{2,\Omega}^{2\alpha} \|u\|_{2,\Omega}^{2(1-\alpha)} \leq \chi \|\nabla u\|_{2,\Omega}^2 + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} C_0^{\frac{2}{1-\alpha}}}{\chi^{\frac{\alpha}{1-\alpha}}} \|u\|_{2,\Omega}^2 ,$$

where $\alpha = \frac{(\sigma-2)N}{2\sigma}, \ \alpha<1, 2<\sigma<\frac{2N}{N-2}, \ N\geq 3.$

Lemma implies the following inequality:

$$\|u\|_{\sigma,\Omega}^{\sigma} \leq C_1 \left(\|u\|_{2,\Omega}^2 + \chi \|\nabla u\|_{2,\Omega}^2\right)^{\frac{\sigma}{2}}, \ 2<\sigma<\frac{2N}{N-2}, \ \ N\geq 3,$$

where $C_1 = \left(\max\left\{1; \ \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} C_0^{\frac{2}{1-\alpha}}}{\chi^{\frac{\alpha}{1-\alpha}}}\right\}\right)^{\frac{\sigma}{2}}.$

3.2 A priori estimates of Galerkin approximations.

We multiply (13) by $v_{mj}(t)$ and sum up both parts of the obtaining equality by $j=1,...,m$. As a result, we get:

$$\begin{aligned}&\int_\Omega u_m D_{0,t}^{\alpha}u_m dx + \chi\int_\Omega \nabla u_m D_{0,t}^{\alpha}\nabla u_m dx + a\int_\Omega |\nabla u_m|^2 dx \\ &+ \int_\Omega c(x,t)|u_m|^2 dx = \int_\Omega b(x,t)|u_m|^p dx + \int_\Omega |\nabla u_m|^q u_m dx + \int_\Omega F(u_m,t)hu_m dx.\end{aligned}$$

By using Lemma 3.4 and the condition (5), we get the inequality:

$$\begin{aligned}&\tfrac{1}{2}D_{0,t}^{\alpha}\int_\Omega [|u_m|^2 + \chi|\nabla u_m|^2]dx + a\int_\Omega |\nabla u_m|^2 dx + c_1'\int_\Omega |u_m|^2 dx \\ &\leq \int_\Omega b(x,t)|u_m|^p dx + \int_\Omega |\nabla u_m|^q u_m dx + \int_\Omega F(u_m,t)hu_m dx.\end{aligned} \quad (18)$$

For the case $0<q\leq 1, \ 2<p<\frac{2N}{N-2}, \ N\geq 3$, we estimate the right-hand side of (18), and apply Lemma 3.5, as well as the Cauchy and Young inequality, we obtain

$$\left|\int_\Omega b(x,t)|u_m|^p dx\right| \leq b_1 \|u_m\|_{p,\Omega}^p \leq C_1 b_1 \left(\|u_m\|_{2,\Omega}^2 + \chi\|\nabla u_m\|_{2,\Omega}^2\right)^{\frac{p}{2}}.$$

$$\begin{aligned}&\left|\int_\Omega |\nabla u_m|^q u_m dx\right| \leq \left(\int_\Omega |\nabla u_m|^2 dx\right)^{\frac{q}{2}} \left(\int_\Omega |u_m|^{\frac{2}{2-q}} dx\right)^{\frac{2-q}{2}} \\ &\leq \|\nabla u_m\|_{2,\Omega}^q |\Omega|^{\frac{1-q}{2}} \|u_m\|_{2,\Omega} \leq C_2' \|\nabla u_m\|_{2,\Omega}^{q+1} \leq \tfrac{a}{6}\|\nabla u_m\|_{2,\Omega}^2 + C_2.\end{aligned}$$

$$\left|\frac{1}{h_1}\int_\Omega D_{0,t}^{\alpha}\varphi(t)h(x,t)u_m dx\right| \leq \frac{|D_{0,t}^{\alpha}\varphi|}{|h_1|}\|h\|_{2,\Omega}\|u_m\|_{2,\Omega} \leq \frac{c_1'}{2}\|u_m\|_{2,\Omega}^2 + \frac{|D_{0,t}^{\alpha}\varphi|^2}{2c_1' h_1^2}\|h\|_{2,\Omega}^2.$$

$$\left|\frac{1}{h_1}a\int_\Omega \nabla u_m \nabla\omega dx\int_\Omega h(x,t)u_m dx\right| \le \frac{1}{|h_1|}\left\|\nabla u_m\right\|_{2,\Omega}\left\|\nabla\omega\right\|_{2,\Omega}\left\|u_m\right\|_{p,\Omega}\left\|h\right\|_{\frac{p}{p-2},\Omega}$$
$$\le \left\|u_m\right\|_{p,\Omega}^p + \frac{(p-2)2^{\frac{2}{p-2}}}{p^{\frac{p}{p-2}}}\cdot\left(\frac{c_2'}{|h_1|}\left\|\omega\right\|_{\frac{p}{p-1},\Omega}\left\|h\right\|_{\frac{p}{p-1},\Omega}\right)^{\frac{p}{p-2}}$$
$$\le C_1\left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right)^{\frac{p}{2}} + \frac{(p-2)2^{\frac{2}{p-2}}}{p^{\frac{p}{p-2}}}\cdot\left(\frac{c_2'}{|h_1|}\left\|\omega\right\|_{\frac{p}{p-1},\Omega}\left\|h\right\|_{\frac{p}{p-1},\Omega}\right)^{\frac{p}{p-2}}.$$

$$\left|\frac{1}{h_1}\int_\Omega b(x,t)|u_m|^{p-2}u_m\omega dx\int_\Omega h(x,t)u_m dx\right| \le \frac{b_1}{|h_1|}\left\|u_m\right\|_{p,\Omega}^p\left\|\omega\right\|_{p,\Omega}\left\|h\right\|_{\frac{p}{p-1},\Omega}$$

$$\le C_1\frac{b_1}{|h_1|}\left\|\omega\right\|_{p,\Omega}\left\|h\right\|_{\frac{p}{p-1},\Omega}\left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right)^{\frac{p}{2}}.$$

$$\left|\frac{1}{h_1}\int_\Omega |\nabla u_m|^q\omega dx\int_\Omega h(x,t)u_m dx\right|$$
$$\le \frac{1}{|h_1|}\left(\int_\Omega |\nabla u_m|^2 dx\right)^{\frac{q}{2}}\left(\int_\Omega |\omega|^{\frac{2}{2-q}}dx\right)^{\frac{2-q}{2}}\left\|h\right\|_{2,\Omega}\left\|u_m\right\|_{2,\Omega}$$
$$\le \left\|\nabla u_m\right\|_{2,\Omega}^q|\Omega|^{\frac{1-q}{2}}\left\|\omega\right\|_{2,\Omega}\left\|h\right\|_{2,\Omega}\left\|u_m\right\|_{2,\Omega}$$

$$\le C_2'\left\|\nabla u_m\right\|_{2,\Omega}^{q+1} \le \frac{a}{6}\left\|\nabla u_m\right\|_{2,\Omega}^2 + C_2.$$

Now, we estimate for the case when $1\le q\le 1+\frac{2}{N},\ 2<p<\frac{2N}{N-2},\ N\ge 3.$

$$\left|\int_\Omega |\nabla u_m|^q u_m dx\right| \le \left(\int_\Omega |\nabla u_m|^{\frac{2Nq}{N+2}}dx\right)^{\frac{N+2}{2N}}\left(\int_\Omega |u_m|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}$$
$$\le C_2'\left\|\nabla u_m\right\|_{2,\Omega}^{q+1} \le \left\|\nabla u_m\right\|_{2,\Omega}^p + C_2 \le \left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right)^{\frac{p}{2}} + C_2.$$

$$\left|\frac{1}{h_1}a\int_\Omega \nabla u_m \nabla\omega dx\int_\Omega h(x,t)u_m dx\right| \le \frac{1}{|h_1|}\left\|\nabla u_m\right\|_{2,\Omega}\left\|\nabla\omega\right\|_{2,\Omega}\left\|u_m\right\|_{p,\Omega}\left\|h\right\|_{\frac{p}{p-2},\Omega}$$
$$\le \frac{a}{2}\left\|\nabla u_m\right\|_{2,\Omega}^2 + C_1\left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right)^{\frac{p}{2}}$$
$$+\frac{(p-2)2^{\frac{2}{p-2}}}{p^{\frac{p}{p-2}}}\cdot\left(\frac{a}{2h_1^2}\left\|\nabla\omega\right\|_{2,\Omega}^2\left\|h\right\|_{\frac{p}{p-2},\Omega}^2\right)^{\frac{p}{p-2}}.$$

$$\left|\frac{1}{h_1}\int_\Omega |\nabla u_m|^q\omega dx\int_\Omega h(x,t)u_m dx\right| \le$$
$$\le \frac{1}{|h_1|}\left(\int_\Omega |\nabla u_m|^{\frac{2Nq}{N+2}}dx\right)^{\frac{N+2}{2N}}\left(\int_\Omega |\omega|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}\left\|h\right\|_{2,\Omega}\left\|u_m\right\|_{2,\Omega}$$
$$\le \left\|\nabla u_m\right\|_{2,\Omega}^q|\Omega|^{\frac{1-q}{2}}\left\|\nabla\omega\right\|_{2,\Omega}\left\|h\right\|_{2,\Omega}\left\|u_m\right\|_{2,\Omega}$$

$$\le \left\|\nabla u_m\right\|_{2,\Omega}^p + C_2 \le \left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right)^{\frac{p}{2}} + C_2.$$

Thus, the obtaining estimates yields the inequality:

$$D_{0,t}^\alpha\int_\Omega[|u_m|^2+\chi|\nabla u_m|^2]dx + C_0\int_\Omega[|u_m|^2+\chi|\nabla u_m|^2]dx$$
$$\le C_3\left(\left\|u_m\right\|_{2,\Omega}^2+\chi\left\|\nabla u_m\right\|_{2,\Omega}^2\right)^{\frac{p}{2}} + C_4. \tag{19}$$

where $C_0 = \min\{a; c_1'\}$.

Denote $y(t) \equiv \|u_m\|_{2,\Omega}^2 + \chi \|\nabla u_m\|_{2,\Omega}^2$, *then* (19) *has the form*

$$D_{0,t}^{\alpha} y(t) + C_0 y(t) \le C_3 [y(t)]^{\frac{p}{2}} + C_4. \tag{20}$$

Hence,

$$\begin{aligned} & y(t) \le E_{\alpha,1}(-C_0 t^{\alpha}) y(0) + C_3 \textstyle\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-C_0 (t-s)^{\alpha}) [y(s)]^{\frac{p}{2}} ds \\ & + \textstyle\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-C_0(t-s)^{\alpha}) C_4 ds, \end{aligned} \tag{21}$$

where $E_{\alpha,\beta}(z)$ *is a Mittag-Leffler function:*

$$\begin{aligned} & E_{\alpha,\beta}(z) = \textstyle\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \\ & E_{\alpha,1}(-\mu t^{\alpha}) = \textstyle\sum_{k=0}^{\infty} \frac{(-1)^k \mu^k t^{\alpha k}}{\Gamma(\alpha k+1)}, \\ & E_{\alpha,\alpha}(-\mu t^{\alpha}) = \textstyle\sum_{k=0}^{\infty} \frac{(-1)^k \mu^k t^{\alpha k}}{\Gamma(\alpha k+\alpha)}. \end{aligned}$$

Lemma 4 *([30]) Mittag-Leffler function has the following properties:*

For $0 < \alpha < 1$ *and* $\mu > 0$, *there exists a constant* $M_1 > 0$, *such that*

$$0 < E_{\alpha,1}(-\mu t^{\alpha}) \le \frac{M_1}{1+\mu t^{\alpha}} \le M_1, \;\; t > 0. \tag{22}$$

For $0 < \alpha < 1$ *and* $\mu > 0$, *there exists a constant* $M > 0$, *such that*

$$0 < t^{\alpha-1} E_{\alpha,\alpha}(-\mu t^{\alpha}) \le M_2 t^{\alpha-1}, \;\; t > 0. \tag{23}$$

Lemma 5 *([45]) Let* $x(t),\ k(t)$ *be positive, continuous functions in* $c \le t \le d$, *and let* $a,\ b$ *be nonnegative constants; further let* $g(u)$ *be a positive nondecreasing function for* $u \ge 0$. *Then the inequality*

$$x(t) \le a + b \int_c^t k(s) g(x(s)) ds, \quad c \le t \le d,$$

implies the inequality

$$x(t) \le H^{-1}\left\{ H(a) + b \int_c^t k(s) ds \right\}, \quad c \le t \le d' \le d,$$

where

$$H(t) = \int_{\varepsilon}^{t} \frac{ds}{g(s)}, \quad (\varepsilon > 0,\ u > 0),$$

and d' *is defined so that* $H(a) + b\int_c^t k(s)ds$ *lies within the domain of definition of* $H^{-1}(u)$, *for* $c \le t \le d'$.

In (21)*, applying* (22) *and* (23)*, we get*

$$y(t) \leq M_1 y(0) + \frac{M_2 C_4}{\alpha} t^{\alpha} + C_3 \int_0^t (t-s)^{\alpha-1} [y(s)]^{\frac{p}{2}} ds.$$

Applying to which the Bihari lemma, we obtain

$$y(t) \leq \frac{C_5}{[1 - C_6 t^{\alpha}]^{\frac{2}{p-2}}}. \tag{24}$$

From this estimate, we can conclude that there exists $T_0 > 0$ *such that*

$$\|u_m\|_{2,\Omega}^2 + \chi \|\nabla u_m\|_{2,\Omega}^2 \leq C_7, \text{ for all } t \in [0,T],\ T < T_0, \tag{25}$$

where the constant C_7 *does not depend on* $m \in \mathrm{N}$.

Now we multiply (13) *by* $D_{0,t}^{\alpha} v_{mj}(t)$ *and sum up by* $j = 1, ..., m$*. Thus, we get*

$$\begin{array}{l} \left\|D_{0,t}^{\alpha} u_m\right\|_{2,\Omega}^2 + \chi \left\|D_{0,t}^{\alpha} \nabla u_m\right\|_{2,\Omega}^2 + a \int_{\Omega} \nabla u_m D_{0,t}^{\alpha} \nabla u_m dx + \int_{\Omega} c(x,t) u_m D_{0,t}^{\alpha} u_m dx \\ = \int_{\Omega} b(x,t) |u_m|^{p-2} u_m D_{0,t}^{\alpha} u_m dx + \int_{\Omega} |\nabla u_m|^q D_{0,t}^{\alpha} u_m dx + \int_{\Omega} F(u_m, t) h D_{0,t}^{\alpha} u_m dx. \end{array}$$

In this identity, we take out the third and fourth terms on the right side, then it will be written in the form

$$\begin{array}{l} \left\|D_{0,t}^{\alpha} u_m\right\|_{2,\Omega}^2 + \chi \left\|D_{0,t}^{\alpha} \nabla u_m\right\|_{2,\Omega}^2 = -a \int_{\Omega} \nabla u_m D_{0,t}^{\alpha} \nabla u_m dx - \int_{\Omega} c(x,t) u_m D_{0,t}^{\alpha} u_m dx \\ + \int_{\Omega} b(x,t) |u_m|^{p-2} u_m D_{0,t}^{\alpha} u_m dx + \int_{\Omega} |\nabla u_m|^q D_{0,t}^{\alpha} u_m dx + \int_{\Omega} F(u_m, t) h D_{0,t}^{\alpha} u_m dx. \end{array} \tag{26}$$

Let's estimate the right side of (26)*, for the case when* $0 < q \leq 1,\ 2 < p < \frac{2N}{N-2},\ N \geq 3$.

$$\left|\int_{\Omega} b(x,t) |u_m|^p dx\right| \leq b_1 \|u_m\|_{p,\Omega}^p \leq C_1 b_1 \left(\|u_m\|_{2,\Omega}^2 + \chi \|\nabla u_m\|_{2,\Omega}^2\right)^{\frac{p}{2}}.$$

$$\begin{array}{l} \left|\int_{\Omega} |\nabla u_m|^q D_{0,t}^{\alpha} u_m dx\right| \leq \left(\int_{\Omega} |\nabla u_m|^2 dx\right)^{\frac{q}{2}} \left(\int_{\Omega} |D_{0,t}^{\alpha} u_m|^{\frac{2}{2-q}} dx\right)^{\frac{2-q}{2}} \\ \leq \|\nabla u_m\|_{2,\Omega}^q |\Omega|^{\frac{1-q}{2}} \left\|D_{0,t}^{\alpha} u_m\right\|_{2,\Omega} \leq \frac{1}{14} \left\|D_{0,t}^{\alpha} u_m\right\|_{2,\Omega}^2 + C_2. \end{array}$$

$$\left|-a \int_{\Omega} \nabla u_m D_{0,t}^{\alpha} \nabla u_m dx\right| \leq \frac{\chi}{4} \left\|D_{0,t}^{\alpha} \nabla u_m\right\|_{2,\Omega}^2 + \frac{a^2}{\chi} \|\nabla u_m\|_{2,\Omega}^2.$$

$$\left|-\int_{\Omega} c(x,t) u_m D_{0,t}^{\alpha} u_m dx\right| \leq \frac{1}{14} \left\|D_{0,t}^{\alpha} u_m\right\|_{2,\Omega}^2 + \frac{7}{2} c_{01}^2 \|u_m\|_{2,\Omega}^2.$$

$$\left|\int_\Omega b(x,t)|u_m|^{p-2}u_m D^\alpha_{0,t}u_m dx\right| \le b_1\int_\Omega |u_m|^{p-1}|D^\alpha_{0,t}u_m|dx$$
$$\le b_1\left(\int_\Omega |u_m|^p dx\right)^{\frac{1}{2}}\left(\int_\Omega |u_m|^{\frac{N(p-2)}{2}}dx\right)^{\frac{1}{N}}\left(\int_\Omega |D^\alpha_{0,t}u_m|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}$$
$$\le b_1\|u_m\|^{\frac{p}{2}}_{p,\Omega}\|u_m\|^{\frac{p-2}{2}}_{\frac{N(p-2)}{2},\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{\frac{2N}{N-2},\Omega}$$
$$\le b_1 C_5^{\frac{3p-4}{4}}\left\|D^\alpha_{0,t}\nabla u_m\right\|_{2,\Omega} \le \frac{b_1^2 C_5^{\frac{3p-4}{2}}}{\chi}+\frac{\chi}{4}\left\|D^\alpha_{0,t}\nabla u_m\right\|^2_{2,\Omega}.$$

$$\left|\frac{1}{h_1}\int_\Omega D^\alpha_{0,t}\varphi(t)h(x,t)D^\alpha_{0,t}u_m dx\right|$$
$$\le \frac{|D^\alpha_{0,t}\varphi|}{|h_1|}\|h\|_{2,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}$$
$$\le \frac{1}{14}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega}+\frac{7|D^\alpha_{0,t}\varphi|^2}{2h_1^2}\|h\|^2_{2,\Omega}.$$

$$\left|\frac{1}{h_1}a\int_\Omega \nabla u_m\nabla\omega dx\int_\Omega h(x,t)D^\alpha_{0,t}u_m dx\right|$$
$$\le \frac{1}{14}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega}+\frac{7\|\nabla\omega\|^2_{2,\Omega}\|h\|^2_{2,\Omega}}{2h_1^2}C_5.$$

$$\left|\frac{1}{h_1}\int_\Omega c(x,t)u_m\omega dx\int_\Omega h(x)D^\alpha_{0,t}u_m dx\right|$$
$$\le \frac{c_{02}}{|h_1|}\|u_m\|_{2,\Omega}\|\omega\|_{2,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}\|h\|_{2,\Omega}$$
$$\le \frac{1}{14}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega}+\frac{7c_{02}^2\|\omega\|^2_{2,\Omega}\|h\|^2_{2,\Omega}}{2h_1^2}\|u_m\|^2_{2,\Omega}$$
$$\le \frac{1}{14}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega}+7C_5\frac{c_{02}^2\|\omega\|^2_{2,\Omega}\|h\|^2_{2,\Omega}}{2h_1^2}.$$

$$\left|\frac{1}{h_1}\int_\Omega b(x,t)|u_m|^{p-2}u_m\omega dx\int_\Omega h(x,t)D^\alpha_{0,t}u_m dx\right|$$
$$\le \frac{b_1}{|h_1|}\|u_m\|^{p-1}_{p,\Omega}\|\omega\|_{p,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}\|h\|_{2,\Omega}$$
$$\le C_1^{\frac{p-1}{p}}\frac{b_1}{|h_1|}\|\omega\|_{p,\Omega}\|h\|_{2,\Omega}\left(\|u\|^2_{2,\Omega}+\chi\|\nabla u\|^2_{2,\Omega}\right)^{\frac{p-1}{2}}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}$$
$$\le \frac{1}{14}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega}+7C_1^{\frac{2p-2}{p}}C_5^{p-1}\frac{b_1^2}{2h_1^2}\|\omega\|^2_{p,\Omega}\|h\|^2_{2,\Omega}.$$

$$\left|\frac{1}{h_1}\int_\Omega |\nabla u_m|^q\omega dx\int_\Omega h(x,t)D^\alpha_{0,t}u_m dx\right|$$

$$\le \frac{1}{|h_1|}\left(\int_\Omega |\nabla u_m|^2 dx\right)^{\frac{q}{2}}\left(\int_\Omega |\omega|^{\frac{2}{2-q}}dx\right)^{\frac{2-q}{2}}\|h\|_{2,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}$$
$$\le \|\nabla u_m\|^q_{2,\Omega}|\Omega|^{\frac{1-q}{2}}\|\omega\|_{2,\Omega}\|h\|_{2,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}\le \frac{1}{14}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega}+C_2.$$

Similarly, we estimate for the case when $1\le q\le 1+\frac{2}{N},\ 2<p<\frac{2N}{N-2},\ N\ge 3.$

$$\left|\int_\Omega |\nabla u_m|^q D^\alpha_{0,t}u_m dx\right| \le \left(\int_\Omega |\nabla u_m|^{\frac{2Nq}{N+2}}dx\right)^{\frac{N+2}{2N}}\left(\int_\Omega |D^\alpha_{0,t}u_m|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}$$
$$\le C_{02}\|\nabla u_m\|^q_{2,\Omega}\left\|\nabla D^\alpha_{0,t}u_m\right\|_{2,\Omega}\le \frac{\chi}{6}\left\|D^\alpha_{0,t}\nabla u_m\right\|^2_{2,\Omega}+C_2.$$

$$\left|-a\int_\Omega \nabla u_m D^\alpha_{0,t}\nabla u_m dx\right| \le \frac{\chi}{6}\left\|D^\alpha_{0,t}\nabla u_m\right\|^2_{2,\Omega} + \frac{3a^2}{2\chi}\left\|\nabla u_m\right\|^2_{2,\Omega}.$$

$$\left|-\int_\Omega c(x,t)u_m D^\alpha_{0,t}u_m dx\right| \le \frac{1}{12}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega} + 3c_{01}^2\left\|u_m\right\|^2_{2,\Omega}.$$

$$\left|\int_\Omega b(x,t)|u_m|^{p-2}u_m D^\alpha_{0,t}u_m dx\right| \le b_1\int_\Omega |u_m|^{p-1}|D^\alpha_{0,t}u_m|dx$$
$$\le b_1\left(\int_\Omega |u_m|^p dx\right)^{\frac{1}{2}}\left(\int_\Omega |u_m|^{\frac{N(p-2)}{2}}dx\right)^{\frac{1}{N}}\left(\int_\Omega |D^\alpha_{0,t}u_m|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}$$
$$\le b_1 C_5^{\frac{3p-4}{4}}\left\|D^\alpha_{0,t}\nabla u_m\right\|_{2,\Omega} \le \frac{3b_1^2C_5^{\frac{3p-4}{2}}}{2\chi} + \frac{\chi}{6}\left\|D^\alpha_{0,t}\nabla u_m\right\|^2_{2,\Omega}.$$

$$\left|\frac{1}{h_1}\int_\Omega D^\alpha_{0,t}\varphi(t)h(x,t)D^\alpha_{0,t}u_m dx\right|$$
$$\le \frac{|D^\alpha_{0,t}\varphi|}{|h_1|}\left\|h\right\|_{2,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega} \le \frac{1}{12}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega} + \frac{3|D^\alpha_{0,t}\varphi|^2}{h_1^2}\left\|h\right\|^2_{2,\Omega}.$$

$$\left|\frac{1}{h_1}a\int_\Omega \nabla u_m\nabla\omega dx\int_\Omega h(x,t)D^\alpha_{0,t}u_m dx\right|$$
$$\le \frac{1}{12}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega} + \frac{3\left\|\nabla\omega\right\|^2_{2,\Omega}\left\|h\right\|^2_{2,\Omega}}{h_1^2}\left\|\nabla u_m\right\|^2_{2,\Omega}$$
$$\le \frac{1}{12}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega} + \frac{3\left\|\nabla\omega\right\|^2_{2,\Omega}\left\|h\right\|^2_{2,\Omega}}{h_1^2}C_5.$$

$$\left|\frac{1}{h_1}\int_\Omega c(x,t)u_m\omega dx\int_\Omega h(x)D^\alpha_{0,t}u_m dx\right|$$
$$\le \frac{c_{02}}{|h_1|}\left\|u_m\right\|_{2,\Omega}\left\|\omega\right\|_{2,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}\left\|h\right\|_{2,\Omega}$$
$$\le \frac{1}{12}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega} + \frac{3c_{02}^2\left\|\omega\right\|^2_{2,\Omega}\left\|h\right\|^2_{2,\Omega}}{h_1^2}\left\|u_m\right\|^2_{2,\Omega}$$
$$\le \frac{1}{12}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega} + 3C_5\frac{c_{02}^2\left\|\omega\right\|^2_{2,\Omega}\left\|h\right\|^2_{2,\Omega}}{h_1^2}.$$

$$\left|\frac{1}{h_1}\int_\Omega b(x,t)|u_m|^{p-2}u_m\omega dx\int_\Omega h(x,t)D^\alpha_{0,t}u_m dx\right|$$
$$\le \frac{b_1}{|h_1|}\left\|u_m\right\|^{p-1}_{p,\Omega}\left\|\omega\right\|_{p,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}\left\|h\right\|_{2,\Omega}$$
$$\le C_1^{\frac{p-1}{p}}\frac{b_1}{|h_1|}\left\|\omega\right\|_{p,\Omega}\left\|h\right\|_{2,\Omega}\left(\left\|u\right\|^2_{2,\Omega} + \chi\left\|\nabla u\right\|^2_{2,\Omega}\right)^{\frac{p-1}{2}}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}$$
$$\le \frac{1}{12}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega} + 3C_1^{\frac{2p-2}{p}}C_5^{p-1}\frac{b_1^2}{h_1^2}\left\|\omega\right\|^2_{p,\Omega}\left\|h\right\|^2_{2,\Omega}.$$

$$\left|\frac{1}{h_1}\int_\Omega |\nabla u_m|^q\omega dx\int_\Omega h(x,t)D^\alpha_{0,t}u_m dx\right|$$

$$\le \frac{1}{|h_1|}\left(\int_\Omega |\nabla u_m|^2dx\right)^{\frac{q}{2}}\left(\int_\Omega |\omega|^{\frac{2}{2-q}}dx\right)^{\frac{2-q}{2}}\left\|h\right\|_{2,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega}$$
$$\le \left\|\nabla u_m\right\|^q_{2,\Omega}|\Omega|^{\frac{1-q}{2}}\left\|\omega\right\|_{2,\Omega}\left\|h\right\|_{2,\Omega}\left\|D^\alpha_{0,t}u_m\right\|_{2,\Omega} \le \frac{1}{12}\left\|D^\alpha_{0,t}u_m\right\|^2_{2,\Omega} + C_2.$$

Substituting the obtained inequalities into (26), *we have*

$$\left\|D_{0,t}^{\alpha}u_m\right\|_{2,\Omega}^2+\chi\left\|D_{0,t}^{\alpha}\nabla u_m\right\|_{2,\Omega}^2\le C_8,\ for\ all\ t\in[0,T],T<T_0. \quad (27)$$

Now we multiply (13) *by* $\lambda_j v_{mj}(t)$ *and* $\lambda_j D_{0,t}^{\alpha}v_{mj}(t)$ *and sum up by* $j=1,...,m$. *Similarly, doing the same calculations as in obtaining estimates* (25) *and* (27), *as a result we have the following estimates*

$$\left\|\Delta u_m\right\|_{2,\Omega}^2+\chi\left\|D_{0,t}^{\alpha}\Delta u_m\right\|_{2,\Omega}^2\le C_7,\ for\ all\ \ t\in[0,T]\,,\ T<T_0. \quad (28)$$

3.3 Limit transition.

The obtained estimates (25), (27) *and* (28) *for* $0<q\le 1+\frac{2}{N}$, $N\ge 3$, *imply the following statements, respectively:*

u_m *is bounded in* $L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega))\cap L_\infty(0,T;W_2^2(\Omega))$,

$D_{0,t}^{\alpha}u_m$*is bounded in* $L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega))\cap L_\infty(0,T;W_2^2(\Omega))$,

Moreover, due to the conditions on p:

$|u_m|^{p-2}u_m$ *is bounded in* $L_\infty(0,T;L_{\frac{p}{p-1}}(\Omega))$, $2<p<\frac{2N}{N-2}$, $N\ge 3$.

The above reasoning allows us to pass to the limit in (13).

4 Uniqueness of a weak generalized solution.

Theorem 2 *Assume that* (5) *holds and* $1\le q\le 1+\frac{2}{N}$, $2<p<\frac{2N}{N-2}$, $N\ge 3$. *Then the generalized solution of the problem* (6) - (7) *on the interval* $(0,T)$ *is unique.*

Proof 3 *Suppose that the problem* (6) - (7) *has two solutions:* $u_1(x,t)$ *and* $u_2(x,t)$. *Then their difference* $u(x,t)=u_1(x,t)-u_2(x,t)$ *satisfies the homogeneous initial condition* $u(x,0)=0$ *and the equality*

$$\begin{aligned}&\int_\Omega\left(D_{0,t}^{\alpha}u\cdot w+\chi\sum_{i=1}^N D_{0,t}^{\alpha}\left(\frac{\partial u}{\partial x_i}\right)\cdot\frac{\partial w}{\partial x_i}+a\nabla u\nabla w+c(x,t)uw\right)dx\\&=\int_\Omega b(x,\tau)\left(|u_1|^{p-2}u_1-|u_1|^{p-2}u_1\right)wdx+\int_\Omega\left(|\nabla u_1|^q-|\nabla u_2|^q\right)wdx\\&+\int_\Omega(F(u_1,\tau)-F(u_2,\tau))hwdx.\end{aligned}$$

Due to $w(x,t)\in L_2(Q_T)$, *as* $w(x,t)$ *we can take* $u(x,t)$, *i.e. put* $w(x,t)=u(x,t)$

$$\begin{aligned}&\int_\Omega\left(D_{0,t}^{\alpha}u\cdot u+\chi D_{0,t}^{\alpha}\nabla u\cdot\nabla u+a|\nabla u|^2+c(x,t)|u|^2\right)dx=\\&=\int_\Omega b(x,\tau)\left(|u_1|^{p-2}u_1-|u_1|^{p-2}u_1\right)udx+\int_\Omega\left(|\nabla u_1|^q-|\nabla u_2|^q\right)udx\\&+\int_\Omega(F(u_1,\tau)-F(u_2,\tau))hudx.\end{aligned} \quad (29)$$

Using the in equality

$$\left||u_1|^m-|u_2|^m\right|\le m\left(|u_1|^{m-1}+|u_2|^{m-1}\right)|u_1-u_2|\,,m>0,$$

$$\left||u_1|^m u_1-|u_2|^m u_2\right|\le(m+1)\left(|u_1|^m+|u_2|^m\right)|u_1-u_2|,\ \ m>0.$$

to the right-hand side of (29), *we have*

$$\left|\int_\Omega b(x,\tau)\left(|u_1|^{p-1}-|u_2|^{p-1}\right)udx\right| \le b_1(p-1)\int_\Omega\left(|u_1|^{p-2}+|u_2|^{p-2}\right)u^2dx$$
$$\le b_1(p-1)\left(\int_\Omega\left(|u_1|^{p-2}+|u_2|^{p-2}\right)^{\frac{N}{2}}dx\right)^{\frac{2}{N}}\left(\int_\Omega u^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}$$
$$\le b_1(p-1)\left(\left(\int_\Omega|u_1|^{\frac{N(p-2)}{2}}dx\right)^{\frac{2}{N}}+\left(\int_\Omega|u_2|^{\frac{N(p-2)}{2}}dx\right)^{\frac{2}{N}}\right)\left(\int_\Omega u^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}.$$

Then by the Sobolev embedding theorem $\overset{0}{W}{}_2^1(\Omega)\subset L_{\frac{2N}{N-2}}(\Omega)$ *and* $\overset{0}{W}{}_2^1(\Omega)\subset L_{\frac{N(p-2)}{2}}(\Omega)$, $2<p<\frac{2N}{N-2}$.

In this case, taking into account smoothness class of solutions $u_1(x,t)$ *and* $u_2(x,t)$, *we get the estimate:*

$$\left|\int_\Omega b(x,\tau)\left(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\right)udx\right| \le C_1\left(\|u\|_{2,\Omega}^2+\chi\|\nabla u\|_{2,\Omega}^2\right). \tag{30}$$

Similarly, we estimate

$$\left|\int_\Omega\left(|\nabla u_1|^q-|\nabla u_2|^q\right)udx\right| \le q\int_\Omega\left(|\nabla u_1|^{q-1}+|\nabla u_2|^{q-1}\right)|\nabla u|\,|u|\,dx$$
$$\le q\left(\int_\Omega\left(|\nabla u_1|^{q-1}+|\nabla u_2|^{q-1}\right)^N dx\right)^{\frac{1}{N}}\left(\int_\Omega|u|^r dx\right)^{\frac{1}{r}}\left(\int_\Omega|\nabla u|^2dx\right)^{\frac{1}{2}}$$
$$\le q\left(\left(\int_\Omega|\nabla u_1|^{N(q-1)}dx\right)^{\frac{1}{N}}+\left(\int_\Omega|\nabla u_2|^{N(q-1)}dx\right)^{\frac{1}{N}}\right)$$
$$\times\left(\int_\Omega u^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}\left(\int_\Omega|\nabla u|^2dx\right)^{\frac{1}{2}} \le C_2\left(\|u\|_{2,\Omega}^2+\chi\|\nabla u\|_{2,\Omega}^2\right).$$

$$\begin{aligned}
&\left|\frac{1}{h_1}\int_\Omega b(x,\tau)\left(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\right)\omega dx\int_\Omega hudx\right|\\
&\le\frac{\omega_1b_1(p-1)}{|h_1|}\|h\|_{2,\Omega}\|u\|_{2,\Omega}\int_\Omega\left(|u_1|^{p-2}+|u_2|^{p-2}\right)|u|dx\\
&\le\frac{\omega_1b_1(p-1)}{|h_1|}|\Omega|^{\frac{1}{2}}\left(\int_\Omega\left(|u_1|^{p-2}+|u_2|^{p-2}\right)^2u^2dx\right)^{\frac{1}{2}}\|h\|_{2,\Omega}\|u\|_{2,\Omega}\\
&\le\frac{\omega_1b_1(p-1)}{|h_1|}|\Omega|^{\frac{1}{2}}\left(\int_\Omega\left(|u_1|^{p-2}+|u_2|^{p-2}\right)^{\frac{N}{2}}dx\right)^{\frac{2}{N}}\left(\int_\Omega u^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}\|h\|_{2,\Omega}\|u\|_{2,\Omega}\\
&\le\frac{\omega_1b_1(p-1)}{|h_1|}|\Omega|^{\frac{1}{2}}\|h\|_{2,\Omega}\left(\left(\int_\Omega|u_1|^{\frac{N(p-2)}{2}}dx\right)^{\frac{2}{N}}+\left(\int_\Omega|u_2|^{\frac{N(p-2)}{2}}dx\right)^{\frac{2}{N}}\right)\\
&\times\left(\int_\Omega u^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}\|u\|_{2,\Omega}\le C_3\left(\|u\|_{2,\Omega}^2+\chi\|\nabla u\|_{2,\Omega}^2\right).
\end{aligned}\tag{31}$$

$$\begin{aligned}
&\left|\tfrac{1}{h_1}a\int_\Omega\nabla u\nabla\omega dx\int_\Omega hudx\right|\le\tfrac{1}{|h_1|}\|\nabla u\|_{2,\Omega}\|\nabla\omega\|_{2,\Omega}\|u\|_{2,\Omega}\|h\|_{2,\Omega}\\
&\le C_4\left(\|u\|_{2,\Omega}^2+\chi\|\nabla u\|_{2,\Omega}^2\right).
\end{aligned}\tag{32}$$

$$\begin{aligned}
&\left|\tfrac{1}{h_1}\int_\Omega c(x,t)u\omega dx\int_\Omega hudx\right|\le\frac{C_2'}{|h_1|}\|u\|_{2,\Omega}\|\omega\|_{2,\Omega}\|u\|_{2,\Omega}\|h\|_{2,\Omega}\\
&\le\tfrac{C_2'}{|h_1|}\|\omega\|_{2,\Omega}\|h\|_{2,\Omega}\|u\|_{2,\Omega}^2\le C_5\left(\|u\|_{2,\Omega}^2+\chi\|\nabla u\|_{2,\Omega}^2\right).
\end{aligned}\tag{33}$$

Due to (30) - (33) *and Lemma 3.4, we obtain*

$$\begin{array}{l}\frac{1}{2}D^{\alpha}_{0,t}\int_{\Omega}[|u|^2+\chi|\nabla u|^2]dx+\dfrac{a}{2}\int_{\Omega}|\nabla u|^2dx+C_{01}\int_{\Omega}|u|^2dx\\ \leq C_6\left(\|u\|^2_{2,\Omega}+\chi\|\nabla u\|^2_{2,\Omega}\right).\end{array}\tag{34}$$

Denote $y(t)=\int_{\Omega}[|u|^2+\chi|\nabla u|^2]dx$, *then inequality* (34) *can be written in the following form*

$$\begin{array}{l}D^{\alpha}_{0,t}y(t)+C_0y(t)\leq C_6y(t),\\ y(0)=0,\end{array}$$

hence, Having applied lemma of Gronwall-Bellman [46], we obtain the validity of the following estimation we get:

$$\begin{array}{l}\|u(x,0)\|^2_{2,\Omega}+\chi\|\nabla u(x,0)\|^2_{2,\Omega}\leq\\ \leq M_1y(0)E_{\alpha,1}\left((C_6\Gamma(\alpha))^{\frac{1}{\alpha}}\,t^{\alpha}\right)=0,\ \ 0\leq t\leq T<\infty,\end{array}$$

which implies $\int_{\Omega}[|u|^2+\chi|\nabla u|^2]dx=0$ *almost everywhere on the time interval* $(0,T)$, *which means uniqueness of the weak generalized solution.*

From Lemma 1, we can establish solvability of the inverse problem (1) - (4). *Let* $u(x,t)$ *be a solution to the initial-boundary value problem* (8) - (9) *from the space* $u\in V_2^{\alpha}(Q_T)$ *(Theorem 1). Obviously, the function* $f(t)$ *from* (10) *belongs to the space* $L_{\infty}(0,T)$. *This proved statement means that the found functions* $u(x,t)$ *and* $f(t)$ *give a weak solution to the inverse problem.*

5 Numerical example. Formulation of the problem

We consider the problem

$$\frac{\partial^{0.5}u}{\partial t^{0.5}}-\frac{\partial^{0.5}}{\partial t^{0.5}}\left(\frac{\partial^2 u}{\partial x^2}\right)-\frac{\partial^2 u}{\partial x^2}=|u|^{p-2}u+\left|\frac{\partial u}{\partial x}\right|^q+f(t)h(x,\ t),$$

$$u(x,0)=0,\ \ u(0,t)=u(1,t)=0,$$

$$\int_0^1 u(x,t)(w(x)-w''(x))dx=e(t),$$

where $p=4$, $q=\frac{1}{2}$, $w(x)=x^2-x^3$, $e(t)=\dfrac{11}{69}\left(\dfrac{t^2}{2}+t\right)$,

$$\begin{array}{l}h(x,t)=\frac{1}{\sqrt{\pi}}\left(\frac{4}{3}t+2\right)(x-x^2)+\frac{2}{\sqrt{\pi}}\left(\frac{4}{3}t+2\right)+2\left(\frac{t^{1.5}}{2}+\sqrt{t}\right)\\ -\left(\frac{t^{\frac{11}{6}}}{2}+t^{\frac{5}{6}}\right)^3(x-x^2)^3-\left|\frac{1}{2}t+1\right|^{\frac{1}{2}}|1-2x|^{\frac{1}{2}}.\end{array}$$

We reduce the inverse problem to the direct problem

$$\frac{\partial^{0.5}u}{\partial t^{0.5}}-\frac{\partial^{0.5}}{\partial t^{0.5}}\left(\frac{\partial^2 u}{\partial x^2}\right)-\frac{\partial^2 u}{\partial x^2}=|u|^{p-2}u+\left|\frac{\partial u}{\partial x}\right|^q+F\left(t,u\right)h\left(x,\ t\right),$$

$$u(x,0)=0,\;\; u(0,t)=u(1,t)=0,$$

where

$$F(t,u)=\frac{D^{\alpha}_{0,t}(t)+\int_0^1 u_x\cdot w'(x)dx-\int_0^1 |u|^{p-2}u\cdot w(x)dx-\int_0^1 \left|\frac{\partial u}{\partial x}\right|^q\cdot w(x)dx}{h_1(t)},$$

$$h_1(t)=\int_0^1 h(x,t)\cdot w(x)dx=\frac{11}{30\sqrt{\pi}}\left(\frac{2}{3}t+1\right)+\frac{1}{6}\left(\frac{t^{1.5}}{2}+\sqrt{t}\right)-$$
$$-\frac{1}{1260}\left(\frac{t^{\frac{11}{6}}}{2}+t^{\frac{5}{6}}\right)^3-\frac{1}{21}\left|\frac{t}{2}+1\right|^{\frac{1}{2}}.$$

To solve the problem, we apply numerical methods. To check the obtained solutions, the numerical solutions are compared with the analytical form, which looks like as $u(x,t)=\left(\frac{t^2}{2}+t\right)(x-x^2)\,, f(t)=\sqrt{t}.$

5.1 Numerical algorithm

For a numerical solution of the posed problem, finite difference approximation is used. To discretize the fractional differential operator, the Grunwald-Letnikov definition is used.

$$\frac{1}{\Delta t^{\alpha}}\sum_{k=0}^{n+1} u_i^{n+1-k}\binom{\alpha}{k}-\left(\frac{1}{\Delta t^{\alpha}}\sum_{j=0}^{n+1}\frac{\left(u_{i+1}^{n+1-j}-2u_i^{n+1-j}+u_{i-1}^{n+1-j}\right)}{\Delta x^2}\binom{\alpha}{j}\right)-$$
$$\frac{\left(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}\right)}{\Delta x^2}=|u_i^n|^{p-2}u_i^n+d\left|\frac{u_i^n-u_{i-1}^n}{\Delta x}\right|^q+F\left(t_n,\;u\right)h\left(x_i,t_n\right),$$

where $\binom{\alpha}{i}$ is a generalization of binomial coefficients.

The basis of the numerical method is the Thomas algorithm, which can be reduced to such an analogue as

$$Au_{i+1}^{n+1}+Bu_i^{n+1}+Cu_{i-1}^{n+1}=d_i,$$

where $A=(-1),\;\; B=2+\Delta x^2),\;\; C=(-1);$

$$d_i=\Delta t^{\alpha}\left(\Delta x^2F(t_n,u)\cdot h(x_i,t_n)+u_{i+1}^n-2u_i^n+u_{i-1}^n+\Delta x^2|u_i^n|^{p-2}u_i^n+\Delta x^2 d\left|\frac{u_i^n-u_{i-1}^n}{\Delta x}\right|^q\right)+$$
$$+\sum_{k=1}^{n+1}\binom{\alpha}{k}\left(u_{i+1}^{n+1-k}-2u_i^{n+1-k}+u_{i-1}^{n+1-k}-\Delta x^2u_i^{n+1-k}\right).$$

The problem is reduced to solving a system of linear equations:

$$\begin{array}{l} Bu_1^{n+1}+Au_2^{n+1}=d_1,\;\; i=1,\\ Cu_{i-1}^{n+1}+Bu_i^{n+1}+Au_{i+1}^{n+1}=d_i,\;\; i=\overline{1,N-1},\\ Cu_{N-1}^{n+1}+Bu_N^{n+1}=d_N,\;\; i=N.\end{array}$$

We solve the numerical solution in the form

$$u_i^{n+1}=\alpha_i u_{i+1}^{n+1}+\beta_i,$$

where $\alpha_i=-\frac{A}{B+C\alpha_{i-1}}$, $\beta_i=\frac{d_i-C\beta_{i-1}}{B+C\alpha_{i-1}}$.

To find the value of $\alpha_0,\;\beta_0,\;u_N^{n+1}$, we use boundary value conditions.

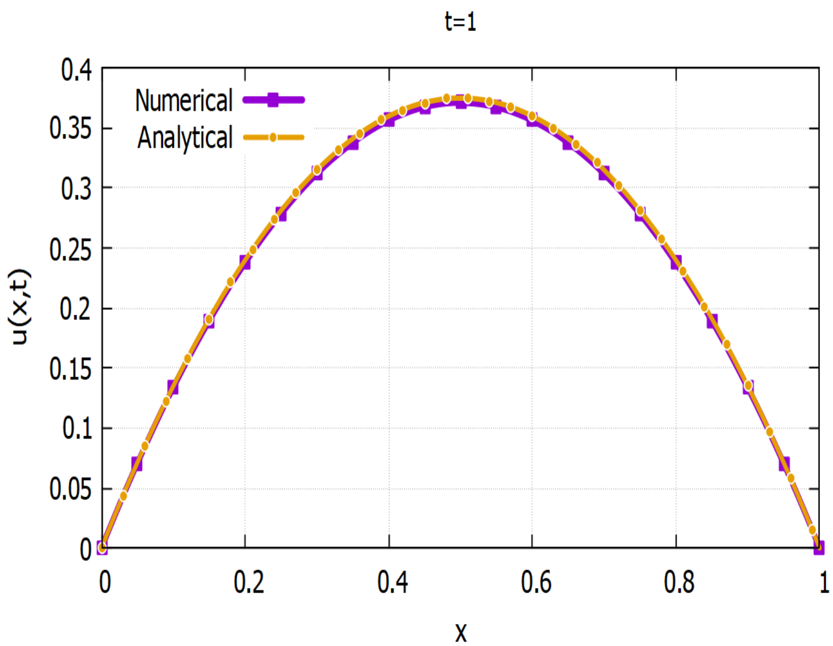

Figure 1: Comparison of the obtained numerical results with the analytical results

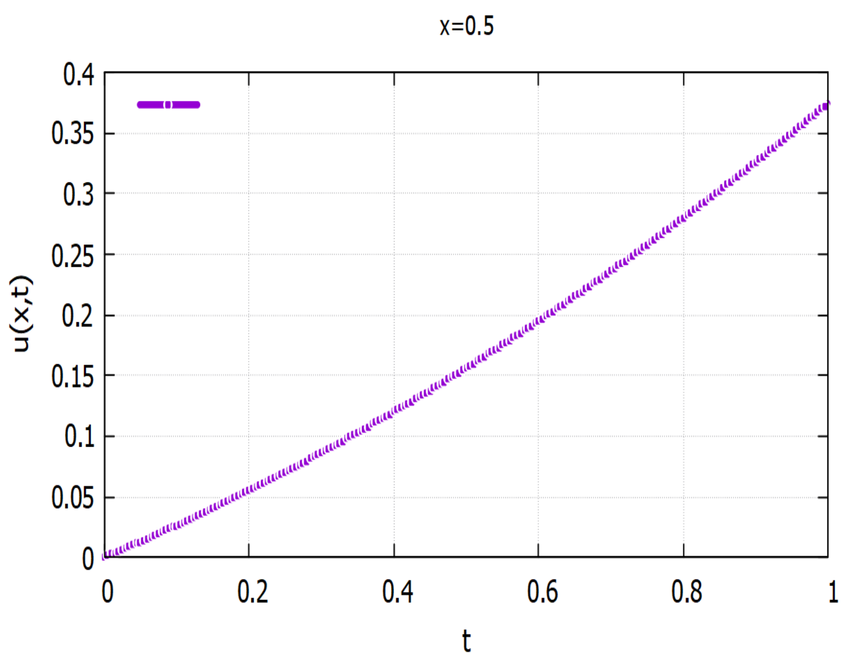

Figure 2: Change in time u(x,t) at x=0.5

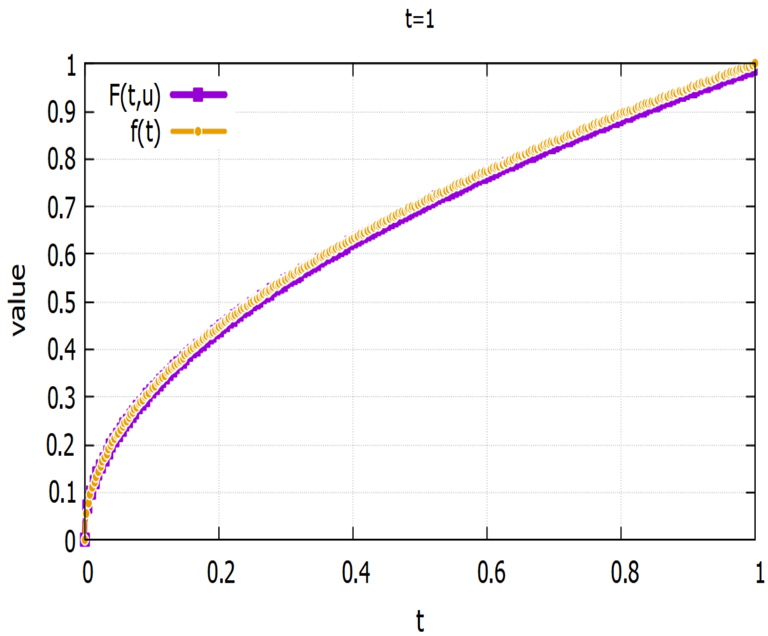

Figure 3: Change in absolute value at over time

$z^i = |u(x_i, t_n) - w(x_i, t_n)|$, where $u(x,t)$ – analytical solution, $w(x,t)$ – numerical solution $n = 1500$

Δt	Δx	$max_{0<i<N+1}[z^i]$
10^{-3}	$1/10$	$0,00028$
	$1/20$	$0,00012$
	$1/40$	$0,00015$
	$1/80$	0.00017
	$1/100$	0.00018
	$1/160$	0.00019
10^{-4}	$1/10$	$2,81 * 10^{-5}$
	$1/20$	$1,14 * 10^{-5}$
	$1/40$	$1,4 * 10^{-5}$
	$1/80$	$1,6 * 10^{-5}$
	$1/100$	$1,65 * 10^{-5}$
	$1/160$	$1,73 * 10^{-5}$
10^{-5}	$1/10$	$2,83 * 10^{-6}$
	$1/20$	$1,14 * 10^{-6}$
	$1/40$	$1,38 * 10^{-6}$
	$1/80$	$1,57 * 10^{-6}$
	$1/100$	$1,62 * 10^{-6}$
	$1/160$	$1,70 * 10^{-6}$

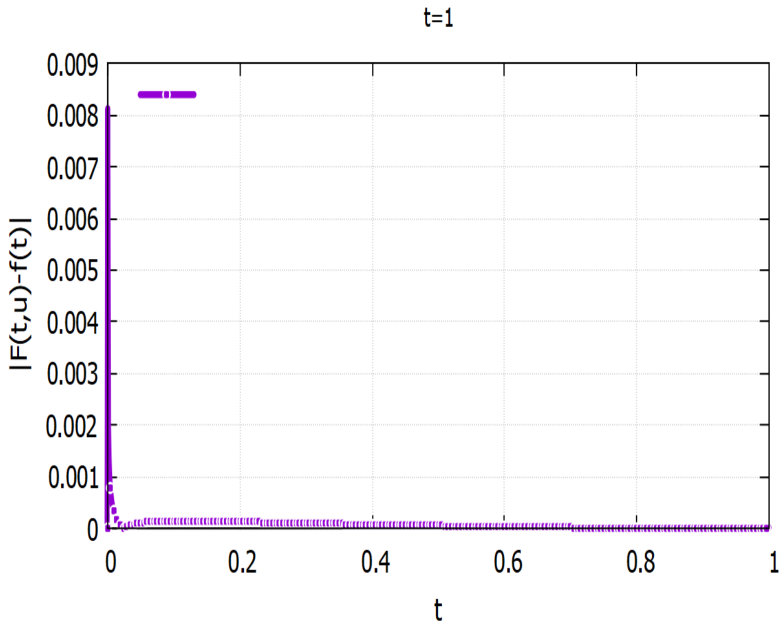

Figure 4: Comparison of F(t,u) and f(t) at versus time

6 Conclusion

The paper combines rigorous mathematical analysis with practical methods of numerical solution, which makes it a significant contribution to the study of inverse problems with fractional derivatives. The proof of the existence and uniqueness of solutions, as well as the development of an algorithm and numerical experiments, provide a comprehensive approach to studying the problem under consideration. The results obtained can be useful for specialists involved in both the theory of fractional equations and their applications in various scientific and engineering problems.

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