1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

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S.K. Kuanysh[1,2*], **A.T. Assanova**[2,3], **Zh.M. Kadirbayeva**[2,3]
[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[2]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[3]Kazakh National Women's Teacher Training University, Almaty, Kazakhstan
*e-mail: saule270898@gmail.com

METHOD OF LINES FOR A LOADED PARABOLIC EQUATION

Loaded parabolic equations belong to a complex yet important class of differential equations and are widely applied in various scientific and engineering problems, as well as in ecology, epidemic propagation modeling, and biological systems. Special analytical and numerical methods are used to solve these equations, taking into account the influence of integral and functional loads. This article examines a two-point boundary value problem for loaded parabolic equations, defined in a closed domain. The solution is approached using the method of lines with respect to the variable x. As a result of this method, a discretized problem is formulated. The obtained discretized problem is represented in a vector-matrix form and is reduced to a two-point boundary value problem for a loaded system of differential equations. The parameterization method proposed by Professor Dzhumabaev is used to solve the boundary value problem. The efficiency of this method lies in the high accuracy of the numerical-analytical solution compared to the exact solution, as well as in the possibility of formulating the solvability conditions of the problem. As a theoretical justification of the method, an additional theorem is proven, based on which the solvability conditions of the problem are determined. The study explores the relationship between the original boundary value problem and its discretized form for the loaded parabolic equation. This relationship is substantiated using an additional theorem derived from the parameterization method.
Key words: loaded parabolic equations, two-point boundary value problem, method of lines, convergence, parameterization method.

С.Қ. Қуаныш[1,2*], А.Т. Асанова[2,3], Ж.М. Қадырбаева[2,3]
[1]Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан
[2]Математика және математикалық моделдеу институты, Алматы, Қазақстан
[3]Қазақ ұлттық қыздар педагогикалық университеті, Алматы, Қазақстан
*e-mail: saule270898@gmail.com

Жүктелген параболалық теңдеу үшін сызықтар әдісі

Жүктелген параболалық теңдеулер күрделі, бірақ маңызды теңдеулер класына жатады және олар әртүрлі ғылыми әрі инженерлік қолданбаларда, экологияда, эпидемиялардың таралуын модельдеуде және биологиялық жүйелерде кеңінен қолданылады. Мұндай теңдеулерді шешу үшін интегралдық және функционалдық жүктемелердің әсерін ескеретін арнайы аналитикалық және сандық әдістер қолданылады. Мақалады тұйық аймақта жүктелген параболалық теңдеулер үшін екі нүктелі шеттік есеп қарастырылады. Бұл есепті шешу мақсатында кеңістік x айнымалысы бойынша сызықтар әдісі қолданылады. Әдіс нәтижесінде дискреттелген есеп алынады. Алынған дискреттелген есеп вектор-матрицалық түрде өрнектеліп, жүктелген дифференциалдық теңдеулер үшін екі нүктелі шеттік есепке келтіріледі. Шеттік есепті шешу үшін профессор Жұмабаевтың параметрлеу әдісі қолданылады. Бұл әдістің тиімділігі – есептің сандық-аналитикалық шешімінің дәл шешімге жуықтау дәлдігінің жоғары болуында және есептің шешілімділік шарттарының алынуы болып табылады. Әдістің теориялық негіздемесі ретінде қосымша теорема дәлелденіп, есептің шешілімділік шарттары анықталады.

Зерттеу барысында жүктелген параболалық теңдеу үшін бастапқы шетті есеп пен оның дискреттелген есеп арасындағы байланыс қарастырылады. Бұл байланыс параметрлеу әдісі негізінде алынған қосымша теоремамен дәлелденеді.
Түйін сөздер: жүктелген параболалық теңдеулер, екі нүктелі шеттік есеп, сынықтар әдісі, жинақтылық, параметрлеу әдісі.

С.К. Куаныш[1,2*], А.Т. Асанова[2,3], Ж.М. Кадирбаева[2,3]
[1]Казахский национальный университет имени аль-Фараби, Алматы, Казахстан
[2]Институт математики и математического моделирования, Алматы, Казахстан
[3]Казахский национальный женский педагогический университет, Алматы, Казахстан
*e-mail: saule270898@gmail.com

Метод прямых для нагруженного параболического уравнения

Нагруженные параболические уравнения относятся к сложному, но важному классу дифференциальных уравнений и широко применяются в различных научных и инженерных задачах, а также в экологии, моделировании распространения эпидемий и биологических системах. Для их решения используются специальные аналитические и численные методы, учитывающие влияние интегральных и функциональных нагрузок. В данной статье рассматривается двухточечная краевая задача для нагруженных параболических уравнений, заданная в замкнутой области. Для ее решения применяется метод прямых по переменной x. В результате этого метода формулируется дискретизированная задача. Полученная дискретизированная задача представляется в векторно-матричной форме и сводится к двухточечной краевой задаче для нагруженной системы дифференциальных уравнений. Для решения краевой задачи используется метод параметризации, предложенный профессором Джумабаевым. Эффективность данного метода заключается в высокой точности численно-аналитического решения по сравнению с точным решением, а также в возможности формулирования условий разрешимости задачи. В качестве теоретического обоснования метода доказывается дополнительная теорема, на основе которой определяются условия разрешимости задачи. В исследовании изучается связь между исходной краевой задачей и ее дискретизированной формой для нагруженного параболического уравнения. Данная связь обосновывается с помощью дополнительной теоремы, полученной на основе метода параметризации.
Ключевые слова: нагруженные параболические уравнения, двухточечная краевая задача, метод прямых, сходимость, метод параметризации.

1 Introduction and preliminaries

The essence of the method of lines, which explains its name, is as follows: for example, in the case of a partial differential equation with respect to a function of two variables, constant values are assigned to one of these variables, and transforming the problem into an ordinary differential equation. Therefore, when addressing boundary and initial-boundary value problems for partial differential equations, existing methods for solving initial and boundary value problems for ordinary differential equations can be effectively applied. In this study, the problem for loaded parabolic equations is transformed into a problem for LDE.

A family of linear and nonlinear parabolic boundary value problems with the first boundary condition are addressed using the method of lines in [1]. It is demonstrated that there is a certain order of error in the approximate solutions produced by this method. The ease of solving the heat conduction equation receives special attention.

A thorough theoretical investigation aimed at proving the convergence and stability of solutions to one-dimensional parabolic equations with Dirichlet boundary conditions is presented in work [2] using the method of lines.

The work [3] approach the method of lines to solve certain quasilinear boundary value problems of parabolic type and establishes theorems proving the convergence and stability of the method.

The work [4] investigates boundary value problems for ordinary and partial differential equations with loading. Estimates for the solutions to both the differential and difference equations are established. These estimates ensure the stability and convergence of the difference schemes for the equations under consideration.

The work [5] analyzes the convergence of one-step schemes of the method of lines (MOL). The primary goal is to establish a general framework for convergence analysis applicable to nonlinear problems. The stability concepts used in this framework are based on the theory of nonlinear stiff ordinary differential equations. In this context, key notions include the norm of the logarithmic matrix and C-stability. To illustrate the proposed ideas, a nonlinear parabolic equation and the cubic Schredinger equation are considered.

The work [6] proposes a parameterization method for finding solutions to a system of ordinary differential equations. The work [7] examines a mixed boundary value problem for a linear parabolic equation with two independent variables. Using the method of lines, estimates for the solutions and their derivatives are obtained in terms of the equation's coefficients and the boundary conditions.

Loaded parabolic equations are widely encountered in mathematical biology, particularly in the mathematical modeling of transfer phenomena in living systems [8], [9]. Different types of boundary value problems for parabolic equations with loading have been explored in the works of T. Yuldashev, M.T. Dzhenaliev, M. I. Ramazanov, V.M. Abdullaev, K.R. Aida-zade and M. Dehghan [10]- [17].

This paper considers the following two-point boundary value problem for a loaded parabolic equation in $\Omega = [0,T]\times[0,\omega]$

$$\frac{\partial u}{\partial t} = a(t,x)\frac{\partial^2 u}{\partial x^2} + b(t,x)u(t,x) + \sum_{j=1}^{m} k_j(t,x)u(t_j,x) + f(t,x), \quad (t,x)\in\Omega = (0,T)\times(0,\omega), \tag{1}$$

$$B(x)u(0,x) + C(x)u(T,x) = \varphi(x), \qquad x\in[0,\omega], \tag{2}$$

$$u(t,0) = \psi_0(t), \quad u(t,\omega) = \psi_1(t), \quad t\in[0,T], \tag{3}$$

where $a(t,x)\geq\rho>0$, $b(t,x)\leq 0$, $k_j(t,x)$, $f(t,x)$ - are continuous in t and Holder continuous in x. We assume that the functions $\varphi(x)$, $\psi_0(t)$, $\psi_1(t)$ are fully smooth and satisfy the following conditions: $B(0)\psi_0(0) + C(0)\psi_0(T) = \varphi(0)$, $B(\omega)\psi_1(0) + C(\omega)\psi_1(T) = \varphi(\omega)$.

The task is to find a function $u(t,x)$ which is continuously differentiable with respect to $t\in[0,T]$ and twice continuously differentiable with respect to $x\in[0,\omega]$, such that it satisfies equation (1) along with the conditions (2), (3).

By discretizing with respect to the spatial variable x, the problem (1)-(3) is transformed into a problem of LDE. The second derivative is approximated using the finite difference method. The finite difference methods are discussed in works [18]- [20]. An auxiliary problem for this system will be investigated, focusing on a two-point boundary value problem for

LDE employing the parameterization method with loaded interval partition points in $[0, T]$ [21]- [25]. A method for determining an approximate solution is proposed, together with sufficient conditions ensuring its convergence to the problem's unique solution. A method for numerically solving the problem in systems of LDE is presented [26]- [34].

2 Materials and methods

We consider $\forall x$ and discretize by setting $x_i = i\tau$, $i = \overline{0, N}$, $N\tau = \omega$, with $u_i(t) = u(t, i\tau)$, $a_i(t) = a(t, i\tau)$, $b_i(t) = b(t, i\tau)$, $k_i^j(t_j) = k_j(t, i\tau)$ and $f_i(t) = f(t, i\tau)$. The problem (1)-(3) is then reformulated in the following form:

$$\frac{du_i}{dt} = a_i(t)\frac{u_{i+1} - 2u_i + u_{i-1}}{\tau^2} + b_i(t)u_i + \sum_{j=1}^{m} k_i^j(t)u_i(t_j) + f_i(t), \qquad i = \overline{1, N-1}, \tag{4}$$

$$B_i u_i(0) + C_i u_i(T) = \varphi_i, \quad i = \overline{1, N-1}, \tag{5}$$

$$u_0(t) = \psi_0(t), \quad u_N(t) = \psi_1(t). \tag{6}$$

The solution to the discretized problem (4)-(6) is $\{u_1(t), u_2(t), \ldots, u_{N-1}(t)\}$ system, where is $u_i(t)$ an approximation to the value of the solution $u(t, x)$ at the spatial grid points x_i. It satisfies the system of equations (4), derived from (1) using finite difference approximations for the spatial derivatives. The conditions (5), (6) ensure that $u_i(t)$ adheres to the physical constraints of the problem.

Caused by the linear of the system, for every $\tau > 0$, there exists solution to problem (4)-(6) defined over the interval $[0, T] : \{u_1(t), u_2(t), \ldots, u_{N-1}(t)\}$.

The following statement holds true.

Theorem 1. *Let $a(t, x) \geq \rho > 0$, $b(t, x) \leq 0$, $k_j(t, x)$, $f(t, x)$ - are continuous in Ω, the functions $\varphi(x)$, $\psi_0(t)$, $\psi_1(t)$ are completely smooth and satisfy the matching conditions. Then the solution of the discretized problem (4)-(6) converges at a rate of $O(\tau^2)$ as $\tau \to 0$ approaches the solution of the two-point boundary value problem for a loaded parabolic equation (1)-(3).*

The main goal of this Theorem is to determine the solution to (4)-(6). Thus, we search the conditions for the existence of a solution to problem (4)-(6).

To do this, we write the discretized problem (4)-(6) in matrix-vector form:

$$\frac{dU}{dt} = A(t)U(t) + \sum_{j=1}^{m} M_j(t)U(t_j) + F(t), \quad U \in R^{N-1}, \tag{7}$$

$$BU(0) + CU(T) = \Phi, \quad t \in [0, T], \quad \Phi \in R^{N-1}. \tag{8}$$

Here, the $A(t), M_j(t)$, where $j = 1, \ldots, m$ are matrices of size $(N-1) \times (N-1)$ and $F(t)$ is a vector function of size $(N-1)$ that remains continuous on the interval $[0, T]$; the

B, C are matrices of size $(N-1)\times(N-1)$, where

$$A(t)=\begin{pmatrix} \frac{-2a_1(t)}{\tau^2}+b_1(t) & \frac{a_1(t)}{\tau^2} & 0 & \dots & 0\\ \frac{a_2(t)}{\tau^2} & \frac{-2a_2(t)}{\tau^2}+b_2(t) & \frac{a_2(t)}{\tau^2} & \dots & 0\\ 0 & \frac{a_3(t)}{\tau^2} & \frac{-2a_3(t)}{\tau^2}+b_3(t) & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \frac{-2a_{N-1}(t)}{\tau^2}+b_{N-1}(t)\end{pmatrix},$$

$$F(t)=\begin{pmatrix} \frac{a_1(t)\psi_0(t)}{\tau^2}+f_1(t)\\ f_2(t)\\ f_3(t)\\ \vdots\\ \frac{a_{N-1}(t)\psi_1(t)}{\tau^2}+f_{N-1}(t)\end{pmatrix},\quad M_j(t)=\begin{pmatrix} k_1^j(t) & 0 & 0 & \dots & 0\\ 0 & k_2^j(t) & 0 & \dots & 0\\ 0 & 0 & k_3^j(t) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & k_{N-1}^j(t)\end{pmatrix},$$

$$B=\begin{pmatrix} B_1 & 0 & \dots & 0\\ 0 & B_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & B_{N-1}\end{pmatrix},\quad C=\begin{pmatrix} C_1 & 0 & \dots & 0\\ 0 & C_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & C_{N-1}\end{pmatrix},\quad \Phi=\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_{N-1}\end{pmatrix}.$$

The solution to the problem (7),(8) is a vector function $U(t)$ that is continuously differentiable on $[0,T]$ that fulfills the system of LDE (7) and possesses values $U(0)$, and $U(T)$ at the points $t=0$, $t=T$ respectively, for which the equality (8) holds.

We define $C([0,T],R^{(N-1)})$ as the space of continuous functions $U:[0,T]\to R^{N-1}$, with $\|U\|_1=\max\limits_{t\in[0,T]}\|U(t)\|$, $\|\Phi\|=\max\limits_{i=\overline{1,N-1}}|\varphi_i|$,

$$\|F\|_1\le\max_{t\in[0,T]}\|F(t)\|=$$
$$=\max_{t\in[0,T]}\Big(\frac{\|a_1(t)\|\cdot\|\psi_0\|}{\tau^2}+\|f_1(t)\|,\max_{i=\overline{2,N-2}}\|f_i(t)\|,\frac{\|a_{N-1}(t)\|\cdot\|\psi_1\|}{\tau^2}+\|f_{N-1}(t)\|\Big)\le$$
$$\le\max_{t\in[0,T]}\Big(\frac{\|a_1(t)\|\cdot\|\psi_0\|}{\tau^2},1,\frac{\|a_{N-1}(t)\|\cdot\|\psi_1\|}{\tau^2}\Big)+\max_{t\in[0,T]}\max_{i=\overline{1,N-1}}\|\|f_i(t)\|.$$

The parametrization method developed by professor Dzhumabaev [21] is applied to solve problem (7),(8).

The given interval $[0,T]$ is divided by loading points as follows:

$[0,T)=\bigcup_{r=1}^{m+1}[t_{r-1},t_r)$, $0=t_0<t_1<t_2<\ldots<t_{m+1}=T$.

We define $C([0,T],t_r,R^{(N-1)(m+1)})$ as the space of function systems $U[t]=(U_1(t),\ldots,U_{m+1}(t))$, where the functions $U_r:[t_{r-1},t_r]\to R^{N-1}$ are continuous and have finite left-hand limits, i.e., $\lim\limits_{t\to t_r-0}U_r(t)$ exists for all $r=\overline{1,m+1}$, with $\|U[\cdot]\|_2=\max\limits_{r=\overline{1,m+1}}\sup\limits_{t\in[t_{r-1},t_r)}\|U_r(t)\|$.

The restriction of the function $U(t)$ to the $r-th$ interval $t\in[t_{r-1},t_r)$ is denoted as

$U_r(t) = U(t)$, $r = \overline{1, m+1}$. Consequently, we have

$$\frac{dU_r}{dt} = A(t)U_r(t) + \sum_{j=1}^{m} M_j(t)U_{j+1}(t_j) + F(t), \quad r = \overline{1, m+1}, \tag{9}$$

$$BU_1(0) + C \lim_{t \to T-0} U_{m+1}(t) = \Phi, \tag{10}$$

$$\lim_{t \to t_s - 0} U_s(t) = U_{s+1}(t_s), \quad s = \overline{1, m}. \tag{11}$$

By introducing parameters $\lambda_r = U_r(t_{r-1})$, $r = \overline{1, m+1}$, $\lambda_{m+2} = \lim\limits_{t \to T-0} U_{m+1}(t)$ and by replacing $U_r(t) = \widetilde{u}_r(t) + \lambda_r$ in each interval $[t_{r-1}, t_r)$, $r = \overline{1, m+1}$, we obtain problem with parameters

$$\frac{d\widetilde{u}_r}{dt} = A(t)(\widetilde{u}_r + \lambda_r) + \sum_{j=1}^{m} M_j(t)\lambda_{j+1} + F(t), \quad t \in [t_{r-1}, t_r), \tag{12}$$

$$\widetilde{u}(t_{r-1}) = 0, \qquad r = \overline{1, m+1}, \tag{13}$$

$$B\lambda_1 + C\lambda_{m+2} = \Phi, \tag{14}$$

$$\lambda_s + \lim_{t \to t_s - 0} \widetilde{u}_s(t) = \lambda_{s+1}, \quad s = \overline{1, m+1}. \tag{15}$$

The solution of the problem (12) – (15) is the pair $(\lambda, \widetilde{u}[t])$ with the elements:

$\lambda = (\lambda_1, \ldots, \lambda_{m+2})' \in R^{(N-1)(m+2)}$, $\widetilde{u}[t] = (\widetilde{u}_1(t), \ldots, \widetilde{u}_{m+1}(t))' \in C([0,T], t_r, R^{(N-1)(m+1)})$ here the functions $\widetilde{u}_r(t)$ are continuously differentiable on the $[t_{r-1}, t_r)$, $r = \overline{1, m+1}$, and the λ_r satisfy the system of ordinary differential equations (12) along with the conditions (13) – (15).

If the pair $(\lambda, \widetilde{u}[t])$, where $\lambda = (\lambda_1, \ldots, \lambda_{m+2})' \in R^{(N-1)(m+2)}$, $\widetilde{u}[t] = (\widetilde{u}_1(t), \ldots, \widetilde{u}_{m+1}(t))' \in C([0,T], t_r, R^{(N-1)(m+1)})$ is a solution to the problem (12) – (15), then the function $U(t)$ defined by the equalities $U_r(t) = \widetilde{u}_r(t) + \lambda_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, m+1}$, $U(T) = \lambda_{m+2}$, is the solution to the problem (7), (8). On the other hand, if $U^*(t)$ is the solution to problem (7), (8) then the pair $(\lambda^*, \widetilde{u}^*[t])$, where $\lambda^* = (U^*(t_0), U^*(t_1), \ldots, U^*(t_{m+1}))$, $\widetilde{u}^*[t] = (U^*(t) - U^*(t_0), U^*(t) - U^*(t_1), \ldots, U^*(t) - U^*(t_m))$ will serve as a solution to the problem (12) – (15).

The emergence of the initial conditions $\widetilde{u}_r(t_{r-1}) = 0$, $r = \overline{1, m+1}$, enables us to ascertain the functions $\widetilde{u}_r(t)$, $r = \overline{1, m+1}$, for constant $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m+2})$ derived from the Volterra integral equations of the second type:

$$\widetilde{u}_r(t) = \int_{t_{r-1}}^{t} A(\xi)(\widetilde{u}_r(\xi) + \lambda_r)\mathrm{d}\xi + \int_{t_{r-1}}^{t} \sum_{j=1}^{m} M_j(\xi)\mathrm{d}\xi\lambda_{j+1} + \int_{t_{r-1}}^{t} F(\xi)\mathrm{d}\xi, \quad r = \overline{1, m+1}. \tag{16}$$

In equation (16), by replacing $\widetilde{u}_r(\xi)$, $r=\overline{1,m+1}$, with the appropriate right-hand side, and by iterating this procedure $\nu, (\nu=1,2,\ldots)$ times, we acquire a depiction of the function $\widetilde{u}_r(t)$, $r=\overline{1,m+1}$, expressed in the form below:

$$\widetilde{u}_r(t)=D_{\nu r}(t)\lambda_r+\sum_{j=1}^{m}H_{\nu r}(t,M_j)\lambda_{j+1}+G_{\nu r}(t,\widetilde{u}_r)+\widetilde{F}_{\nu r}(t)\quad r=\overline{1,m+1}, \tag{17}$$

where

$$D_{\nu r}(t)=\int_{t_{r-1}}^{t}A(\xi_1)\mathrm{d}\xi_1+\int_{t_{r-1}}^{t}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}A(\xi_\nu)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1,$$

$$H_{\nu r}(t,M_j)=\int_{t_{r-1}}^{t}M_j(\xi_1)\mathrm{d}\xi_1+\int_{t_{r-1}}^{t}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}M_j(\xi_\nu)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1,$$

$$G_{\nu r}(t,\widetilde{u}_r)=\int_{t_{r-1}}^{t}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}A(\xi_\nu)\widetilde{u}_r(\xi_\nu)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1,$$

$$\widetilde{F}_{\nu r}(t)=\int_{t_{r-1}}^{t}F(\xi_1)\mathrm{d}\xi_1+\int_{t_{r-1}}^{t}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}F(\xi_\nu)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1.$$

From (17) we find

$$\lim_{t\to t_r-0}\widetilde{u}_r(t)=D_{\nu r}(t_r)\lambda_r+\sum_{j=1}^{m}H_{\nu r}(t_r,M_j)\lambda_{j+1}+G_{\nu r}(t_r,\widetilde{u}_r)+\widetilde{F}_{\nu r}(t_r),\quad r=\overline{1,m+1}.$$

Substituting the appropriate right-hand side from (17) into the conditions (14), (15) and multiplying (14) by $l=\max_s(t_s-t_{s-1})$, $s=\overline{1,m+1}$, we obtain the following:

$$B\lambda_1\cdot l+C\lambda_{m+2}\cdot l=\Phi\cdot l, \tag{18}$$

$$[I+D_{\nu s}(t_s)]\lambda_s+\sum_{j=1}^{m}H_{\nu s}(t_s,M_j)\lambda_{j+1}-\lambda_{s+1}=-G_{\nu s}(t_s,\widetilde{u}_s)-\widetilde{F}_{\nu s}(t_s),\quad s=\overline{1,m+1}. \tag{19}$$

where I is an identity matrix size of $((N-1)\times(N-1))$. Let $Q_\nu(l)$ represent the matrix relating to the left-hand side of the system (18), (19), we obtain

$$Q_\nu(l)\lambda=-\widetilde{F}_\nu(l)-G_\nu(\widetilde{u},l), \tag{20}$$

where $\widetilde{F}_\nu(l)=(-\Phi l,\widetilde{F}_{\nu 1}(t_1),\ldots,\widetilde{F}_{\nu m+1}(T))$,

$G_\nu(\widetilde{u},l)=(0,G_{\nu 1}(\widetilde{u}_1,t_1),\ldots,G_{\nu m+1}(\widetilde{u}_{m+1},T))$.

Therefore, to identify the unknown pairs $(\lambda,\widetilde{u}(t))$ that solve the problem (12) –(15) we possess a self-contained set of equations (16), (20). The pairs $(\lambda,\widetilde{u}(t))$ that solves the problem (12) – (15) results in sequences of pairs $(\lambda^k,\widetilde{u}^k(t)), k=0,1,2,\ldots$, determined by the subsequent algorithm:

Step 0: a) Supposing that for the selected $l\in R^+,\nu\in N$, the matrix $Q_\nu(l)$ is invertible, we establish the initial approximation concerning the parameters$\lambda^{(0)}=(\lambda_1^{(0)},\ldots,\lambda_{m+2}^{(0)})\in R^{(N-1)(m+2)}$ from the equation $Q_\nu(l)\lambda^{(0)}=-\widetilde{F}_\nu(l)$, producing $\lambda^{(0)}=-[Q_\nu(l)]^{-1}\widetilde{F}_\nu(l)$.

b) Utilizing the elements of the vector $\lambda^{(0)} \in R^{(N-1)(m+2)}$ and solving the Cauchy problems (12), (13) with $\lambda_r = \lambda_r^{(0)}$ on the $[t_{r-1}, t_r]$, we determine the functions $\widetilde{u}_r^{(0)}(t)$, $r = \overline{1, m+1}$.

Step 1: a) By inserting the obtained $\widetilde{u}_r^{(0)}(t), r = \overline{1, m+1}$, into the right-hand side of (20), we establish $\lambda^{(1)} = (\lambda_1^{(1)}, \dots, \lambda_{m+2}^{(1)}) \in R^{(N-1)(m+2)}$ from $Q_\nu(l)\lambda = -\widetilde{F}_\nu(l) - G_\nu(\widetilde{u}^{(0)}, l)$. b) On the $[t_{r-1}, t_r]$, we address the Cauchy problems (12), (13) by using $\lambda_r = \lambda_r^{(1)}$ and determine the functions $\widetilde{u}_r^{(1)}(t)$, $r = \overline{1, m+1}$, etc.

Proceeding with the procedure, at the k - th step, we obtain a system of pairs $(\lambda^{(k)}, \widetilde{u}^{(k)}[t])$, $k = 0, 1, 2, \dots$. Observe that in point b), for constant values of the parameter λ_r, the solution to the Cauchy problem is determined individually for each interval $t \in [t_{r-1}, t_r]$, $r = \overline{1, m+1}$.

3 Conditions for Convergence of Algorithms and Unique Solution of the Problem (7), (8)

Suppose $\|A(t)\| \le \alpha = const$, $\|M_j(t)\| \le \beta_j = const$, $j = 1, \dots, m$. The requirement for the algorithm's convergence and the uniqueness of the solution to problem (7),(8) lead to the following statement.

Theorem 2. *Let the matrix $Q_\nu(l) : \mathbb{R}^{(N-1)(m+2)} \to \mathbb{R}^{(N-1)(m+2)}$ be invertible, for $l \in \mathbb{R}^+, \nu \in N$, and let the following conditions be satisfied:*

a) $\|[Q_\nu(l)]^{-1}\| \le \varepsilon_\nu(l)$,

b) $g_\nu(l) = \varepsilon_\nu(l) \cdot \Big[e^{\alpha l} - \sum_{j=0}^{\nu} \frac{(\alpha l)^j}{j!} + \sum_{j=1}^{m} \beta_j l \cdot \Big(e^{\alpha l} - \sum_{k=0}^{\nu-1} \frac{(\alpha l)^k}{k!}\Big)\Big] < 1.$

Consequently, the two-point boundary value problem for the LDE (7), (8) has a unique solution $U^*(t)$ and the evaluation is just for its:

$$\|U^*\|_1 \le \widetilde{K}_\nu(l) \max(\|F\|_1, \|\Phi\|),$$

$$\widetilde{K}_\nu(l) = \Big\{\Big(e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^{m} \beta_j l\Big) \cdot \frac{\varepsilon_\nu(l)}{1 - g_\nu(l)} \cdot \frac{(\alpha l)^\nu}{\nu!} + \frac{\varepsilon_\nu(l)}{1 - g_\nu(l)} \cdot \frac{(\alpha l)^\nu}{\nu!} + 1\Big\} \times$$
$$\times \Big\{\Big(e^{\alpha l} - 1 + e^{\alpha l} \cdot \sum_{j=1}^{m} \beta_j l\Big)\varepsilon_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\Big) + e^{\alpha l}\Big\} l + \gamma_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\Big) l.$$

Proof. Under the assumptions of the theorem from step zero of the algorithm, we define and estimate $\lambda^{(0)}$:

$$\|\lambda^{(0)}\| = \max_{r=\overline{1,m+1}} \|\lambda_r^{(0)}\| \le \|[Q_\nu(l)]^{-1}\| \cdot \|\widetilde{F}_\nu(l)\| \le \varepsilon_\nu(l) \cdot \|\widetilde{F}_\nu(l)\|,$$

$$\|\widetilde{F}_\nu(l)\| \le \max(\|\Phi\| l, \max_{r=\overline{1,m+1}} \|\widetilde{F}_{\nu r}(t_r)\|),$$

$$\|\widetilde{F}_{\nu r}(t_r)\| \le \Big\| \int_{t_{r-1}}^{t_r} F(\xi_1)\mathrm{d}\xi_1 \Big\| + \Big\| \int_{t_{r-1}}^{t_r} A(\xi_1) \int_{t_{r-1}}^{\xi_1} F(\xi_2)\mathrm{d}\xi_2 d\xi_1 \Big\| + \ldots +$$
$$+ \Big\| \int_{t_{r-1}}^{t_r} A(\xi_1) \int_{t_{r-1}}^{\xi_1} A(\xi_2) \ldots \int_{t_{r-1}}^{\xi_{\nu-1}} F(\xi_\nu)\mathrm{d}\xi_\nu \ldots \mathrm{d}\xi_1 \Big\| \le$$
$$\le \|F\|_1 l + \alpha l \|F\|_1 l + \ldots + \frac{(\alpha l)^{\nu-1}}{(\nu-1)!}\|F\|_1 l = \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\|F\|_1 l.$$

Then

$$\|\lambda^{(0)}\| \le \varepsilon_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\Big) \max(\|\Phi\|, \|F\|_1) l. \tag{21}$$

Functions $\widetilde{u}_r^{(0)}(t)$ we determine from the following integral systems equations
$\|\widetilde{u}_r^{(0)}(t)\| \le \int_{t_{r-1}}^{t} \alpha \|\widetilde{u}_r^{(0)}\|\mathrm{d}\xi + \int_{t_{r-1}}^{t} \alpha \mathrm{d}\xi \|\lambda_r^{(0)}\| + \int_{t_{r-1}}^{t} \sum_{j=1}^{m} \beta_j \mathrm{d}\xi \|\lambda_{j+1}^{(0)}\| + \int_{t_{r-1}}^{t} \|F(\xi)\|\mathrm{d}\xi,$
adding $\|\lambda_r^{(0)}\|$, $r = \overline{1, m+1}$, and using the Gronwall-Bellman inequality, we obtain:
$\|\widetilde{u}_r^0(t)\| + \|\lambda_r^{(0)}\| \le e^{\alpha(t-t_{r-1})} \cdot \max_{r=\overline{1,m+1}} \Big(\int_{t_{r-1}}^{t_r} \sum_{j=1}^{m} \beta_j \mathrm{d}\xi \|\lambda_{j+1}^{(0)}\| + \int_{t_{r-1}}^{t_r} \|F(\xi)\|\mathrm{d}\xi + \|\lambda_r^{(0)}\| \Big),$

$$\|\widetilde{u}^{(0)}[\cdot]\|_2 = \max_{r=\overline{1,m+1}} \sup_{t\in[t_{r-1},t_r)} \|\widetilde{u}_r^{(0)}(t)\| \le (e^{\alpha l} - 1)\|\lambda^{(0)}\| +$$
$$+ e^{\alpha l} \cdot \max_{r=\overline{1,m+1}} \Big(\int_{t_{r-1}}^{t_r} \sum_{j=1}^{m} (\beta_j \|\lambda_{j+1}^{(0)}\| + \|F(\xi)\|)\mathrm{d}\xi \Big).$$

Now we get

$$\|\widetilde{u}^{(0)}[\cdot]\|_2 \le \Big(e^{\alpha l} - 1 + e^{\alpha l} \cdot \sum_{j=1}^{m} \beta_j l\Big)\|\lambda^{(0)}\| + e^{\alpha l}\|F\|_1 l. \tag{22}$$

from where, taking into account (21) we obtain:
$\|\widetilde{u}^{(0)}[\cdot]\|_2 \le K_\nu(l) \max(\|\Phi\|, \|F\|_1),$
$K_\nu(l) = \Big(e^{\alpha l} - 1 + e^{\alpha l} \cdot \sum_{j=1}^{m} \beta_j l\Big)\varepsilon_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\Big) l + e^{\alpha l} l.$

Using the first step of the algorithm, we determine $\lambda^{(1)}$ and estimate the norm of the difference $\|\lambda^{(1)} - \lambda^{(0)}\|$:
$\|\lambda^{(1)} - \lambda^{(0)}\| \le \gamma_\nu(l) \cdot \|G_\nu(t_r, \widetilde{u}^{(0)})\|,$
$\|G_\nu(t_r, \widetilde{u}^{(0)})\| \le \max_{r=\overline{1,m+1}} \|G_{\nu r}(t_r, \widetilde{u}_r^{(0)})\| \le \frac{(\alpha l)^\nu}{\nu!}\|\widetilde{u}^{(0)}[\cdot]\|_2,$

$\|\lambda^{(1)}-\lambda^{(0)}\| \leq \varepsilon_\nu(l)\cdot \frac{(\alpha l)^\nu}{\nu!}\|\widetilde{u}^{(0)}[\cdot]\|_2 \leq \varepsilon_\nu(l)\cdot \frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|,\|F\|_1)$,

Substituting $\lambda=\lambda^{(1)}$ to the right side of (16) and solving the Cauchy problem we determine $\widetilde{u}^{(1)}[t]=(\widetilde{u}_1^{(1)}(t),\widetilde{u}_2^{(1)}(t),\dots,\widetilde{u}_{m+1}^{(1)}(t))\in C([0,T],t_r,R^{(N-1)(m+1)})$.

By persisting with the iterative process at the $k-th$ step, we derive a pair $(\lambda^{(k)},\widetilde{u}^{(k)}[t])$, $k=0,1,2,\dots$.

$\widetilde{u}^{(k)}[t]=(\widetilde{u}_1^{(k)}(t),\widetilde{u}_2^{(k)}(t),\dots,\widetilde{u}_{m+1}^{(k)}(t))\in C([0,T],t_r,R^{(N-1)(m+1)})$,

$\lambda^{(k)}=(\lambda_1^{(k)},\dots,\lambda_{m+2}^{(k)})\in R^{(N-1)(m+2)}$.

Since $\lambda^{(k+1)}$, $\lambda^{(k)}$ are solutions to equation (20) with the corresponding right-hand sides, then their difference holds the following inequality:

$\|\lambda^{(k+1)}-\lambda^{(k)}\|\leq\varepsilon_\nu(l)\cdot\|G_\nu(t_r,\widetilde{u}^{(k)})-G_\nu(t_r,\widetilde{u}^{(k-1)})\|$,

$$G_{\nu r}(t_r,\widetilde{u}_r^{(k)}-\widetilde{u}_r^{(k-1)})=$$
$$=\int_{t_{r-1}}^{t_r}A(\xi_1)\dots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}A(\xi_\nu)\Big(\widetilde{u}_r^{(k)}(\xi_\nu)-\widetilde{u}_r^{(k-1)}(\xi_{\nu-1})\Big)d\xi_\nu\dots d\xi_1. \quad (23)$$

Using the Gronwall-Bellman inequality again, we estimate the difference in solutions of the Cauchy problems through the difference in parameters:

$$\|\widetilde{u}_r^{(k)}(t)-\widetilde{u}_r^{(k-1)}(t)\|\leq(e^{\alpha(t-t_{r-1})}-1)\cdot\|\lambda_r^{(k)}-\lambda_r^{(k-1)}\|+$$
$$+e^{\alpha(t-t_{r-1})}\int_{t_{r-1}}^{t_r}\sum_{j=1}^m\beta_j d\xi\|\lambda_{j+1}^{(k)}-\lambda_{j+1}^{(k-1)}\|,\quad r=\overline{1,m+1} \quad (24)$$

Substituting (24) into the right side of (23) and calculating the repeated integrals, we obtain:

$$\|G_{\nu r}(t_r,\widetilde{u}^{(k)}-\widetilde{u}^{(k-1)})\|\leq\int_{t_{r-1}}^{t_r}\alpha\dots\int_{t_{r-1}}^{\xi_{\nu-2}}\alpha\int_{t_{r-1}}^{\xi_{\nu-1}}\alpha\Big[(e^{\alpha\xi_\nu}-1)\cdot\|\lambda_r^{(k)}-\lambda_r^{(k-1)}\|+$$
$$+e^{\alpha(t-t_{r-1})}\cdot\max_{r=\overline{1,m+1}}\int_{t_{r-1}}^{t_r}\sum_{j=1}^m\beta_j d\xi\|\lambda_{j+1}^{(k)}-\lambda_{j+1}^{(k-1)}\|\Big]d\xi_\nu\dots d\xi_1=$$
$$=\Big[e^{\alpha l}-1-\alpha l-\dots-\frac{(\alpha l)^\nu}{\nu!}+e^{\alpha l}\sum_{j=1}^m\beta_j l-\sum_{j=1}^m\beta_j l-\dots-\sum_{j=1}^m\beta_j\frac{(\alpha l)^{\nu-1}}{(\nu-1)!}\Big]\|\lambda^{(k)}-\lambda^{(k-1)}\|,$$

$$\|\lambda^{(k+1)}-\lambda^{(k)}\|\leq\varepsilon_\nu(l)\cdot\Big[e^{\alpha l}-\sum_{j=0}^\nu\frac{(\alpha l)^j}{j!}+\sum_{j=1}^m\beta_j l\cdot\Big(e^{\alpha l}-\sum_{\mu=0}^{\nu-1}\frac{(\alpha l)^\mu}{\mu!}\Big)\Big]\|\lambda^{(k)}-\lambda^{(k-1)}\|=$$
$$=g_\nu(l)\cdot\|\lambda^{(k)}-\lambda^{(k-1)}\|\quad k=1,2,\dots \quad (25)$$

$$g_\nu(l)=\varepsilon_\nu(l)\cdot\Big[e^{\alpha l}-\sum_{j=0}^\nu\frac{(\alpha l)^j}{j!}+\sum_{j=1}^m\beta_j l\cdot\Big(e^{\alpha l}-\sum_{\mu=0}^{\nu-1}\frac{(\alpha l)^\mu}{\mu!}\Big)\Big]$$

$\|\lambda^{(k+1)}-\lambda^{(k)}\|\leq g_\nu(l)\cdot g_\nu(l)\cdot\|\lambda^{(k-1)}-\lambda^{(k-2)}\|\leq\dots\leq g_\nu^k(l)\cdot\|\lambda^{(1)}-\lambda^{(0)}\|$,

$$\|\lambda^{(k+p)}-\lambda^{(k)}\|\leq\|\lambda^{(k+p)}-\lambda^{(k+p-1)}\|+\lambda^{(k+p-1)}-\lambda^{(k+p-2)}\|+\dots+\|\lambda^{(k+1)}-\lambda^{(k)}\|\leq$$
$$\leq g_\nu^p(l)\cdot\|\lambda^{(k+1)}-\lambda^{(k)}\|+g_\nu^{p-1}(l)\cdot\|\lambda^{(k+1)}-\lambda^{(k)}\|+\dots+g_\nu(l)\cdot\|\lambda^{(k+1)}-\lambda^{(k)}\|+\|\lambda^{(k+1)}-\lambda^{(k)}\|=$$
$$=(g_\nu^p(l)+g_\nu^{p-1}(l)+\dots+g_\nu(l)+1)\|\lambda^{(k+1)}-\lambda^{(k)}\|.$$

Due to the condition $g_\nu(l) < 1$ and inequalities (22), (23) as $p \to \infty$ the sequence $\lambda^{(k)}$ converges to λ^*, sequence of systems of function $\widetilde{u}^{(k)}[t]$ by the norm of the space $C([0,T], t_r, R^{(N-1)(m+1)})$ converges to $\widetilde{u}^*[t]$ and the following estimates are valid:

$\|\lambda^* - \lambda^{(k)}\| \le \frac{1}{1-g_\nu(l)}\|\lambda^{(k+1)} - \lambda^{(k)}\| \le \frac{1}{1-g_\nu(l)} \cdot g_\nu^k(l)\gamma_\nu(l) \cdot \frac{(\alpha l)^\nu}{\nu!} K_\nu(l) \max(\|\Phi\|, \|F\|_1) l,$

$$\|\widetilde{u}_r^{(k)}(t) - \widetilde{u}_r^{(k-1)}(t)\| \le (e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^m \beta_j l) \cdot \|\lambda_r^{(k)} - \lambda_r^{(k-1)}\|,$$

$$\|\widetilde{u}^*[\cdot] - \widetilde{u}^{(k)}[\cdot]\|_2 \le$$
$$\le \left(e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^m \beta_j l\right)\frac{1}{1-g_\nu(l)} g_\nu^k(l)\varepsilon_\nu(l)\frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1) \quad k = 1, 2, \ldots$$

Using these inequalities for $k = 0$ and taking into account the established estimates (21), (22) we obtain:

$\|\lambda^* - \lambda^{(0)}\| \le \frac{1}{1-g_\nu(l)}\varepsilon_\nu(l)\frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1),$

$$\|U^*\|_1 = \|\lambda^* + \widetilde{u}^*[\cdot]\|_2 \le \|\lambda^* - \lambda^{(0)}\| + \|\widetilde{u}^*[\cdot] - \widetilde{u}^{(0)}[\cdot]\|_2 + \|\lambda^{(0)}\| + \|\widetilde{u}^{(0)}[\cdot]\|_2 \le$$
$$\le \frac{1}{1-g_\nu(l)}\varepsilon_\nu(l)\frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1) +$$
$$+ \left(e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^m \beta_j l\right)\frac{1}{1-g_\nu(l)}\varepsilon_\nu(l)\frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1) +$$
$$+ \varepsilon_\nu(l)\max\left(1, \sum_{j=0}^{\nu-1}\frac{(\alpha l)^j}{j!}\right)\max(\|\Phi\|, \|F\|_1) l + K_\nu(l)\max(\|\Phi\|, \|F\|_1) = \widetilde{K}_\nu(l).$$

Uniqueness. Let $U^*(t)$, $U^{**}(t)$ two solutions to problem (7),(8). Then the corresponding systems of pairs $(\lambda^*, \widetilde{u}^*[t])$, $(\lambda^{**}, \widetilde{u}^{**}[t])$, where $\lambda^* = (U^*(t_0), U^*(t_1), \ldots, U^*(t_{m+1}))$, $\widetilde{u}^*[t] = (U^*(t) - U^*(t_0), U^*(t) - U^*(t_1), \ldots, U^*(t) - U^*(t_m))$, $\lambda^{**} = U^{**}(t_0), U^{**}(t_1), \ldots, U^{**}(t_{m+1}))$, $\widetilde{u}^{**}[t] = (U^{**}(t) - U^{**}(t_0), U^{**}(t) - U^{**}(t_1), \ldots, U^{**}(t) - U^{**}(t_m))$ are solutions to the boundary value problem with parameters (12) – (15) and satisfy relations (16), (20). Similar to estimates (24), (25) the following estimates are established:

$$\|\widetilde{u}^*[\cdot] - \widetilde{u}^{**}[\cdot]\|_2 \le \left(e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^m \beta_j l\right) \cdot \|\lambda^* - \lambda^{**}\|,$$

$\|\lambda^* - \lambda^{**}\| \le g_\nu(l)\|\lambda^* - \lambda^{**}\|.$

Since $g_\nu(l) < 1$ these estimates imply $\lambda^* = \lambda^{**}$, $\widetilde{u}^*[t] = \widetilde{u}^{**}[t]$ i.e. $U^*(t) = U^{**}(t)$ when $t \in [0, T]$.

The Theorem 2 is proved.

Let us now return to problem (4)-(6). Since there is a connection between problems (4)-(6) and (7)-(8), the following theorem holds.

Theorem 3. *Let $U^*(t)$ represent the solution to the problem (7)-(8) and $u^*(t) = (u_1^*(t), u_2^*(t), \ldots, u_{N-1}^*(t))$ represent the solution of the discretized problem (4)-(6). The following conditions hold:*

a) The coefficients $a(t,x)$, $b(t,x)$, $k_j(t,x)$, $f(t,x)$ are smooth and satisfy the constraints $a(t,x) \geq \rho > 0$, $b(t,x) \leq 0$ and Holder continuity in x,

b) The functions $\psi_0(t)$, $\psi_1(t)$, $\varphi(x)$ are sufficiently smooth and satisfy compatibility conditions,

c) The requirements of Theorem 2 are met.

Thus, the discretized problem (4)-(6) has a unique solution $u^*(t) = (u_1^*(t), u_2^*(t), \ldots, u_{N-1}^*(t))$ and the assessment is fair for its:

$\|u_i^*(t)\| \leq \|U^*(t)\| \leq \widetilde{K}_\nu(l) \max(\|F\|_1, \|\Phi\|)$, $i = \overline{1, N-1}$.

Now we can prove Theorem 1. Let $u_i(t)$ - the solution of the discretized problem (4)-(6) and $u(t,x_i)$ - the solution at the grid points problem (1)-(3).

The solution $u(t,x)$ at x_i satisfies:

$\frac{\partial u(t,x_i)}{\partial t} = a(t,x_i) \frac{\partial^2 u}{\partial x^2}\Big|_{x=x_i} + b(t,x_i)u(t,x_i) + \sum\limits_{j=1}^{m} k_j(t,x_i)u(t_j,x_i) + f(t,x_i)$

Using the finite difference approximation: $\frac{\partial^2 u}{\partial x^2}\Big|_{x=x_i} = \frac{u(t,x_{i+1}) - u(t,x_i) + u(t,x_{i-1})}{\tau^2} + O(\tau^2)$, we get

$$\frac{\partial u(t,x_i)}{\partial t} = a(t,x_i)\frac{u(t,x_{i+1}) - u(t,x_i) + u(t,x_{i-1})}{\tau^2} + b(t,x_i)u(t,x_i) + \\ + \sum_{j=1}^{m} k_j(t,x_i)u(t_j,x_i) + f(t,x_i) + O(\tau^2), \quad (26)$$

$$B(x_i)u(0,x_i) + C(x_i)u(T,x_i) = \varphi(x_i), \quad i = \overline{1, N-1}, \quad (27)$$

$$u_0(t) = \psi_0(t), \quad u_N(t) = \psi_1(t), \quad t \in [0,T]. \quad (28)$$

Subtracting the discretized problem (4)-(6) from (26)-(28) gives the error evolution equation $\delta_i(t) = u(t,x_i) - u_i(t)$:

$$\frac{\partial \delta_i}{\partial t} = a(t,x_i)\frac{\delta_{i+1} - \delta_i + \delta_{i-1}}{\tau^2} + b(t,x_i)\delta_i + \sum_{j=1}^{m} k_j(t,x_i)\delta(t_j) + R_i(t), \quad (29)$$

$$B(x_i)\delta(0) + C(x_i)\delta(T) = 0, \quad i = \overline{1, N-1}, \quad (30)$$

$$\delta_0(t) = 0, \quad \delta_N(t) = 0, \quad t \in [0,T]. \quad (31)$$

$R_i(t) = O(\tau^2)$ represents the truncation error from the finite difference approximation. (29)-(31) equation is similar to (4)-(6). This means we can use the estimate from Theorem 2:

$\max\limits_i \|\delta_i(t)\| \leq \widetilde{K}_\nu(l)|R_i(t)|$, $i = \overline{1, N-1}$.

So, we have proven Theorem 1.

4 Conclusion

In this study, the method of lines is utilized to address the two-point boundary value problem for loaded parabolic equations. Assuming that the solution is sufficiently smooth to the initial problem, and according to Theorem 2, the interrelation between the two-point boundary value problem for loaded parabolic equations (1)-(3) and the discretized problem (4)-(6) is demonstrated. The stability and error of this problem will be studied in future works.

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Авторлар туралы мәлімет:

Қуаныш Сәуле Қалдарбекқызы (корреспондент автор) – әл-Фараби атындағы Қазақ ұлттық университетінің Математика кафедрасының докторанты, оқытушысы (Алматы, Қазақстан, электрондық пошта: saule270898@gmail.com);

Асанова Анар Тұрмағанбетқызы – Физика-математика ғылымдарының докторы, Математика және математикалық моделдеу институтының профессоры, бас ғылыми қызметкері. Қазақ ұлттық қыздар педагогикалық университетінің Математика кафедрасының профессоры (Алматы, Қазақстан, электрондық пошта: assanova@math.kz);

Кадирбаева Жазира Муратбековна – Физика-математика ғылымдарының кандидаты, Математика және математикалық моделдеу институтының бас ғылыми қызметкері, профессоры. Қазақ ұлттық қыздар педагогикалық университетінің Математика кафедрасының профессоры (Алматы, Қазақстан, электрондық пошта: zhkadirbayeva@gmail.com);

Сведения об авторах:

Куаныш Сауле Калдарбековна (корреспондент автор) — PhD докторант, преподаватель кафедры математики Казахского национального университета имени аль-Фараби (Алматы, Казахстан, электронная почта: saule270898@gmail.com);

Асанова Анар Турмаганбетовна— доктор физико-математических наук, профессор, главный научный сотрудник отдела дифференциальных уравнений и динамических систем Института математики и математического моделирования. Профессор кафедры математики Казахского национального женского педагогического университета (Алматы, Казахстан, электронная почта: assanova@math.kz);

Кадирбаева Жазира Муратбековна — кандидат физико-математических наук, профессор, главный научный сотрудник отдела дифференциальных уравнений и динамических систем Института математики и математического моделирования. Профессор кафедры математики Казахского национального женского педагогического университета (Алматы, Казахстан, электронная почта: zhkadirbayeva@gmail.com);

Information about authors:

Kuanysh Saule Kaldarbekkyzy (corresponding author) — Doctoral student, Teacher at the Department of Mathematics, Al-Farabi Kazakh National University, (Almaty, Kazakhstan, email: saule270898@gmail.com);

Assanova Anar Turmaganbetkyzy — Doctor of physical and mathematical sciences, Professor, Chief Researcher of the Department of Differential Equations and Dynamical Systems, Institute of Mathematics and Mathematical Modeling. Professor of Department of Mathematics, Kazakh National Women's Teacher Training University (Almaty, Kazakhstan, e-mail: assanova@math.kz);

Kadirbayeva Zhazira Muratbekovna — Candidate of physical and mathematical sciences, Professor, Chief Researcher of the Department of Differential Equations and Dynamical Systems, Institute of Mathematics and Mathematical Modeling. Professor of Department of Mathematics, Kazakh National Women's Teacher Training University (Almaty, Kazakhstan; e-mail: zhkadirbayeva@gmail.com);

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