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SYMMETRIC BANACH-KANTOROVICH SPACES

Let B be a complete Boolean algebra, let $Q(B)$ be the Stone compact of B, let $C_\infty(Q(B))$ be the commutative unital algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, possibly assuming the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$. We consider Maharam measure m defined on B, which takes values in the algebra L^0 of all real measurable functions. With the help of the property of equimeasurability of elements from $C_\infty(Q(B))$, associated with such a measure m, the notion of a symmetric Banach-Kantorovich space $(E, \|\cdot|_E)$ over L^0 is introduced and studied in detail. Here $E \subset C_\infty(Q(B))$, and $\|\cdot|_E$ – L^0-valued norm in E, endowing it with the structure of a Banach-Kantorovich space. Examples of symmetric Banach-Kantorovich spaces are given, which are vector-valued analogues of classical L^p-spaces for $1 \le p \le \infty$, associated with a numerical σ-finite measure.
Key words:: the Banach-Kantorovich space, order complete vector lattice, vector-valued measure, vector integration, symmetric space.

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Симметриялық Банах-Канторович кеңiстiктерi

B толық бульдiк алгебра, $Q(B)$ B бульдiк алгебраға сәйкес келушi стоундiк компакт болсын және $L^0(B) := C_\infty(Q(B))$ $Q(B)$- да анықталған және $\pm$ мәндердi $Q(B)$-дағы еш жерде тығыз болмаған жиындарда ғана қабылдайтын $x : Q(B) \to [-\infty, +\infty]$ барлық өзiлiссiз функциялардың алгебрасы болсын. Магарамның вектор мәндi $m : B \to L^0(\Omega)$ өлшемдерi қарастырылады, олардың мәндерi өлшемi σ-ақырлы болған (Ω, Σ, μ) өлшемдi кеңiстiктегi нақты өлшемдi функцияларға дерлiк барлық жерде тең барлық класстардың $L^0(\Omega)$ алгебрасында болады. m өлшеммен байланысқан $L^0(B)$ - дағы элементтердiң теңөлшемдiлiк қасиетi көмегiмен $(E, \|\cdot|_E)$ симметриялық Банах-Канторович кеңiстiгi түсiнiгi енгiзiледi, мұнда $E \subset L^0(B)$, $\|\cdot|_E$ $E-$дегi $L^0(\Omega)$-мәндi норма, бұл норма оған Банах-Канторович кеңiстiгiнiң құрылымын бередi. Симметриялық Банах-Канторович кеңiстiгiне мысалдар келтiрiледi, олар сандық σ-ақырлы өлшеммен байланысқан классикалық L^p, $1 \le p \le \infty$ кеңiстiктердiң вектор мәндi аналогтары болады.
Түйiн сөздер: векторлық интегралдау, Магарам өлшемi, тең өлшем, Банах – Канторович кеңiстiгi.

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Симметричные пространства Банаха-Канторовича

Пусть B произвольная полная булева алгебра, $Q(B)$ стоуновский компакт, соответствующий булевой алгебре B и $L^0(B) := C_\infty(Q(B))$ алгебра всех непрерывных функций $x : Q(B) \to [-\infty, +\infty]$, определенных на $Q(B)$ и принимающих значения $\pm\infty$ лишь на нигде не плотных множествах из $Q(B)$. Рассматриваются векторнозначные меры Магарам $m : B \to L^0(\Omega)$ со значениями в алгебре $L^0(\Omega)$ всех классов равных почти всюду действительных измеримых функций на измеримом пространстве (Ω, Σ, μ) с σ-конечной мерой. С помощью свойства равноизмеримости элементов из $L^0(B)$, ассоциированного с такой мерой m, вводится понятие симметричного пространства Банаха-Канторовича $(E, \|\cdot|_E)$ над $L^0(\Omega)$, где $E \subset L^0(B)$, и $\|\cdot|_E$ – $L^0(\Omega)$-значная норма в E, наделяющая его структурой пространства Банаха-Канторовича.

Приводятся примеры симметричных пространств Банаха-Канторовича, являющихся векторнозначными аналогами классических L^p-пространств, $1 \le p \le \infty$, ассоциированных с числовой σ-конечной мерой.
Ключевые слова: векторное интегрирование, мера Магарам, равноизмеримость, пространство Банаха - Канторовича.

Introduction

The development of the theory of Banach-Kantorovich space theory began with the construction of integration for measures with the values in order complete vector lattice (K-spaces), in particular, in the algebra $L^0(\Omega)$ of all classes of almost everywhere equal real measurable functions on the measurable space (Ω, Σ, μ) with a σ-finite numerical measure μ. Important examples of the Banach-Kantorovich spaces include the "vector-valued" analogues of the L_p-spaces, $1 \le p < \infty$ [1], [2], and the Orlicz spaces [3], [4], [5]. If Ω is a singleton, then the class of Banach-Kantorovich spaces coincides with the class of real Banach spaces, important examples of which are functional symmetric spaces. The theory of symmetric spaces contains many profound results and has important applications in a wide variety of fields of function theory and functional analysis, in particular, in the theory interpolation of linear operators, ergodic theory and harmonic analysis (see for example, [6], [7], [8]). The development of the theory of Banach-Kantorovich spaces naturally involves the introduction and study of symmetric Banach-Kantorovich spaces. In this paper, we consider a measure m defined on a complete Boolean algebra B, which takes on value in the algebra $L^0(\Omega)$. With the help of this measure, the associated distribution function for elements of the algebra $L^0(B) := C_\infty(Q(B))$ of all continuous functions $x : Q(B) \to \overline{\mathbb{R}} = [-\infty, +\infty]$, defined on the Stone compact $Q(B)$ of a Boolean algebra B, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(B)$, is determined. Then the notion of a symmetric Banach-Kantorovich space $(E, \|\cdot|_E)$ over $L^0(\Omega)$ is introduced, where $E \subset L^0(B)$ and $\|\cdot\|_E - L^0(\Omega)$-valued norm in E, endowing it with the structure of the Banach-Kantorovich space. Examples of symmetric Banach-Kantorovich spaces are given, which are vector-valued analogues of classical L^p-spaces, $1 \le p \le \infty$, associated with a numerical σ-finite measure. Throughout the paper, we use the terminology and notation of the theory of Boolean algebras [9], an order complete vector lattice [10], the theory of vector integration and the theory of Banach-Kantorovich spaces [1], as well as the terminology of the general theory of symmetric spaces [6].

1 Preliminaries

Let (Ω, Σ, μ) be a measurable space with σ-finite measure μ, and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the algebra of all real measurable functions on (Ω, Σ, μ) (functions coinciding almost everywhere are identified). $L^0(\Omega)$ is an order complete vector lattice with respect to the natural partial order $(f \le g \Leftrightarrow g - f \ge 0$ almost everywhere). The weak unit is $\mathbf{1}(\omega) \equiv 1$, and the set $B(\Omega)$ of all idempotents in $L^0(\Omega)$ is a complete Boolean algebra. Denote $L^0(\Omega)_+ = \{f \in L^0(\Omega) : f \ge 0\}$.

Let X be a vector space over the field $\mathbb{R}$ of real numbers. A mapping $\|\cdot\| : X \to L^0(\Omega)$ is called an $L^0(\Omega)$-valued norm on X if the following relations hold for any $x, y \in X$ and $\lambda \in \mathbb{R}$:

(1) $\|x\| \ge 0$, $\|x\| = 0 \Leftrightarrow x = 0$;

(2) $\|\lambda x\| = |\lambda| \; \|x\|$;

(3) $\|x+y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a lattice-normed space over $L^0(\Omega)$. A lattice-normed space X is said to be d-decomposable if for any $x \in X$ and any decomposition $\|x\| = f_1 + f_2$ into a sum of nonnegative disjoint elements $f_1, f_2 \in L^0(\Omega)$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$, $\|x_1\| = f_1$ and $\|x_2\| = f_2$.

A net $\{x_\alpha\}_{\alpha \in A}$ of elements of $(X, \|\cdot\|)$ is said to (bo)-converge to $x \in X$ if the net $\{\|x - x_\alpha\|\}_{\alpha \in A}$ (o)-converges to zero in $L^0(\Omega)$, that is, there exists a decreasing net $\{f_\gamma\}_{\gamma \in \Gamma}$ in $L^0(\Omega)$ such that $f_\gamma \downarrow 0$ and for each $\gamma \in \Gamma$ there is $\alpha(\gamma) \in A$ with $\|x - x_\alpha\| \leq f_\gamma$ $(\alpha \geq \alpha(\gamma))$ [1, 1.3.4] (note, that the o-convergence of a net in $L^0(\Omega)$ is equivalent to its convergence almost everywhere). A net $\{x_\alpha\}_{\alpha \in A} \subset X$ is called (bo)-fundamental if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta) \in A \times A}$ (bo)-converges to zero.

The Banach-Kantorovich space over $L^0(\Omega)$ is defined as a (bo)-complete d-decomposable lattice-normed space over $L^0(\Omega)$. If a Banach Kantorovich space $(X, \|\cdot\|)$ is in addition a vector lattice and the norm $\|\cdot\|$ is monotone (i.e. the conditions $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in X$) then it is called a Banach-Kantorovich lattice over $L^0(\Omega)$ (see [1], [2]). Useful examples of Banach-Kantorovich lattices are constructed using vector integration theory. Let us recall some basic notions of the theory of vector integration (see [1], [2]).

Let B be a arbitrary complete Boolean algebra with zero $\mathbf{0}$ and unit $\mathbf{1}$. A mapping $m : B \to L^0(\Omega)$ is called a $L^0(\Omega)$*-valued measure* if it satisfies the following conditions:

1) $m(e) \geq 0$ for all $e \in B$;

2) $m(e \vee g) = m(e) + m(g)$ for any $e, g \in B$ with $e \wedge g = \mathbf{0}$;

3) $m(e_\alpha) \downarrow 0$ for any net $e_\alpha \downarrow \mathbf{0}$, $\{e_\alpha\} \subset B$.

A measure m is said to be *strictly positive*, if $m(e) = 0$ implies $e = \mathbf{0}$. In this case, B is a Boolean algebra of countable type, thus, in condition 3) above, instead of the net $e_\alpha \downarrow \mathbf{0}$, one can take a sequence $e_n \downarrow \mathbf{0}$, $\{e_n\}_{n \in \mathbb{N}} \subset B$.

A strictly positive $L^0(\Omega)$-valued measure m is said to be *decomposable*, if for any $e \in B$ and a decomposition $m(e) = f_1 + f_2$, $f_1, f_2 \in L^0(\Omega)_+$ there exist $e_1, e_2 \in B$, such that $e = e_1 \vee e_2$, $m(e_1) = f_1$ and $m(e_2) = f_2$. A measure m is decomposable if and only if it is a Maharam measure, that is, the measure m is strictly positive and for any $e \in B$, $0 \leq f \leq m(e)$, $f \in L^0(\Omega)$, there exist $q \in B$, $q \leq e$, such that $m(q) = f$ [11].
The following statement shows that, in the case of the Maharam measure m, there is a natural embedding of the Boolean algebra $B(\Omega)$ into the Boolean algebra B.

Proposition 1 ([12], Proposition 2.3) . *For each $L^0(\Omega)$-valued Maharam measure $m : B \to L^0(\Omega)$ there exists a unique injective completely additive Boolean homomorphism $\varphi : B(\Omega) \to B$ such that $\varphi(B(\Omega))$ is a regular Boolean subalgebra of B, and $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$.*

Let $Q(B)$ be the Stone compact of a complete Boolean algebra B, and let $L^0(B) := C_\infty(Q(B))$ be the algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(B)$. Denotes by $C(Q(B))$ the Banach algebra of all continuous real functions on $Q(B)$ with the uniform norm $\|x\|_\infty = \sup\limits_{t \in Q(B)} |x(t)|$.

We denote by $s(x) := \sup_{n\geq 1}\{|x| > n^{-1}\}$, the support of an element $x \in L^0(B)$, where $\{|x| > \lambda\} \in B$ is the characteristic function χ_{E_λ} of the set E_λ which is the closure in $Q(B)$ of the set $\{t \in Q(B) : |x(t)| > \lambda\},\ \lambda \in \mathbb{R}$.

Let $m : B \to L^0(\Omega)$ be a Maharam measure. We identify B with the complete Boolean algebra of all idempotents in $L^0(B)$, i.e., we assume $B \subset L^0(B)$. By Proposition 1, there exists a regular Boolean subalgebra $\nabla(m)$ in B and a Boolean isomorphism φ from $B(\Omega)$ onto $\nabla(m)$ such that $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$. In this case, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) = C_\infty(Q(\nabla(m)))$ (the corresponding isomorphism will also be denoted by φ), and the algebra $C_\infty(Q(\nabla(m)))$ itself can be considered as a subalgebra and as a regular vector sublattice in $L^0(B) = C_\infty(Q(B))$ (this means that the exact upper and lower bounds for bounded subsets of $L^0(\nabla(m))$ are the same in $L^0(B)$ and in $L^0(\nabla(m))$. In addition, $L^0(B)$ is an $L^0(\nabla(m))$-module.

Denote by $\mathcal{S}(B)$ the vector sublattice of $L^0(B)$ comprising all B-simple (finite-valued) elements, i.e. $x \in \mathcal{S}(B)$ means, that there is a representation $x = \sum_{i=1}^{n} \alpha_i e_i$, where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $e_1, \dots, e_n \in B$ are pairwise disjoint. Let $m : B \to L^0(\Omega)$ be a strongly positive measure on a complete Boolean algebra B. If $x \in \mathcal{S}(B)$ then we put by definition

$$I_m(x) := \int x\,dm := \sum_{k=1}^{n} \alpha_k m(e_k)\ (x \in \mathcal{S}(B)).$$

This formula correctly defines a linear order continuous operator $I_m : \mathcal{S}(B) \to L^0(\Omega)$ [1, 6.1.1, 6.1.2].

We say that a positive element $x \in L^0(B)$ is *integrable* by m, or *m-integrable* if there is an increasing sequence $(x_n)_{n\in\mathbb{N}}$ of positive elements in $\mathcal{S}(B)$ (o)-converging in $L^0(B)$ to x and the supremum $\sup_{n\in\mathbb{N}} \int x_n\,dm$ existing in L^0. In this case, the sequence of integrals $(I_m(x_n))_{n\in\mathbb{N}}$ is (o)-fundamental sequence (see [1, 6.1.3]). Therefore, we may define the integral of x by putting

$$I_m(x) := \int x\,dm := \text{(o)-}\lim_{n\to\infty} \int x_n\,dm.$$

An element $x \in L^0(B)$ is *integrable* (= *m-integrable*), if its positive part x_+ and the negative part x_- are integrable. Denote by $L^1(B,m)$ the set of all integrable elements and, given $x \in L^1(B,m)$, put

$$I_m(x) := \int x\,dm := \int x_+\,dm - \int x_-\,dm.$$

It is known, that $L^1(B,m)$ is an order-dense ideal in $L^0(B)$ and $I_m : L^1(B,m) \to L^0(\Omega)$ is a linear operator. For each $x \in L^1(B,m)$, let $\|x\|_{1,m} := \int |x|\,dm$. Then $(L^1(B,m), \|x\|_{1,m})$ is a lattice-normed space over $L^0(\Omega)$ (see [1, 6.1.3]).

Let $p > 1$, and let

$$L^p(B,m) = \{x \in L^0(B) : |x|^p \in L^1(B,m)\},$$

$$\|x\|_{p,m} := \Big[\int |x|^p dm\Big]^{\frac{1}{p}}, \quad x \in L^p(B,m).$$

The following is known

Theorem 1 *([1], [2]). Let* $m : B \to L^0(\Omega)$ *be a Maharam measure. Then*
(i). $(L^1(B,m), \|x\|_{1,m})$ *is a Banach-Kantorovich space over* $L^0(\Omega)$*, moreover,*

$$L^0(\nabla(m)) \cdot L^1(B,m) \subset L^1(B,m), \int \varphi(\alpha) x dm = \alpha \int x dm,$$

for every $x \in L^1(B,m)$, $\alpha \in L^0(\Omega)$*;*
(ii). $(L^p(B,m), \|x\|_{p,m})$ *is a Banach-Kantorovich space over* $L^0(\Omega)$*.*

In what follows we identify $\varphi(L^0(\Omega))$ and $L^0(\nabla(m))$, and instead of $\varphi(f)$ we will write $f \in L^0(\Omega)$.

The element $x \in L^0(B)$ is called $L^0(\Omega)$-*bounded,* if there exists an element $f \in L^0(\Omega)_+$ such that $|x| \le f$. Denote by $L^\infty(B, L^0(\Omega))$ the set of all $L^0(\Omega)$-bounded elements from $L^0(B)$. It is clear that $L^\infty(B, L^0(\Omega))$ is a subalgebra in $L^0(B)$, as well as order complete vector sublattice in $L^0(B)$, moreover, $L^0(\Omega) \subset L^\infty(B, L^0(\Omega))$, $C(Q(B)) \subset L^\infty(B, L^0(\Omega))$.

For each $x \in L^\infty(B, L^0(\Omega)$ put

$$\|x\|_{\infty, L^0(\Omega)} = \inf\{f \in L^0(\Omega)_+ : |x| \le f\}.$$

It follows directly from the definition of element $\|x\|_{\infty, L^0(\Omega)} \in L^0(\Omega)_+$ that $|x| \le \|x\|_{\infty, L^0(\Omega)}$.

Proposition 2 ([13], Propositions 4 and 5) *. The map*

$$\| \cdot \|_{\infty, L^0(\Omega)} : L^\infty(B, L^0(\Omega)) \to L^0(\Omega)$$

is a $L^0(\Omega)$*-valued d-decomposable norm on* $L^\infty(B, L^0(\Omega))$*. Moreover, if* $|y| \le |x|$, $y, x \in L^\infty(B, L^0(\Omega))$*, then* $\|y\|_{\infty, L^0(\Omega)} \le \|x\|_{\infty, L^0(\Omega)}$*.*

Theorem 2 *.* $(L^\infty(B, L^0(\Omega)), \| \cdot \|_{\infty, L^0(\Omega)})$ *is the Banach-Kantorovich lattice over* $L^0(\Omega)$

Proof. According to Proposition 2, $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty, L^0(\Omega)})$ is a d-decomposable lattice-normed space over $L^0(\Omega)$.

It remains to show that this lattice-normed space is (bo)-complete. Take an (bo)-fundamental net $\{x_\alpha\}_{\alpha \in A} \subset L^\infty(B, L^0(\Omega))$. Then by the definition of fundamentality, net $(x_\alpha - x_\beta)_{(\alpha,\beta) \in A \times A}$ (bo)-converges to zero. Hence, there exists a net $\{h_\gamma\}_{\gamma \in \Gamma} \downarrow 0$, $h_\gamma \in L^\infty(B, L^0(\Omega))$ such that for any h_γ there is $\alpha(\gamma) \in A$, that

$$|x_\alpha - x_\beta| \le \|x_\alpha - x_\beta\|_{\infty, L^0(\Omega)} \le h_\gamma \text{ for all } \alpha \ge \alpha(\gamma),\ \beta \ge \alpha(\gamma). \tag{1}$$

This means that the net $\{x_\alpha\}_{\alpha \in A}$ is a (o)-fundamental net from order complete vector lattice $L^\infty(B, L^0(\Omega))$. Consequently, this net (o)-converges in $L^\infty(B, L^0(\Omega))$, i.e., there exists an element $x \in L^\infty(B, L^0(\Omega))$, for which $x_\alpha \xrightarrow{(o)} x$. In particular, for each fixed $\beta \ge \alpha(\gamma)$ the net $\{x_\alpha - x_\beta\}_{\alpha \in A,\ \alpha \ge \alpha(\gamma)}$ (o)-converges in $L^\infty(B, L^0(\Omega))$ to element $x - x_{\alpha(\gamma)}$. Hence by (1) we have $|x - x_\beta| \le h_\gamma \in L^\infty(B, L^0(\Omega))$ for all $\beta \ge \alpha(\gamma)$, which implies the inequality

$$\|x - x_\beta\|_{\infty, L^0(\Omega)}) \le h_\gamma \text{ for all } \beta \ge \alpha(\gamma).$$

This means that the net $\{x_\alpha\}_{\alpha \in A}$ (bo)-converges to the element $x \in L^\infty(B, L^0(\Omega))$. $\square$

2 Symmetric spaces of Banach-Kantorovich

Denote by $L^0(\Omega)_{++}$ the set of all positive elements $\lambda \in L^0_+(\Omega)$ such that $s(\lambda) = \mathbf{1}$. It is clear that for any $\lambda \in L^0(\Omega)_{++}$ there exists $\lambda^{-1} \in L^0(\Omega)_{++}$ such that $\lambda \cdot \lambda^{-1} = \mathbf{1}$.

Let m be $L^0(\Omega)$-valued Maharam measure on a complete Boolean algebra B. In the rest of this section we assume that $m(\mathbf{1}) = \mathbf{1}$.

Definition 1 . *Let $0 \le x \in L^0(B)$ and $h \in L^0_{++}(\Omega)$. The $L^0(\Omega)$-valued distribution function $\eta_x : L^0_{++}(\Omega) \to L^0(\Omega)_+$ is defined by setting*

$$\eta_x(h) := m(\{x > h\}),$$

where $\{x > h\} \in B$ is the idempotent in the algebra $L^0(B)$, which is the characteristic function $\chi_{E_h(x)}$ of the closure $E_h(x)$ of the set $\{s \in Q(B) : x(s) > h(s)\}$.

Proposition 3 . *A mapping η_x is decreasing, and right-continuous, that is, if $h_n \in L^0_{++}(\Omega), n = 0, 1, ...,$ and $h_n \downarrow h_0$, then $\eta_x(h_0) = \sup\limits_{n \ge 1} \eta_x(h_n)$*

Proof. The first statement follows from the following implications

$$h_1 \le h_2 \Rightarrow E_{h_1}(x) \supseteq E_{h_2}(x) \Rightarrow \{x > h_1\} \ge \{x > h_2\} \Rightarrow \eta_x(h_1) \ge \eta_x(h_2).$$

To establish right-continuity, let $q_h = \{x > h\}$, $(h \in L^0_{++}(\Omega))$, and fix $h_0 \in L^0_{++}(\Omega)$. The sets $E_{h_n}(x)$ increase as h_n decreases, and $E_{h_0}(x) = \bigcup\limits_{n=1}^{\infty} E_{h_n}(x)$. Hence, by the monotone convergence property of measure m,

$$\eta_x(h_n) = m(q_{h_n}) \uparrow m(q_{h_0}) = \eta_x(h_0). \qquad \square$$

Proposition 4 . *Suppose $x, y, x_n \, (n = 1, 2, \ldots)$ belong to $L^0(B)$, and let $h, g \in L^0_{++}(\Omega)$. Then*

(i) if $|x| \le |y|$, then $\eta_{|x|}(h) \le \eta_{|y|}(h)$;

(ii) $\eta_{g|x|}(h) = \eta_{|x|}(\frac{h}{g})$;

(iii) if $x \ge 0, y \ge 0$, $h_1, h_2 \in L^0_{++}(\Omega)$, then $\eta_{x+y}(h_1 + h_2) \le \eta_x(h_1) + \eta_y(h_2)$;

(iv) if $|x_n| \uparrow |x|$, then $\eta_{|x_n|}(h) \uparrow \eta_{|x|}(h)$ for every $h \in L^0_{++}(\Omega)$.

Proof. (i). If $|x| \le |y|$, then $\{|x| > h\} \le \{|y| > h\}$. Consequently, $\eta_{|x|}(h) = m(\{|x| > h\}) \le m(\{|y| > h\}) = \eta_{|y|}(h)$.

(ii). $\eta_{g|x|}(h) = m(\{g|x| > h\}) = m(\{|x| > \frac{h}{g}\}) = \eta_{|x|}(\frac{h}{g})$.

(iii). If $s \in Q(B)$ and $x(s) + y(s) > h_1(s) + h_2(s)$, then either $x(s) > h_1(s)$ or $y(s) > h_2(s)$. Therefore $\{x + y > h_1 + h_2\} \le \{x > h_1\} \vee \{y > h_2\}$. Consequently,

$$\eta_{x+y}(h_1 + h_2) = m(\{x + y > h_1 + h_2\}) \le m(\{x > h_1\}) + m(\{y > h_2\}) = \eta_x(h_1) + \eta_y(h_2).$$

(iv). We fix $h \in L^0_{++}(\Omega)$ and put $G_h(x) = \{s \in Q(B) : |x(s)| > h(s)\}$, $G_h(x_n) = \{s \in Q(B) : |x_n(s)| > h(s)\}$, $(n = 1, 2, ...)$. Since $|x_n| \le |x_{n+1}|$, then $G_h(x_n) \subset G_h(x_{n+1})$.

Furthermore, the condition $|x_n| \uparrow |x|$ imply that $G_h(x) = \bigcup_{n=1}^{\infty} G_h(x_n)$. Hence, by the monotone convergence property of measure m,

$$\eta_{|x_n|}(h) = m(\{|x_n| > h\}) \uparrow m(\{|x| > h\}) = \eta_{|x|}(h). \qquad \square$$

Examples 1 . *1. Let $x = e \in B$ and $h \in L^0_{++}(\Omega)$. Then $\eta_e(h) = m(\{e > h\}) = m(e)$, if $h < \mathbf{1}$, and $\eta_e(h) = \mathbf{0}$, if $h \geq \mathbf{1}$.*

2. It will be worthwhile to formally compute the $L^0(\Omega)$-valued distribution function η_x of a positive B-simple element $x \in \mathcal{S}(B)$. Suppose

$$x = \sum_{k=1}^{n} \alpha_k e_k,$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+ = (0, \infty)$, and $e_1, \ldots, e_n$ are pairwise disjoint elements of B. Without loss of generality we may assume that $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$. If $h \in L^0_{++}(\Omega)$ and $h \geq \alpha_1 \cdot \mathbf{1}$, then clearly $\eta_x(h) = 0$. However, if $\alpha_2 \cdot \mathbf{1} \leq h < \alpha_1 \cdot \mathbf{1}$, then $\{x > h\} = e_1$, and so $\eta_x(h) = m(e_1)$. Similarly, if $\alpha_3 \cdot \mathbf{1} \leq h < \alpha_2 \cdot \mathbf{1}$, then $\{x > h\} = e_1 \vee e_2$, and so $\eta_x(h) = m(e_1 \vee e_2) = m(e_1) + m(e_2)$. In general, we have

$$\eta_x(h) = \sum_{i=1}^{k} m(e_i) \quad \textit{if} \;\; \alpha_{k+1} \cdot \mathbf{1} \leq h < \alpha_k \cdot \mathbf{1} \; (h \in L^0_{++}(\Omega)),$$

where $k = 1, 2, \ldots, n$, and $\alpha_{n+1} = 0$.

Definition 2 . *Positive elements $x, y \in L^0(B)$ are called m-equimeasurable, if $\eta_x = \eta_y$, i.e.,*

$$m\{x > h\} = m\{y > h\}$$

for all $h \in L^0_{++}(\Omega)$.

Examples 2 . *1. Two idempotents $e_1, e_2 \in B$ are m-equimeasurable if and only if $m(e_1) = m(e_2)$ (see Example 1.1.)*

2. Let $x, y \in \mathcal{S}(B)_+$, $x = \sum_{k=1}^{n} \alpha_k e_k$ and $y = \sum_{k=1}^{n} \beta_k g_k$, where $\alpha_k, \beta_k \in \mathbb{R}_+$, $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$, $\beta_1 > \beta_2 > \ldots > \beta_n > 0$, and $\{e_k\}$, respectively, $\{g_k\}$ are pairwise disjoint elements of B. By Example 1.2, we have

$$\eta_x(h) = \sum_{i=1}^{k} m(e_i) \quad \textit{if} \; \alpha_{k+1} \cdot \mathbf{1} \leq h < \alpha_k \cdot \mathbf{1},$$

$$\eta_y(h) = \sum_{i=1}^{k} m(g_i) \quad \textit{if} \; \beta_{k+1} \cdot \mathbf{1} \leq h < \beta_k \cdot \mathbf{1} \; (h \in L^0_{++}(\Omega)),$$

where $k = 1, 2, \ldots, n$, and $\alpha_{n+1} = \beta_{n+1} = 0$.

From equality $\eta_x(t) = \eta_y(t)$ we get $\alpha_k = \beta_k$ and $\sum_{i=1}^{k} m(e_i) = \sum_{i=1}^{k} m(g_i)$ for all $k = 1, \ldots, n$. Of the last equalities by $k = 1$ we have $m(e_1) = m(g_1)$. Further, if $k = 2$ the equality

$m(e_1)+m(e_2) = m(g_1)+m(g_2)$ is true, thus $m(e_2) = m(g_2)$, etc., when $k = n$ we get $m(e_n) = m(g_n)$.

Thus the elements x and y m-equimeasurable if and only if $\alpha_k = \beta_k$ and $m(e_k) = m(g_k)$ for all $k = 1,\dots,n$.

3. If x and y are positive elements of $L^0(B)$ and $m\{x > t\cdot\mathbf{1}\} = m\{y > t\cdot\mathbf{1}\}$ for all $t > 0$, then

$$m\{x \le t\cdot\mathbf{1}\} = m\{y \le t\cdot\mathbf{1}\} \quad \text{and} \quad m\{s\cdot\mathbf{1} < x \le t\cdot\mathbf{1}\} = m\{s\cdot\mathbf{1} < y \le t\cdot\mathbf{1}\}$$

for any $0 < s < t$.

Indeed, $m\{x \le t\cdot\mathbf{1}\} = m(\mathbf{1}) - m\{x > t\cdot\mathbf{1}\} = m(\mathbf{1}) - m\{y > t\cdot\mathbf{1}\} = m\{y \le t\cdot\mathbf{1}\}$.

Further, using the equalities $\{s\cdot\mathbf{1} < x \le t\cdot\mathbf{1}\} = \{x > s\cdot\mathbf{1}\} - \{x > t\cdot\mathbf{1}\}, \{s\cdot\mathbf{1} < y \le t\cdot\mathbf{1}\} = \{y > s\cdot\mathbf{1}\} - \{y > t\cdot\mathbf{1}\}$, we get

$$m\{s\cdot\mathbf{1} < x \le t\cdot\mathbf{1}\} = m\{x > s\cdot\mathbf{1}\} - m\{x > t\cdot\mathbf{1}\} =$$

$$m\{y > s\cdot\mathbf{1}\} - m\{y > t\cdot\mathbf{1}\} = m\{s\cdot\mathbf{1} < y \le t\cdot\mathbf{1}\}.$$

The following theorem establishes the equality of integrals for m-equimeasurable elements.

Theorem 3 . *If x, y are m-equimeasurable, where $y \in L^1(B,m)$, then $x \in L^1(B,m)$ and $\int xdm = \int ydm$.*

Proof. Let $x, y \in S(B)_+$, $x = \sum\limits_{k=1}^{n}\alpha_k e_k$ and $y = \sum\limits_{k=1}^{n}\beta_k g_k$. Then by m-equimeasurable x and y (see Example 2.2.),

$$\int xdm = \sum_{k=1}^{n}\alpha_k m(e_k) = \sum_{k=1}^{n}\beta_k m(g_k) = \int ydm.$$

Let now $x \in L^0(B)_+$, $0 \le y \in L^1(B,m)$ and $\eta_x = \eta_y$. Let us, first assume that $y \in C(Q(B))$. Recall that by assumption $m(\mathbf{1}) = \mathbf{1}$, and therefore $C(Q(B)) \subset L^1(B,m)$), in this case, $\|y\|_{1,m} \le \|y\|_\infty\mathbf{1}$. Without loss of generality we may assume that $\|y\|_\infty \le 1$. Since $\eta_x = \eta_y$, then $m\{x > \mathbf{1}\} = m\{y > \mathbf{1}\} = \mathbf{0}$, that is $\|x\|_\infty \le 1$.

Consider the following two increasing sequences of positive simple elements

$$x_n = (\sum_{k=1}^{2^n}\frac{k-1}{2^n}e_k) \uparrow x, \quad y_n = (\sum_{k=1}^{2^n}\frac{k-1}{2^n}g_k) \uparrow y,$$

where $e_k = \{\frac{k-1}{2^n}\cdot\mathbf{1} < x \le \frac{k}{2^n}\cdot\mathbf{1}\}$, $g_k = \{\frac{k-1}{2^n}\cdot\mathbf{1} < y \le \frac{k}{2^n}\cdot\mathbf{1}\}$. Since $\eta_x = \eta_y$, then $m(e_k) = m(g_k)$ (see example 2.3.), and therefore

$$\int x_n dm = \sum_{k=1}^{2^n}\frac{k-1}{2^n}m(e_k) = \sum_{k=1}^{2^n}\frac{k-1}{2^n}m(g_k) = \int y_n dm \uparrow \int ydm.$$

Hence,

$$\int xdm = (o)\text{-}\lim_{n\to\infty}\int x_n dm = \int ydm.$$

Now let y be an arbitrary positive element $L^1(B,m)$. Consider two increasing sequences of positive elements of $C(Q(B))$

$$x_n = xp_n \uparrow x, \quad y_n = yq_n \uparrow y,$$

where $p_n = \{x \le n\cdot\mathbf{1}\}$, $q_n = \{y \le n\cdot\mathbf{1}\}$. Using example 2.3., we obtain

$$m\{x_n > t\cdot\mathbf{1}\} = m\{xp_n > t\cdot\mathbf{1}\} = m\{t\cdot\mathbf{1} < x \le n\cdot\mathbf{1}\} = m\{t\cdot\mathbf{1} < y \le n\cdot\mathbf{1}\} =$$

$$= m\{yq_n > t\cdot\mathbf{1}\} = m\{y_n > t\cdot\mathbf{1}\}$$

for any $t \in \mathbb{R}^+$. Since y_n is an integrable element of $C(Q(B))$, it follows from the above, that $\int x_n dm = \int y_n dm$. At the same time, there is a limit

$$(o)\text{-}\lim_{n\to\infty}\int x_n dm = (o)\text{-}\lim_{n\to\infty}\int y_n dm = \int y dm.$$

Hence $x \in L^1(B,m)$ and $\int x dm = \int y dm$. $\square$

Corollary 1 . *Let $0 \le x \in L^0(B)$ and $0 \le y \in L^p(B,m)$, $p > 1$. If x and y m-equimeasurable, then $x \in L^p(B,m)$ and $\|x\|_{p,m} = \|y\|_{p,m}$.*

Proof. Since $y^p \in L^1(B,m)$ and

$$m\{x^p > t\cdot\mathbf{1}\} = m\{x > t^{\frac{1}{p}}\cdot\mathbf{1}\} = m\{y > t^{\frac{1}{p}}\cdot\mathbf{1}\} = m\{y^p > t\cdot\mathbf{1}\}$$

for any $t \in \mathbb{R}^+$, $p > 1$, then for the elements x^p and y^p the proof of Theorem 3 is preserved, by virtue of which we obtain

$$x^p \in L^1(B,m) \text{ and } \int x^p dm = \int y^p dm,$$

i.e. $x \in L^p(B,m)$ and $\|x\|_{p,m} = \|y\|_{p,m}$. $\square$

Definition 3 . *Let E – be a nonzero linear subspace in $L^0(B)$ with the property of ideality, i.e. for $x \in L^0(B)$ and $y \in E$, from $|x| \le |y|$ it follows that $x \in E$. Consider the $L^0(\Omega)$-valued norm $\|\cdot\|_E$ on E, which endows E with the structure of a Banach-Kantorovich lattice. We say that E is a symmetric Banach-Kantorovich space over $L^0(\Omega)$, if m-equimeasurablity of the elements x and y, where $x \in L^0(B)_+$, $0 \le y \in E$, implies that $x \in E$ and $\|x\|_E = \|y\|_E$.*

The main and most important examples of symmetric Banach-Kantorovich spaces are the spaces $L^p(B,m)$, $1 \le p < \infty$, and $L^\infty(B, L^0(\Omega))$.

Theorem 4 . *(i). $(L^p(B,m), \|\cdot\|_{p,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$ for every $1 \le p < \infty$.*

(ii). $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty, L^0(\Omega)})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

Proof. (i). According to [1, Section 6.1], linear subspace $L^1(B,m) \subset L^0(B)$ has the ideality property, moreover, the norm $\|\cdot\|_{1,m}$ is monotone, and the space $L^1(B,m)$, equipped with this norm, is a Banach-Kantorovich lattice. It remains to apply theorem 3, by virtue of which the pair $(L^1(B,m), \|\cdot\|_{1,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

Now let $|x| \leq |y|$, $x \in L^0(B)$, $y \in L^p(B,m)$, where $1 < p < \infty$. Since $|x|^p \leq |y|^p \in L^1(B,m)$, then $|x|^p \in L^1(B,m)$ and

$$\|x\|_{p,m}^p = \||x|^p\|_{1,m} \leq \||y|^p\|_{1,m} = \|y\|_{p,m}^p,$$

and therefore $\|x\|_{p,m} \leq \|y\|_{p,m}$, i.e. $\|\cdot\|_{p,m}$ is $L^0(\Omega)$-valued monotone norm on $L^p(B,m)$, which endows $L^p(B,m)$ with the structure of a Banach-Kantorovich lattice over $L^0(\Omega)$. It remains to apply Corollary 1, by virtue of which the pair
$(L^p(B,m), \|\cdot\|_{p,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

(ii). By Theorem 2, the pair $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a Banach-Kantorovich lattice, moreover, it is clear that $L^\infty(B, L^0(\Omega))$ has the ideality property and the norm $\|\cdot\|_{\infty,L^0(\Omega)}$ is monotone on $L^\infty(B, L^0(\Omega))$.
Let $x \in L^0(B)$, $y \in L^\infty(B, L^0(\Omega))$, and let x and y be m-equimeasurable. Assign $h(\varepsilon) = \|y\|_{\infty,L^0(\Omega)} + \varepsilon \cdot \mathbf{1}$, $\varepsilon > 0$. Since $h(\varepsilon) \in L^0_{++}(\Omega)$, then

$$m\{|x| > h(\varepsilon)\} = m\{|y| > h(\varepsilon)\} = \mathbf{0}.$$

Hence, $|x| \leq h(\varepsilon)$, and therefore $x \in L^\infty(B, L^0(\Omega)$, moreover, $\|x\|_{\infty,L^0(\Omega)} \leq h(\varepsilon)$ for every $\varepsilon > 0$. From this it follows that $\|x\|_{\infty,L^0(\Omega)} \leq \|y\|_{\infty,L^0(\Omega)}$.

Let's put now $h_1(\varepsilon) = \|x\|_{\infty,L^0(\Omega)} + \varepsilon \cdot \mathbf{1} \in L^0_{++}(\Omega)$, $\varepsilon > 0$. Using equalities

$$m\{|y| > h_1(\varepsilon)\} = m\{|x| > h_1(\varepsilon)\} = \mathbf{0},$$

we get that $\|y\|_{\infty,L^0(\Omega)} \leq h_1(\varepsilon)$ for every $\varepsilon > 0$. This means that $\|y\|_{\infty,L^0(\Omega)} \leq \|x\|_{\infty,L^0(\Omega)}$. Thus, $\|x\|_{\infty,L^0(\Omega)} = \|y\|_{\infty,L^0(\Omega)}$.

Consequently, $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$. □
Following the general theory of functional symmetric spaces, consider a space $L^1(B,m) \cap L^\infty(B, L^0(\Omega)$ with a norm

$$\|x\|_{L^1 \cap L^\infty} = \|x\|_{1,m} \vee \|x\|_{\infty,L^0(\Omega)}, x \in L^1(B,m) \cap L^\infty(B, L^0(\Omega)).$$

Proposition 5 . *$(L^1(B,m) \cap L^\infty(B, L^0(\Omega)), \|\cdot\|_{L^1 \cap L^\infty})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.*

Proof. Since $m(\mathbf{1}) = \mathbf{1}$, and for every $x \in L^\infty(B, L^0(\Omega))$ the inequality $|x| \leq \|x\|_{\infty,L^0(\Omega)}$ is true, then $L^\infty(B, L^0(\Omega)) \subset L^1(B,m)$, moreover, $\|x\|_{1,m} \leq \|x\|_{\infty,L^0(\Omega)}$. Hence, $L^1(B,m) \cap L^\infty(B, L^0(\Omega) = L^\infty(B, L^0(\Omega))$ and $\|x\|_{1,m} \vee \|x\|_{\infty,L^0(\Omega)} = \|x\|_{\infty,L^0(\Omega)}$. Thus, the pair

$$(L^1(B,m) \cap L^\infty(B, L^0(\Omega)), \|\cdot\|_{L^1 \cap L^\infty}(B, L^0(\Omega)) = (L^\infty(B, L^0(\Omega), \|\cdot\|_{\infty,L^0(\Omega)})$$

is a symmetric Banach-Kantorovich space over $L^0(\Omega)$ (see Theorem 4 (ii)). □

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