IRSTI 27.39.21 DOI: https://doi.org/10.26577/JMMCS2025125105

K.N. Ospanov[1], **Ye.O. Moldagali** [1*]
[1]L.N. Gumilyov Eurasian National University, Faculty of Mechanics and Mathematics, Astana, Kazakhstan
[*]e-mail: yerka2998@gmail.com

CONDITIONS FOR SOLVABILITY AND COERCIVENESS OF A FOURTH-ORDER DIFFERENTIAL EQUATION WITH AN INTERMEDIATE COEFFICIENT

The article considers a three-term fourth-order differential equation with unbounded coefficients. The coefficient of the intermediate term of the equation is assumed to be a smooth and rapidly increasing function at infinity. This intermediate term, as an operator, does not obey the differential operator formed by the extreme terms of the equation. This is precisely what makes the work unique. Using functional methods, sufficient conditions are obtained for a generalized solution of the equation to exist, be unique and maximally regular. These conditions characterize the relationship between intermediate and small coefficients. The differential equation under consideration is caused by problems of practical processes of stochastic analysis, shaft oscillations, etc. The article uses such methods as obtaining an a priori estimate of the solution, reducing the problem above to the problem of invertibility of a third-order differential operator with a potential of constant sign, and constructing a pseudo-resolvent using correct local operators. In general, the article substantiates an effective method for solving the main problems posed for differential equations on an infinite interval in the case of a new equation with an unbounded intermediate coefficient. Although the coefficients are assumed to be smooth, the work does not impose restrictions on the variation of their derivatives. This, in turn, allows us to cover a wide class of fourth-order equations. The methods developed in the work and the results obtained can be used in the study of the qualitative properties of higher-order differential equations.

Key words: differential equation, variable coefficient, strong solution, correctness, regularity.

Қ.Н. Оспанов[1], Е.Ө. Молдағали[1*]
[1]Л.Н. Гумилев атындағы Еуразия ұлттық университеті, механика-математика факультеті, Астана, Қазақстан
[*]e-mail: yerka2998@gmail.com

Аралық коэффициенті бар төртінші ретті бір дифференциалдық теңдеудің шешілу және коэрцитивтілік шарттары

Мақалада коэффициенттері шенелмеген төртінші ретті үшмүшелі дифференциалдық теңдеу қарастырылған. Теңдеудің аралық мүшесінің коэффициенті тегіс және шексіздікте жылдам өсетін функция деп есептеледі. Бұл аралық мүше оператор ретінде теңдеудің шеткі мүшелерінен құралған дифференциалдық операторға бағынбайды. Жұмыстың ерекшелігі осында. Біз функционалдық әдістерді қолдана отырып, теңдеудің жалпыланған шешімінің табылуы, жалғыз және максималды регулярлы болуы үшін жеткілікті шарттар алдық. Бұл шарттар аралық және кіші коэффициенттердің өзара байланысын сипаттайды. Қарастырылған дифференциалдық теңдеуге стохастикалық талдаудың, біліктің тербелісінің және т.б. практикалық процестердің мәселелері алып келеді. Мақалада шешімнің априорлық бағасын алу, қойылған есепті потенциалы тұрақты таңбалы бір үшінші ретті дифференциалдық оператордың қайтымдылық мәселесіне келтіру, корректілі локальды операторлар арқылы псевдорезольвентаны тұрғызу сияқты тың амалдар қолданылды. Жалпы, мақала шексіз аралықта берілген дифференциалдық теңдеулер үшін қойылатын негізгі есептерді жаңа, аралық коэффициенті шенелмеген теңдеу жағдайында шешудің бір тиімді әдісін негіздейді. Коэффициенттер тегіс деп есептелсе де, олардың туындыларының өзгеруіне жұмыста шектеулер қойылмайды. Бұл, өз кезегінде, зерттеумен төртінші ретті теңдеулердің кең класын қамтуға мүмкіндік береді. Жұмыста жасалған әдістер мен алынған нәтижелерді жоғарғы ретті дифференциалдық теңдеулерді сапалық зерттеу кезінде пайдалануға болады.

Түйін сөздер: дифференциалдық теңдеу, айнымалы коэффициент, қатаң шешім, қисындылық, регулярлық.

К.Н. Оспанов[1], Е.О. Молдагали[1*]
[1] Евразийский национальный университет имени Л.Н. Гумилева,
Механико-математический факультет, Астана, Казахстан
*e-mail: yerka2998@gmail.com

Условия разрешимости и коэрцитивности одного дифференциального уравнения четвертого порядка с промежуточным коэффициентом

В статье рассматривается трехчленное дифференциальное уравнение четвертого порядка с неограниченными коэффициентами. Коэффициент промежуточного члена уравнения предполагается гладкой и быстро возрастающей функцией на бесконечности. Этот промежуточный член, как оператор, не подчиняется дифференциальному оператору, образованному крайними членами уравнения. Именно в этом и заключается уникальность работы. Используя функциональные методы, получены достаточные условия для того, чтобы обобщенное решение уравнения существовало, было единственным и максимально регулярным. Эти условия характеризуют взаимосвязь промежуточного и младшего коэффициентов. К рассматриваемому дифференциальному уравнению приводят проблемы практических процессов стохастического анализа, колебаний вала и др. В статье используются такие методы, как получение априорной оценки решения, сведение поставленного вопроса к задаче обратимости дифференциального оператора третьего порядка со знакопостоянным потенциалом и построение псевдорезольвенты с использованием корректных локальных операторов. В целом, в статье обоснован эффективный метод решения основных задач, поставленных для дифференциальных уравнений на бесконечном интервале, в случае нового уравнения с неограниченным промежуточным коэффициентом. Хотя коэффициенты предполагаются гладкими, работа не накладывает ограничений на изменение их производных. Это, в свою очередь, позволяет охватить исследованием широкий класс уравнений четвертого порядка. Разработанные в работе методы и полученные результаты могут быть использованы при исследовании качественных свойств дифференциальных уравнений высших порядков.
Ключевые слова: дифференциальное уравнение, переменный коэффициент, сильное решение, корректность, регулярность.

1 Introduction

Let's consider the equation

$$L_0 y = y^{(4)} + r(x)y' + q(x)y = F(x), \tag{1}$$

where we assume that $x \in \mathbb{R} = (-\infty, \infty)$, $r(x) > 0$, $r(x) \in C^{(1)}_{\mathrm{loc}}(\mathbb{R})$, $q(x)$ is a continuous function, and $F(x) \in L_2(\mathbb{R})$. Let L denote the closure in the $L_2(\mathbb{R})$ norm of the operator L_0, defined by the equality $L_0 y = y^{(4)} + r(x)y' + q(x)y$ on the set $C_0^{(4)}(\mathbb{R})$ of continuously differentiable functions up to the fourth order with compact support. An element $y \in D(L)$ satisfying the equality $Ly = F$ is called a solution to equation (1). The coefficients of equation (1) can grow infinitely. It is known that fourth-order differential equations with variable coefficients are of great importance in physics and engineering [1, 2]. The features of their research and the obtained results in the case of singular coefficients are presented in the article [3]. The solvability conditions and maximal regularity of various differential equations with intermediate coefficients in an infinite interval are considered in the works [4-9].

2 Research method

The paper uses methods such as obtaining an a priori estimate for the solution of a fourth-order differential equation with variable coefficients, reducing the problem to the problem of invertibility of a third-order differential operator with a potential of constant sign, and constructing a pseudo-resolvent using some correct local operators.

3 Case $q(x) = 0$

We take a binomial operator $L_0 y = y^{(4)} + r(x)y'$ with $D(L_0) = C_0^{(4)}(\mathbb{R})$, and denote its closure in $L_2(\mathbb{R})$ by L. We introduce the following notation for continuous functions $\rho(t)$ and $v(t) \neq 0$:

$$\alpha_{\rho,v}(x) = \sup_{x>0} \|\rho\|_{L_2(0,x)} \|v^{-1}\|_{L_2(x,\infty)}, \quad \beta_{\rho,v}(x) = \sup_{x<0} \|\rho\|_{L_2(x,0)} \|v^{-1}\|_{L_2(-\infty,x)},$$

$$\gamma_{\rho,v} = \max(\alpha_{\rho,v}(x), \beta_{\rho,v}(x)).$$

Lemma 1. If the following conditions are satisfied for the coefficient $r(x)$:

$$r(x) \geq 1, \quad \gamma_{1,\sqrt{r}} < \infty, \tag{2}$$

then the operator L is invertible and for $y \in D(L)$, the inequality

$$\|\sqrt{r}y'\|_2 + \|y\|_2 \leq C\|Ly\|_2 \tag{3}$$

is true.

Proof. Transforming the functional $(L_0 y, y')$, we obtain

$$\|\sqrt{r}y'\|_2 \leq \left\|\frac{L_0 y}{\sqrt{r}}\right\|_2.$$

According to condition (2) and the results of [5],

$$\|\sqrt{r}y'\|_2 + \|y\|_2 \leq (1 + 2\gamma_{1,\sqrt{r}}) \left\|\frac{L_0 y}{\sqrt{r}}\right\|_2.$$

Closing this inequality, we obtain (3). The lemma is proved.

Let us now consider equation

$$ly = y^{(4)} + r(x)y' = f(x). \tag{4}$$

The following statement follows directly from (3).

Lemma 2. Let $r(x)$ satisfy the conditions of Lemma 1. Then the solution of equation (4) is a unique .

Suppose that the conditions of Lemma 1 are satisfied. From estimate (3) we obtain the relation $\sqrt{r}\,y' \in L_2(\mathbb{R})$ for each $y \in D(L)$. According to conditions (2) and estimate (3), $y' \in L_2(\mathbb{R})$ and $\|y'\|_2 \leq C\|Ly\|_2$. If we make the notation $y' = z$, then in view of (4), we have:

$$z^{(3)} + r(x)z = f(x). \tag{5}$$

Let L be the closure in the space $L_2(\mathbb{R})$ of the operator $L_0 : L_0 v = v^{(3)} + r(x)v$, $D(L_0) = C_0^{(3)}(\mathbb{R})$. A solution of equation (5) is a function $z \in D(L)$ satisfying the equality $Lz = f$. If the conditions of Lemma 1 are satisfied and a solution to equation (4) exists, then it is clear that equation (5) also has a solution. The converse is also true, namely:

Lemma 3. Suppose that $r(x)$ satisfies the conditions of Lemma 1 and equation (5) has a solution. Then equation (4) is also solvable.

Proof. If $z \in L_2(\mathbb{R})$ is a solution of equation (5), then there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset C_0^{(3)}(\mathbb{R})$ such that $\|z_n - z\|_2 \to 0$ and $\|L_0 z_n - f\|_2 \to 0$ as $n \to \infty$. Let us take the functional (Lz_n, z_n). Repeating the method of Lemma 1, we will see that $\|\sqrt{r}\, z_n\|_2 \le \|L_0 z_n\|_2$. Let $y_n(x)$ be a function such that $y_n' = z_n$. Then y_n is four times continuously differentiable, and, according to [5], from the inequality $\|\sqrt{r}\, y_n'\|_2 \le \|L_0 z_n\|_2$, we obtain the estimate

$$\|y_n\|_2 \le \|L_0 z_n\|_2, \quad z_n \in C_0^{(3)}(\mathbb{R}). \tag{6}$$

That is, $y_n \in L_2(\mathbb{R})$. Then, since $y_n' \in C_0^{(3)}(\mathbb{R})$, there is a number $a > 0$ such that the equality $y_n(x) = 0$ holds for all $|x| > a$ (otherwise the relation $y_n \in L_2(\mathbb{R})$ is violated). Therefore, $y_n(x) \in C_0^{(4)}(\mathbb{R}) = D(L_0)$. From estimate (6) it follows that there exists an element $\bar{y} \in L_2(\mathbb{R})$ and the relations $\|y_n - \bar{y}\|_2 \to 0$, $\|L_0 y_n - f\|_2 \to 0$ as $n \to \infty$ are satisfied. Consequently, $\bar{y}$ is a solution to equation (4). The lemma is proven.

4 Separability of a third-order differential operator

Let us assume that the function $r(x)$, in addition to the conditions of Lemma 1, also satisfies the relation

$$\sup_{x,\eta\in\mathbb{R},|x-\eta|\le 1} \frac{r(x)}{r(\eta)} < \infty. \tag{7}$$

We choose sequences of intervals $\{\Delta_j\}_{j=1}^{\infty}$, $\{\Omega_j\}_{j=1}^{\infty}$, and functions $\varphi_j(x) \in C_0^{\infty}(\Omega_j)$ as follows:
(a) $\Delta_j = [j, j+1)$, $\Omega_j = (j - \frac{1}{2}, j + \frac{3}{2})$ for $j \in \mathbb{Z}$,
(b) $0 \le \varphi_j \le 1$, $\quad \varphi_j(x) = 1\, \forall x \in \Delta_j$ for $j \in \mathbb{Z}$, $\sup\limits_{j\in\mathbb{Z}} \max\limits_{x\in\Omega_j} (|\varphi_j'(x)|, |\varphi_j''(x)|, |\varphi_j^{(3)}|) \le M$.
Then:

$$|\Omega_j| = 2, \quad \overline{\Delta_j} \subset \Omega_j \subset \Delta_{j-1} \cup \Delta_j \cup \Delta_{j+1}, \quad \Delta_j \cap \Delta_k = \emptyset \quad (j \ne k), \quad \bigcup_{j=-\infty}^{\infty} \Delta_j = \mathbb{R},$$

$$\Omega_j \cap \Omega_m = \emptyset \quad (|j - m| \ge 2), \quad \sum_j \varphi_j(x)\chi_{\Delta_j}(x) = 1,$$

where χ_{Δ_j} is the characteristic function of Δ_j. Recall that the sequence $\{\varphi_j(x)\}_{j=1}^{\infty}$ satisfying relations (b) always exists.

We extend the restriction of the function $r(x)$ to the interval Ω_j on all $\mathbb{R}$ so that the resulting extension $r_j(x)$ (for $j \in \mathbb{Z}$) turns out to be a continuously differentiable function and satisfies the inequalities

$$\frac{1}{2} \inf_{z\in\Omega_j} r(z) \le r_j(x) \le 2 \sup_{z\in\Omega_j} r(z), \quad x \in \mathbb{R}.$$

According to (7), such an extension $r_j(x)$ exists. Let $\theta_{\lambda j} (\lambda \geq 0)$ denote the closure in $L_2(\mathbb{R})$ of the differential operator

$$\theta_{0\lambda j} z = z^{(3)} + [r_j(x) + \lambda] z(x), \quad D(\theta_{0\lambda j}) = C_0^{(3)}(\mathbb{R}).$$

Then, it is easy to see that $D(\theta_{\lambda j}) = W_2^3(\mathbb{R})$. Therefore, for an element $z \in D(\theta_{\lambda j})$, the relations $z \in C^{(2)}(\mathbb{R})$ and $z(-\infty) = z(+\infty) = z^{(k)}(-\infty) = z^{(k)}(+\infty) = 0$ $(k = 1, 2)$ are satisfied. Taking these equalities into account, we transform the scalar product $(\theta_{\lambda j} z, z)$ $(z \in D(\theta_{\lambda j}))$. Then we have

$$\|z\|_2 \leq \frac{2}{\delta + 2\lambda} \|\theta_{\lambda j} z\|_2.$$

Consequently, $R(\theta_{\lambda j})$ is a closed set.

Lemma 4. If the function $r(x)$ satisfies conditions (2) and (7), then $R(\theta_{\lambda j}) = L_2(\mathbb{R})$ for $\lambda \geq 0$.

Proof. If $R(\theta_{\lambda j}) \neq L_2(R)$, then there exists an element $w \in L_2(R) \setminus R(\theta_{\lambda j})$, $w \neq 0$, such that the equality

$$\theta_{\lambda j}^* w = -w''' + (r_j(x) + \lambda) w = 0 \tag{8}$$

holds, where $\theta_{\lambda j}^*$ is the operator conjugate to $\theta_{\lambda j}$. Since the function $r_j(x)$ is bounded, by (8) and condition (2), $w \in W_2^3(R)$. Consequently, $w \in C^2(R)$ and $w(-\infty) = w(+\infty) = w^{(k)}(-\infty) = w^{(k)}(+\infty) = 0$ $(k = 1, 2)$. Taking these equalities into account and transforming the functional $(\theta_{\lambda j}^* w, w)$, we obtain the estimate $\delta \|w\|_2 \leq 2 \|\theta_{\lambda j}^* w\|_2$. According to (8), $w = 0$. The lemma is proved.

If the conditions of Lemma 4 are met, then, similarly to the proof of Lemma 1, we obtain the inequality

$$\|z\|_2 \leq \frac{2}{\delta + 2\lambda} \|\theta_{\lambda j} z\|_2 \quad (j \in \mathbb{Z}, z \in D(\theta_{\lambda j}), \lambda \geq 0). \tag{9}$$

Consider the following operator $L_{0\lambda} = L_0 + \lambda E$, $D(L_{0\lambda}) = C_0^3(R)$, where $\lambda \in \mathbb{R}_+ = [0, +\infty)$, and E is the identity operator. Denote the closure of $L_{0\lambda}$ in $L_2(R)$ as L_λ. Repeating the method of Lemma 1, we obtain the estimate

$$\sqrt{1 + \lambda} \|z\|_2 \leq \|L_\lambda z\|_2 \tag{10}$$

for each $z \in D(L_\lambda)$. Therefore, there exists an inverse operator L_λ^{-1} $(\lambda \geq 0)$.

Lemma 5. Let the coefficient r satisfy conditions (2) and (7). Then the operator L_λ is continuously invertible, and for each $z \in D(L_\lambda)$ the following inequality is true:

$$\|z'''\|_2 + \|rz\|_2 + \lambda \|z\|_2 \leq C \|L_\lambda z\|_2. \tag{11}$$

Proof. Let $K^2 = \sup\limits_{x,t \in \Omega_j, |x-t| \leq 1} \frac{r(x)}{r(t)}$. For $z \in D(\theta_{\lambda j})$, we have

$$(i)\ \|z'''\|_2 \leq \left(3 + 2 \sup_{x,t \in \Omega_j, |x-t| \leq 1} \frac{r(x)}{r(t)} \right) \|\theta_{\lambda j} z\|_2 \leq (3 + 2K^2) \|\theta_{\lambda j} z\|_2.$$

Using (9) and simple calculations, we get

$$(ii)\ \|z'\|_2 \le \left(\frac{2}{\delta+2\lambda}\right)^{2/3}(3+2K^2)^{1/3}\|\theta_{\lambda j}z\|_2,$$

$$(iii)\ \|z''\|_2 \le \left(\frac{2}{\delta+2\lambda}\right)^{1/3}(3+2K^2)^{2/3}\|\theta_{\lambda j}z\|_2, \quad z\in D(\theta_{0\lambda j})=C_0^3(R).$$

Let $f\in C_0^3(R)$, and M_λ and B_λ be operators acting according to the following formulas:

$$M_\lambda f=\sum_j \phi_j\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f),$$

$$B_\lambda f=\sum_j \phi_j^{(3)}\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f)+3\sum_j \phi_j''(\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f))'+3\sum_j \phi_j'(\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f))''.$$

It is easy to see that the equality

$$L_\lambda(M_\lambda f)=(B_\lambda+E)f \tag{12}$$

holds. Considering that $\Omega_j\cap\Omega_k=\emptyset$ if $|j-k|\ge 2$, we obtain the following inequality:

$$\|B_\lambda f\|_2^2\le 9M^2\left(\sum_j\|\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f)\|_2^2+\sum_j\|9(\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f))'\|_2^2+\sum_j\|9(\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f))''\|_2^2\right).$$

According to $(i),(ii),(iii)$, we have the following estimates:

$$\|\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f)\|_2^2\le\left(\frac{2}{\delta+2\lambda}\right)^2\|\chi_{\Delta_j}f\|_2^2,$$

$$\|(\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f))'\|_2^2\le\left(\left(\frac{2}{\delta+2\lambda}\right)^{2/3}(3+2K^2)^{1/3}\right)^2\|\chi_{\Delta_j}f\|_2^2,$$

$$\|(\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f))''\|_2^2\le\left(\left(\frac{2}{\delta+2\lambda}\right)^{1/3}(3+2K^2)^{2/3}\right)^2\|\chi_{\Delta_j}f\|_2^2 \quad (j\in\mathbb{Z}).$$

Hence,

$$\|B_\lambda f\|_2^2\le 9M^2\left(\frac{2}{\delta+2\lambda}+9\left(\frac{2}{\delta+2\lambda}\right)^{2/3}(3+2K^2)^{1/3}+9\left(\frac{2}{\delta+2\lambda}\right)^{1/3}(3+2K^2)^{2/3}\right)^2\|f\|_2^2. \tag{13}$$

If we choose the number λ_0 so that

$$9M^2\left(\frac{2}{\delta+2\lambda_0}+9\left(\frac{2}{\delta+2\lambda_0}\right)^{2/3}(3+2K^2)^{1/3}+9\left(\frac{2}{\delta+\lambda_0}\right)^{1/3}(3+2K^2)^{2/3}\right)^2\le\beta^2,$$

$(0 < \beta < 1)$ then $\|B_\lambda\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} \le \beta$ $(\lambda \ge \lambda_0)$, where $\|\cdot\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}$ is the operator norm. So, for any $\lambda \ge \lambda_0$, the operator $E + B_\lambda$ is boundedly invertible, and its inverse $(E + B_\lambda)^{-1}$ satisfies the estimates

$$(1+\beta)^{-1} \le \|(E+B_\lambda)^{-1}\| \le (1-\beta)^{-1} \quad (\lambda \ge \lambda_0). \tag{14}$$

From (12) it follows that

$$L_\lambda^{-1} = M_\lambda (E+B_\lambda)^{-1}, \quad \lambda \ge \lambda_0. \tag{15}$$

By (10), $\|z\|_2 \le \|L_\lambda z\|_2$ for $z \in D(L_\lambda)$ and $\lambda \ge 0$. Then, according to the well-known statement [10] (p. 350), the operator L_λ is continuously invertible for any $\lambda \ge 0$.

Let us prove estimate (11). It suffices to show that $\|(r+\lambda)z\|_2 \le C\|L_\lambda z\|_2$ $(\lambda \ge \lambda_0 > 0, z \in D(L_\lambda))$. By (14) and (15), $\|(r+\lambda)L_\lambda^{-1}\| \le (1-\beta)^{-1}\|(r+\lambda)M_\lambda\|$ $(\lambda \ge \lambda_0)$, and

$$\|(r+\lambda)M_\lambda f\|_2^2 \le 3\sum_{j=-\infty}^{\infty}\left(\sup_{x\in\Omega_j}(r(x)+\lambda)^2\int_{-\infty}^{\infty}|\phi_j(x)\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f)|^2\,dx\right).$$

Taking into account the property (b) of the sequence $\{\phi_j(x)\}_{j=1}^{\infty}$, we have

$$\|(r+\lambda)M_\lambda f\|_2^2 \le 12(K^2+1)\|f\|_2^2 \quad (\lambda \ge \lambda_0). \tag{16}$$

If $z \in D(L_\lambda)$, $L_\lambda z = f$ $(\lambda \ge \lambda_0)$, then $z = L_\lambda^{-1} f$. Therefore, according to (15) and (16),

$$\|(r+\lambda)z\|_2 \le 2\sqrt{3}(K^2+1)\|(E+B_\lambda)^{-1}\|\|f\|_2 \le 2\sqrt{3}(K^2+1)(1-\beta)^{-1}\|f\|_2. \tag{17}$$

Then

$$\|z'''\|_2 \le \left(2\sqrt{3}(K^2+1)(1-\beta)^{-1}+1\right)\|f\|_2. \tag{18}$$

By (18) and (17), $\|z'''\|_2 + \|rz\|_2 + \|\lambda z\|_2 \le \left(6\sqrt{3}(K^2+1)(1-\beta)^{-1}+1\right)\|f\|_2$. The lemma is proved.

Thus, if conditions (2) and (7) are satisfied, then, according to Lemma 3, the following estimate is valid for the solution y of equation (4):

$$\|y^{(4)}\|_2 + \|ry'\|_2 \le \left(6\sqrt{3}(K^2+1)(1-\beta)^{-1}+1\right)\|f\|_2. \tag{19}$$

Remark 1. The statement of Lemma 5 remains true if condition (2) is replaced by $r(x) \ge \delta > 0$.

5 Main Result

Theorem 1. If the functions $r(x)$ and $q(x)$ satisfy conditions (2), (7), and $\gamma_{q,r} < \infty$, then for any $f \in L_2(\mathbb{R})$, there exists a unique solution y of equation (1). Furthermore, the following inequality holds for y:

$$\|y^{(4)}\|_2 + \|ry'\|_2 + \|qy\|_2 \le C\|f\|_2. \tag{20}$$

Proof. Let us replace $x = t/a$ $(a > 0)$ in (1). Denoting $\tilde{y}(t) = y(a^{-1}t)$, $\tilde{r}(t) = r(a^{-1}t)$, $\tilde{q}(t) = q(a^{-1}t)$, $\tilde{F}(t) = a^{-4}F(a^{-1}t)$, equation (1) becomes

$$\tilde{L}_{0a}\tilde{y} = \tilde{y}^{(4)}(t) + a^{-3}\tilde{r}(t)\tilde{y}'(t) + a^{-4}\tilde{q}(t)\tilde{y}(t) = \tilde{F}(t). \tag{21}$$

Denote the closure of the operator $l_{0a}\tilde{y} = \tilde{y}^{(4)}(t) + a^{-3}\tilde{r}(t)\tilde{y}'(t)$, $\tilde{y} \in C_0^{(4)}(\mathbb{R})$, in the norm of $L_2(\mathbb{R})$ as l_a. From the condition $\tilde{r}(t) \geq 1$, it follows that $1 \geq \tilde{r}^{-1} \geq \tilde{r}^{-2}$. The coefficient $a^{-3}\tilde{r}(t)$ of the operator l_a satisfies the conditions of Lemma 3. Consequently, l_a is a continuously invertible operator, and

$$\|\tilde{y}^{(4)}\|_2 + \|a^{-3}\tilde{r}\tilde{y}'\|_2 \leq C_a\|l_a\tilde{y}\|_2, \quad \tilde{y} \in D(l_a). \tag{22}$$

It is straightforward to calculate that $\gamma_{a^{-4}\tilde{q},a^{-3}\tilde{r},1} = \gamma_{q,r,1} < \infty$. Therefore,

$$\|a^{-4}\tilde{q}\tilde{y}\|_2 \leq 2\gamma_{a^{-4}\tilde{q},a^{-3}\tilde{r},1}\|a^{-3}\tilde{r}\tilde{y}'\|_2 \leq 2\gamma_{a^{-4}\tilde{q},a^{-3}\tilde{r},1}C_a\|l_a\tilde{y}\|_2. \tag{23}$$

Using the substitution $x = a^{-1}\tau$, we obtain that $\gamma_{a^{-4}\tilde{q},a^{-3}\tilde{r},1} = \tilde{\gamma}_{a^{-4}\tilde{q},a^{-3}\tilde{r},1}$. Let us prove the equality

$$\lim_{a\to\infty} \tilde{\gamma}_{(a^{-4}\tilde{q},a^{-3}\tilde{r},1)}(a) = 0. \tag{24}$$

Let $\tilde{y} \in D(l_a)$. Since l_a is a closed operator, there exists a sequence $\{\tilde{y}_n\}_{n=1}^{\infty} \subseteq C_0^{(4)}(\mathbb{R})$ such that $\|\tilde{y}_n - \tilde{y}\|_2 \to 0$, $\|l_a\tilde{y}_n - l_a\tilde{y}\|_2 \to 0$ $(n \to \infty)$. Let $\{\tilde{y}_n\}_{n=1}^{\infty}$ and a number N_0 be such that $\operatorname{supp}\tilde{y}_n \subseteq [-N_0, N_0]$. Denote

$$\tilde{q}_{N_0}(t) = \begin{cases} \tilde{q}(t), & t \in [-N_0, N_0], \\ 0, & t \notin [-N_0, N_0], \end{cases} \quad \tilde{r}_{N_0}(t) = \begin{cases} \tilde{r}(t), & t \in [-N_0, N_0], \\ 0, & t \notin [-N_0, N_0]. \end{cases}$$

According to (23), for each $\tilde{y} \in C_0^{(4)}[-N_0, N_0]$, we obtain the estimate:

$$\|a^{-4}\tilde{q}_{N_0}\tilde{y}\|_{L_2[-N_0,N_0]} \leq 2\gamma_{a^{-4}\tilde{q}_{N_0},a^{-3}\tilde{r}_{N_0},1}(a)\|a^{-3}\tilde{r}_{N_0}\tilde{y}'\|_{L_2[-N_0,N_0]}.$$

Further,

$$\alpha_{a^{-4}\tilde{q}_{N_0},a^{-3}\tilde{r}_{N_0}}(a) = \sup_{x>0}\left(\int_0^{a^{-1}x} a^{-8}\tilde{q}_{N_0}^2(t)\,dt\right)^{1/2}\left(\int_{a^{-1}x}^{N_0} \frac{a^6}{\tilde{r}_{N_0}^2(t)}\,dt\right)^{1/2} =$$

$$= \sup_{0<x\leq N_0}\left(\int_0^{a^{-1}x} a^{-7}q^2(t)\,dt\right)^{1/2}\left(\int_{a^{-1}x}^{N_0} \frac{a^7}{r^2(t)}\,dt\right)^{1/2} \leq \alpha_{q,r} < \infty. \tag{26}$$

Therefore,

$$\lim_{a\to\infty}\sup_{0<x\leq N_0}\left(\int_0^{a^{-1}x} q^2(t)\,dt\right)^{1/2}\left(\int_{a^{-1}x}^{N_0} \frac{1}{r^2(t)}\,dt\right)^{1/2} =$$

$$= \sup_{0<x\leq N_0}\lim_{a\to\infty}\left(\int_0^{a^{-1}x} q^2(t)\,dt\right)^{1/2}\left(\int_{a^{-1}x}^{N_0} \frac{1}{r^2(t)}\,dt\right)^{1/2} = 0.$$

Similarly, we have:

$$\lim_{a\to\infty}\sup_{-N_0\le x<0}\left(\int_{a^{-1}x}^{0}q^2(t)\,dt\right)^{1/2}\left(\int_{-N_0}^{a^{-1}x}\frac{1}{r^2(t)}\,dt\right)^{1/2}=0.$$

From the last two equalities follows (24). Therefore, there exists $a_0 > 0$ such that $4C_a\gamma_{(a^{-4}\tilde{q},a^{-3}\tilde{r},1)}(a)\le 1$ for all $a\ge a_0$. From (23), it follows that

$$\|a^{-4}\tilde{q}\tilde{y}\|_2\le\frac{1}{2}\|l_a\tilde{y}\|_2\quad(a\ge a_0). \tag{24}$$

By theorem on small perturbations of linear operators, the operator $\tilde{L}_a\tilde{y}=l_a\tilde{y}+a^{-4}\tilde{q}(t)\tilde{y}(t)$ is closed and boundedly invertible. Thus, for each $\tilde{F}(t)\in L_2(\mathbb{R})$, equation (21) has a unique solution $\tilde{y}$. It remains to show the validity of estimate (20). It is easy to see that $\|l_a\tilde{y}\|_2\le 2\|\tilde{L}_a\tilde{y}\|_2$. According to (22) and (24), $\|\tilde{y}^{(4)}\|_2+\|a^{-3}\tilde{r}\tilde{y}'\|_2+\|a^{-4}\tilde{q}\tilde{y}\|_2\le(C_a+1/2)\|l_a\tilde{y}\|_2$. From the last two inequalities we have $\|\tilde{y}^{(4)}(t)\|_2+\|a^{-3}\tilde{r}\tilde{y}'\|_2+\|a^{-4}\tilde{q}\tilde{y}\|_2\le(2(C_a+1)\|\tilde{L}_a\tilde{y}\|_2$, or equivalently,

$$\|a^{-4}y^{(4)}(a^{-1}t)\|_2+\|a^{-4}r(a^{-1}t)y'(a^{-1}t)\|_2+\|a^{-4}q(a^{-1}t)y(a^{-1}t)\|_2\le(2C_a+1)\|a^{-4}F(a^{-1}t)\|_2.$$

Putting $t=ax$, we obtain inequality (20), where $C=2C_a+1$. The theorem is proved.

6 Conclusion

The paper studies one fourth order three-term differential equation (1) with an intermediate term. A special case is considered where the intermediate term as an operator does not obey the differential operator formed by the extreme terms of the equation. Sufficient conditions for the existence of a strong solution to the equation, its uniqueness, and maximal regularity are shown. For this purpose, such methods as obtaining an a priori estimate of the solution, reducing the problem to the study of properties of one of the third-order differential operator with potential of constant sign, and estimation using local operators were used. The obtained conditions are specified in the form of an integral relation between the intermediate and small coefficients of the equation and allow us to cover a wide class of differential equations of the fourth order. The methods developed in the paper and the results obtained can be used in a qualitative study of singular differential equations of higher orders.

7 Acknowledgments

This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP23488049).

References

[1] Ostrovskii L.A., Potapov I.A. Vvedenie v teoriyu modulirovannykh voln. // Moscow: Fizmatlit. – 2003.

[2] Umarov Kh.G. Zadacha Koshi dlya uravneniya izgibnyh kolebanii nelineino – uprugogo sterzhnya beskonechnoi dliny. // Vladikavkaz Math. J. – 2017. – Vol. 19:3. – Pp. 59-69.

[3] Moldagali Ye.O., Ospanov K.N. On the correctness of one class of fourth – order singular differential equations. //Journal of Mathematics, Mechanics and Computer Science. – 2024. – No. 3(123). – Pp. 21-32.

[4] Yesbayev A.N., Ospanov K.N. Maximal regularity estimate for a differential equation with oscillating coefficients. // Journal of Mathematics, Mechanics and Computer Science. – 2021. – No.1(109). – Pp. 25-34.

[5] Ospanov K.N., Yesbayev A.N. Solvability and maximal regularity results for a differential equation with diffusion coefficient. // Turk. J. Math. – 2020. – Vol. 44:4. – Pp. 1304 – 1316.

[6] Akhmetkaliyeva R.D., Mukasheva T.D., Ospanov, K.N. Correct and coercive solvability conditions for a degenerate high order differential equation. // Eurasian Math. J. – 2023. – Vol. 14:4. – Pp. 9–14.

[7] Muratbekov M., Ospanov K., Igisinov, S. Solvability of a class of mixed type second rder equations and nonlocal estimates. // Appl. Math. Lett. – 2012. – Vol. 25:11. – Pp. 1661–1665.

[8] Muratbekov M.B., Muratbekov M.M. One-dimensional Schrodinger operator with a negative parameter and its applications to the study of the approximation numbers of a singular hyperbolic operator. // Filomat. – 2018. – Vol. 32:3. – Pp. 785–790.

[9] Akhmetkalieva R.D., Persson L.-E., Ospanov K.N., Woll, P. Some new results concerning a class of third order differential equations. // Applicable Analysis. – 2015. – Vol. 94, Iss. 2. – Pp. 419-434.

[10] Akhiezer N.I., Glazman I.M. Theory of linear operators in Hilbert space. // Moscow: Nauka. – 1966.

Авторлар туралы мәлімет:

Оспанов Қордан Наурызханұлы – Физика-математика ғылымдарының докторы, Л.Н. Гумилев атындағы Еуразия ұлттық университетінің профессоры (Астана, Қазақстан, электрондық пошта: kordan.ospanov@gmail.com);

Молдағали Еркебұлан Өміргалиұлы (корреспондент автор) – Л.Н. Гумилев атындағы Еуразия ұлттық университетінің Іргелі математика кафедрасының докторанты (Астана, Қазақстан, электрондық пошта: Yerka2998@gmail.com);

Сведения об авторах:

Оспанов Қордан Наурызханович – Доктор физико-математический наук, профессор Евразийского национального университета имени Л.Н. Гумилева (Астана, Казахстан, электронная почта: kordan.ospanov@gmail.com);

Молдағали Еркебұлан Өміргалиевич (корреспондент автор) – докторант кафедры фундаментальной математики, Евразийский национальный университет имени Л.Н. Гумилева (Астана, Казахстан, электронная почта: Yerka2998@gmail.com);

Information about authors:

Kordan Nauryzkhanovich Ospanov – Doctor of Physical and Mathematical Sciences, Professor of L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, e-mail: kordan.ospanov@gmail.com);

Yerkebulan Omirgaliuly Moldagali (corresponding author) – PhD student of the Department of Fundamental Mathematics of L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, e-mail: Yerka2998@gmail.com);

Received: January 22, 2025
Accepted: February 23, 2025