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CAUCHY PROBLEM FOR A DEGENERATE HYPERBOLIC EQUATION OF THE SECOND KIND WITH THE TWO LINES AND IDENTICAL ORDER OF DEGENERACY

In scientific literature, degenerate hyperbolic equations are usually divided into equations of the first and second kind. In the case of an equation of the first kind, the line of parabolic degeneracy is the locus of cusps of the equation's characteristics, and in the case of an equation of the second kind, it is simultaneously a special line (characteristic), i.e., it is the envelope of the family of characteristics. Therefore, degenerate hyperbolic equations of the second kind have been studied relatively little in all respects than equations of the first kind. At present, a solution to the Cauchy problem for a degenerate hyperbolic equation of the second kind with two lines and different orders of degeneracy is known. Further studies have shown that if a hyperbolic equation of the second kind degenerates with identical order in two lines, then special studies are required to solve the Cauchy problem. In this paper, using the Gauss hypergeometric function, new properties of the Riemann function are established for the named equation, due to which the unique regular solution to the Cauchy problem for a hyperbolic equation of the second kind with the identical order of degeneracy is constructed explicitly.

Key words: Gauss hypergeometric function, Cauchy problem, Riemann function, Riemann method, method of introducing an auxiliary function.

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Өзгешеленудің екі сызығы бар және реті бірдей екінші текті өзгешеленетін гиперболалық теңдеу үшін Коши есебі

Ғылыми әдебиеттегі өзгешеленетін гиперболалық теңдеулер әдетте бірінші және екінші текті теңдеулерге бөлінеді. Бірінші текті теңдеу жағдайында параболалық өзгешелену сызығы теңдеудің сипаттамаларын қайтару нүктелерінің геометриялық орны болып табылады, ал екінші текті теңдеу жағдайында ол бір уақытта ерекше сызық (сипаттама) болып табылады, яғни ол сипаттамалар тобының орайжанаушысы болып табылады. Сондықтан екінші текті өзгешеленетін гиперболалық теңдеулер бірінші текті теңдеулерге қарағанда барлық жағынан салыстырмалы түрде аз зерттелген. Қазіргі уақытта Коши есебінің шешімі өзгешеленудің екі сызығы бар және реті әр түрлі екінші текті өзгешеленетін гиперболалық үшін белгілі. Одан әрі зерттеулер көрсеткендей, егер екінші текті гиперболалық теңдеу екі сызықта бірдей өзгешеленетін болса, онда Коши есебін шешу үшін арнайы зерттеулер қажет. Бұл жұмыста Гаустың гипергеометриялық функциясы арқылы аталған теңдеу үшін Риман функциясының жаңа қасиеттері орнатылады, соның көмегімен өзгешелену реті бірдей екінші текті гиперболалық теңдеу үшін Коши есебінің жалғыз регуляр шешімі айқын түрде қарастырылады.

Түйін сөздер: Гаустың гипергеометриялық функциясы, Коши есебі, Риман функциясы, Риман әдісі, көмекші функцияны енгізу әдісі.

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Задача Коши для вырождающегося гиперболического уравнения второго рода с двумя линиями и одинаковым порядком вырождения

Вырождающиеся гиперболические уравнения в научной литературе принято делить на уравнения первого и второго родов. В случае уравнения первого рода линия параболического вырождения является геометрическим местом точек возврата характеристик уравнения, а в случае уравнения второго рода одновременно является особой линией (характеристикой), т.е. является огибающей семейства характеристик. Поэтому вырождающиеся гиперболические уравнения второго рода во всех отношениях относительно мало исследованы, чем уравнения первого рода. В настоящее время известно решение задачи Коши для вырождающегося гиперболического уравнения второго рода с двумя линиями и различными порядками вырождения. Дальнейшие исследования показали, что, если гиперболическое уравнение второго рода в двух линиях одинаково вырождается, то для решения задачи Коши требуются особые исследования. В этой работе с помощью гипергеометрической функции Гаусса устанавливаются новые свойства функции Римана для названного уравнения, благодаря которым единственное регулярное решение задачи Коши для гиперболического уравнения второго рода с одинаковым порядком вырождения строится в явном виде.

Ключевые слова: гипергеометрическая функция Гаусса, задача Коши, функция Римана, метод Римана, метод введения вспомогательной функции.

1 Introduction

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of one, two and several variables) is motivated essentially by the fact that solutions of many applied problems involving thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, and quantum mechanics and potential theory are obtainable with the help of hypergeometric (higher and special or transcendental) functions [1, 2]. Such kinds of functions are often referred to as special functions of mathematical physics.

The study of applied problems modeled by degenerate or singular partial differential equations is closely related to the properties of hypergeometric functions of one or more variables. Explicit solutions to problems posed for degenerate or singular partial differential equations are expressed through hypergeometric functions, the number of variables of which depends on the number of degeneracy lines or the number of singular coefficients [3–5].

In particular, the degenerate hyperbolic equations are encountered in the solution of various problems of gas dynamics [6], in the computer tomography [7], etc. In the scientific literature, degenerate hyperbolic equations are usually divided into the first and second kinds. If a hyperbolic equation degenerates along a straight line, which is at the same time a characteristic, then such equation is a degenerate equation of the second kind, in contrast to equations of the first kind, when the degeneracy line consists of cusp points of the family of characteristics of the degenerate hyperbolic equation. Therefore, equations of the second kind are difficult to study and they are relatively little studied with respect to equations of the first kind.

Karol [8], considering a model equation of the second kind with one line of degeneracy

$$y^m u_{xx} - u_{yy} = 0, \quad y > 0, \quad -1 < m < 0, \quad (1)$$

proved the correctness of the Cauchy problem in the ordinary statement. Volkodavov and Nosov [9] solved this problem for a general linear equation with one line of degeneracy. In [10–12], boundary-value problems for degenerate equations of the hyperbolic and parabolic-hyperbolic types with a spectral parameter and a mixed-type equation of the second kind are investigated.

In a recent paper [13], for a hyperbolic equation of the second kind with two lines and different orders of degeneration

$$y^m u_{xx} - x^n u_{yy} = 0, \quad x > 0, \quad y > 0, \quad -1 < m < n \leq 0, \quad (2)$$

the Cauchy problem is solved only under the condition $m \neq n$.

In papers [14,15] a potential theory is constructed for the elliptic analogues of the equation (2) in the forms

$$y^m u_{xx} + x^n u_{yy} = 0, \quad x > 0, \quad y > 0, \quad m > n \geq 0,$$

$$\sum_{i=1}^p u_{x_i x_i} + \frac{2\alpha}{x_1} u_{x_1} = 0, \quad x_1 > 0, \quad 0 < 2\alpha < 1, \quad p > 2.$$

In this paper the Cauchy problem for a degenerate hyperbolic equation of the second kind with the identical order of degeneracy, i.e. $m = n$ in equation (2).

2 Cauchy problem for the second kind degenerate hyperbolic equation

Consider the following degenerating hyperbolic equation of second kind with the same order of degeneration

$$y^m U_{xx} - x^m U_{yy} = 0, \quad -1 < m < 0 \quad (3)$$

in a finite simply connected domain D , bounded by characteristics

$$AB : y = 0, \quad AC : x - y = 0, \quad BC : x^{(m+2)/2} + y^{(m+2)/2} = 1$$

of equation (3) for $x \geq 0$ and $y \geq 0$, where $A(0, 0)$, $B(1, 0)$, $C((1/2)^{2/(m+2)}, (1/2)^{2/(m+2)})$.

Cauchy problem. Find a function $u(x, y) \in C^2(D) \cap C(\bar{D})$ satisfying equation (3) and the following initial-value conditions

$$U(x, y)|_{y=0} = \tau(x), \quad 0 \leq x \leq 1, \quad (4)$$

$$\left. \frac{\partial U(x, y)}{\partial y} \right|_{y=0} = \nu(x), \quad 0 < x < 1, \quad (5)$$

where $\tau(x)$ and $\nu(x)$ are sufficiently smooth given functions.

It is known, that with the help of a non-singular change of variables

$$\xi = x^{(m+2)/2} - y^{(m+2)/2}, \eta = x^{(m+2)/2} + y^{(m+2)/2}, \quad (6)$$

the equation (3) is transformed into the generalized Euler-Poisson-Darboux equation with the identical negative parameters

$$u_{\xi\eta} + \frac{p}{\eta + \xi} (u_{\xi} + u_{\eta}) + \frac{p}{\eta - \xi} (u_{\xi} - u_{\eta}) = 0, \quad p = \frac{m}{2(m+2)}, \quad -\frac{1}{2} < p < 0, \quad (7)$$

where

$$u(\xi, \eta) = U [((\eta + \xi)/2)^{1-2p}, ((\eta - \xi)/2)^{1-2p}],$$

Under the change (6), the domain D is mapped into a triangle Δ in the plane $\xi O\eta$ with sides $\xi = 0, 0 \leq \eta \leq 1; \eta = 1, 0 \leq \xi \leq 1; \eta = \xi$ whose vertices are at the points $P(0,0), Q(1,1)$ and $M(0,1)$, and the conditions (4) and (5) take the form

$$\lim_{\eta \rightarrow \xi} u(\xi, \eta) = \tau(\xi^{2/(2+m)}) = \tilde{\tau}(\xi), \quad 0 \leq \xi \leq 1,$$

$$\lim_{\eta \rightarrow \xi} \left(\frac{\eta - \xi}{2(1-2p)} \right)^{2p} \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) = \nu(\xi^{2/(2+m)}) = \tilde{\nu}(\xi), \quad 0 < \xi < 1.$$

Let's to solve the Cauchy problem. The Riemann function for equation (7) is known [16]

$$R(\xi, \eta; \xi_0, \eta_0) = \frac{(\eta^2 - \xi^2)^{2p}}{(\eta^2 - \xi_0^2)^p (\eta_0^2 - \xi^2)^p} F(p, p; 1; \sigma), \quad \sigma = \frac{(\eta^2 - \eta_0^2)(\xi^2 - \xi_0^2)}{(\eta^2 - \xi_0^2)(\xi^2 - \eta_0^2)}, \quad (8)$$

where $F(a, b; c; z)$ is a Gaussian hypergeometric function defined by [17, Ch.2, Eq. 2.8(1)]

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}, \quad |z| < 1, \quad (9)$$

a, b, c are independent of z . We call a, b, c the parameters of the hypergeometric function; they are arbitrary complex numbers with $c \neq 0, -1, -2, \dots$. Here $(\nu)_n$ is a Pochhammer symbol:

$$(\nu)_0 := 1, \quad (\nu)_n := \nu(\nu+1)\dots(\nu+n-1) = \frac{\Gamma(\nu+n)}{\Gamma(\nu)};$$

$\Gamma(z)$ is a well-known gamma function.

Applying the Riemann method to the domain Δ_ε bounded by line segments $P_\varepsilon Q_\varepsilon: \eta = \xi + \varepsilon$ ($\varepsilon > 0$), $MQ_\varepsilon: \eta = \eta_0$ and $MP_\varepsilon: \xi = \xi_0$, we obtain

$$u(\xi_0, \eta_0) = \frac{1}{2}(uR)_{P_\varepsilon} + \frac{1}{2}(uR)_{Q_\varepsilon} + \int_{\xi_0}^{\eta_0 - \varepsilon} [\mu(R)u]_{\eta = \xi + \varepsilon} d\xi$$

$$-\frac{1}{2} \int_{\xi_0}^{\eta_0 - \varepsilon} \left[R \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) \right]_{\eta = \xi + \varepsilon} d\xi, \quad (10)$$

where

$$\mu(R) = \frac{1}{2} \left(\frac{\partial R}{\partial \xi} - \frac{\partial R}{\partial \eta} \right) + \frac{2pR}{\eta - \xi}.$$

Let us transform the integrand $\mu(R)$. First, we calculate the difference $\frac{\partial R}{\partial \xi} - \frac{\partial R}{\partial \eta}$. To this end, using the autotransformation formula for the Gauss hypergeometric function [17, Ch.2, Eq. 2.9(2)]

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x), \quad (11)$$

we transform the Riemann function into the form

$$R(\xi, \eta; \xi_0, \eta_0) = (\eta^2 - \xi^2)^p (\eta_0^2 - \xi_0^2)^{-p} (1 - \sigma)^{1-p} F(1-p, 1-p; 1; \sigma).$$

Next, applying the well-known differentiation formula [17, Ch.2, Eq. 2.8(25)]

$$\frac{d}{dx} [(1-x)^a F(a, b; c; x)] = -\frac{a(c-b)}{c} (1-x)^{a-1} F(a+1, b; c+1; x), \quad (12)$$

after elementary transformations, we obtain

$$\frac{\partial R}{\partial \xi} - \frac{\partial R}{\partial \eta} = -\frac{2pR}{\eta - \xi} - p(1-p) \frac{(\eta^2 - \xi^2)^{2p-1} (\eta_0^2 - \xi_0^2)^{-1}}{(\eta^2 - \xi_0^2)^{p-1} (\eta_0^2 - \xi^2)^{p-1}} F(p, 1+p; 2; \sigma) \left(\frac{\partial \sigma}{\partial \xi} - \frac{\partial \sigma}{\partial \eta} \right).$$

It is easy to calculate the partial derivatives $\frac{\partial \sigma}{\partial \xi}$, $\frac{\partial \sigma}{\partial \eta}$ and their difference:

$$\frac{\partial \sigma}{\partial \xi} = -\frac{2\xi(\eta^2 - \eta_0^2)(\eta_0^2 - \xi_0^2)}{(\eta^2 - \xi_0^2)(\eta_0^2 - \xi^2)^2}, \quad \frac{\partial \sigma}{\partial \eta} = -\frac{2\eta(\xi^2 - \xi_0^2)(\eta_0^2 - \xi_0^2)}{(\eta^2 - \xi_0^2)^2(\eta_0^2 - \xi^2)},$$

$$\frac{\partial \sigma}{\partial \xi} - \frac{\partial \sigma}{\partial \eta} = \frac{2(\eta_0^2 - \xi_0^2)}{(\eta^2 - \xi_0^2)^2(\eta_0^2 - \xi^2)^2} [\eta(\xi^2 - \xi_0^2)(\eta_0^2 - \xi^2) - \xi(\eta^2 - \eta_0^2)(\eta^2 - \xi_0^2)].$$

Thus, we finally have

$$\begin{aligned} \frac{\partial R}{\partial \xi} - \frac{\partial R}{\partial \eta} &= -\frac{2pR}{\eta - \xi} - \frac{2p(1-p)(\eta^2 - \xi^2)^{2p-1}}{(\eta^2 - \xi_0^2)^{p+1}(\eta_0^2 - \xi^2)^{p+1}} \times \\ &\times [\eta(\xi^2 - \xi_0^2)(\eta_0^2 - \xi^2) - \xi(\eta^2 - \eta_0^2)(\eta^2 - \xi_0^2)] F(p, 1+p; 2; \sigma). \end{aligned}$$

Hence, due to the obvious equality

$$\frac{2\xi(\eta^2 - \eta_0^2)(\eta^2 - \xi_0^2) - 2\eta(\xi^2 - \xi_0^2)(\eta_0^2 - \xi^2)}{(\eta + \xi)(\eta - \xi)^2}$$

$$= (\eta_0^2 + \xi_0^2 + 2\xi\eta) + \frac{(\xi^2 - \xi_0^2)(\eta^2 - \eta_0^2) + (\xi^2 - \eta_0^2)(\eta^2 - \xi_0^2)}{(\eta - \xi)^2},$$

we get

$$\begin{aligned} \mu(R) &= \frac{p(1-p)(\eta - \xi)^{2p+1}(\eta_0^2 + \xi_0^2 + 2\xi\eta)}{2(\eta + \xi)^{-2p}(\eta^2 - \xi_0^2)^{p+1}(\eta_0^2 - \xi^2)^{p+1}} F(p, 1+p; 2; \sigma) \\ &+ \frac{p(\eta + \xi)(\eta_0^2 - \xi_0^2)^{1-2p}}{(\eta^2 - \xi_0^2)^{1-p}(\eta_0^2 - \xi^2)^{1-p}} F(1-p, 1-p; 1; \sigma) \\ &- \frac{p(1-p)(\sigma + 1)(\eta + \xi)(\eta_0^2 - \xi_0^2)^{1-2p}}{2(\eta^2 - \xi_0^2)^{1-p}(\eta_0^2 - \xi^2)^{1-p}} F(2-p, 1-p; 2; \sigma). \end{aligned}$$

Now we substitute the function $\mu(R)$ into (10). Taking into account the easily verifiable equality for the Gaussian hypergeometric function (9):

$$F(1-p, 1-p; 1; \sigma) - \frac{(1-p)(\sigma + 1)}{2} F(2-p, 1-p; 2; \sigma) = \frac{1+p}{2} F(1-p, -p; 2; \sigma) \quad (13)$$

and the definition of the Riemann function, we obtain

$$u(\xi_0, \eta_0) = \frac{1}{2}(uR)_P + \frac{1}{2}(uR)_Q + I_1 + I_2 + I_3, \quad (14)$$

where

$$\begin{aligned} I_1 &= \frac{p(1-p)}{2} \int_{\xi_0}^{\eta_0 - \varepsilon} \left[\frac{(\eta - \xi)^{2p+1}(\eta_0^2 + \xi_0^2 + 2\xi\eta) F(p, 1+p; 2; \sigma) u(\xi, \eta)}{(\eta + \xi)^{-2p}(\eta^2 - \xi_0^2)^{p+1}(\eta_0^2 - \xi^2)^{p+1}} \right]_{\eta=\xi+\varepsilon} d\xi, \\ I_2 &= + \frac{p(1+p)}{2} \int_{\xi_0}^{\eta_0 - \varepsilon} \left[\frac{(\eta + \xi)(\eta_0^2 - \xi_0^2)^{1-2p} F(1-p, -p; 2; \sigma) u(\xi, \eta)}{(\eta^2 - \xi_0^2)^{1-p}(\eta_0^2 - \xi^2)^{1-p}} \right]_{\eta=\xi+\varepsilon} d\xi, \\ I_3 &= - \frac{1}{2} \int_{\xi_0}^{\eta_0 - \varepsilon} \left[\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) \frac{(\eta^2 - \xi_0^2)^{-p}(\eta_0^2 - \xi^2)^{-p}}{(\eta^2 - \xi^2)^{-2p}} F(p, p; 1; \sigma) \right]_{\eta=\xi+\varepsilon} d\xi. \end{aligned}$$

It follows from the formula (8) that the function R on the line $\eta = \xi$ tends to infinity of order $-2p$ ($0 < -2p < 1$). Therefore, unlike the degenerate hyperbolic equations of the first kind [?], the terms in (14) containing $u(\eta_0 - \varepsilon, \eta_0)$ and $u(\xi_0, \xi_0 + \varepsilon)$ do not disappear at $\varepsilon \rightarrow 0$ (and even more: they are infinitely large). Moreover, I_2 becomes a divergent integral. Therefore, we first transform the integrand in I_2 .

We first prove the very important lemma.

Lemma 1. The following differentiation formula

$$d \left[\frac{F(-p, -p; 1; \sigma)}{[(\eta^2 - \xi_0^2)(\eta_0^2 - \xi^2)]^{-p}} \right] = p(1-p) \frac{(\eta - \xi)(\eta_0^2 - \xi_0^2)^p}{(\eta^2 - \xi^2)^{1-p}} (1 - \sigma)^{1+p} F(1+p, p; 2; \sigma)$$

$$+ \frac{p(1+p)(\eta+\xi)(\eta_0^2+\xi_0^2-2\eta\xi)}{[(\eta^2-\xi_0^2)(\eta_0^2-\xi^2)]^{1-p}} F(1-p, -p; 2; \sigma) \quad (15)$$

is valid.

Proof. Let us introduce the notation:

$$\psi(\xi, \eta) = \frac{F(-p, -p; 1; \sigma)}{[(\eta^2-\xi_0^2)(\eta_0^2-\xi^2)]^{-p}}. \quad (16)$$

Applying the differentiation formula (12), the autotransformation formula (11) and the relation (13), we obtain

$$\begin{aligned} d\psi(\xi, \eta) &= p(1-p) \frac{(\eta-\xi)(\eta_0^2-\xi_0^2)^p}{(\eta^2-\xi^2)^{1-p}} (1-\sigma)^{1+p} F(1+p, p; 2; \sigma) \\ &+ p(1+p) \frac{(\eta-\xi)(\eta_0^2-\xi_0^2)^p}{(\eta^2-\xi^2)^{1-p}} (1-\sigma)^{-p} (\sigma+1) F(1-p, -p; 2; \sigma) \\ &+ p(1+p) \frac{(\eta_0^2-\xi_0^2)^p}{(\eta^2-\xi^2)^{-p}} (1-\sigma)^{-p-1} F(1-p, -p; 2; \sigma) \left(\frac{\partial\sigma}{\partial\xi} + \frac{\partial\sigma}{\partial\eta} \right). \end{aligned}$$

Using the following identities successively

$$\begin{aligned} \sigma+1 &= \frac{(\eta^2-\xi_0^2)(\eta_0^2-\xi^2) + (\eta_0^2-\eta^2)(\xi^2-\xi_0^2)}{(\eta^2-\xi_0^2)(\eta_0^2-\xi^2)}, \\ \frac{\partial\sigma}{\partial\xi} + \frac{\partial\sigma}{\partial\eta} &= \frac{2(\eta_0^2-\xi_0^2)[\xi(\eta_0^2-\eta^2)(\eta^2-\xi_0^2) + \eta(\xi^2-\xi_0^2)(\xi^2-\eta_0^2)]}{(\eta^2-\xi_0^2)^2(\eta_0^2-\xi^2)^2}, \\ \xi(\eta_0^2-\eta^2)(\eta^2-\xi_0^2) + \eta(\xi^2-\xi_0^2)(\xi^2-\eta_0^2) \\ &= (\eta-\xi)[\eta_0^2\xi_0^2 + \eta\xi\xi_0^2 + \eta\xi\eta_0^2 - \eta^3\xi - \eta^2\xi^2 - \eta\xi^3], \\ (\eta^2-\xi_0^2)(\eta_0^2-\xi^2) + (\eta_0^2-\eta^2)(\xi^2-\xi_0^2) \\ &+ 2(\eta_0^2\xi_0^2 + \eta\xi\xi_0^2 + \eta\xi\eta_0^2 - \eta^3\xi - \eta^2\xi^2 - \eta\xi^3) = (\eta+\xi)^2(\eta_0^2+\xi_0^2-2\eta\xi), \end{aligned}$$

we get

$$\begin{aligned} d\psi(\xi, \eta) &= p(1-p) \frac{(\eta-\xi)(\eta_0^2-\xi_0^2)^p}{(\eta^2-\xi^2)^{1-p}} (1-\sigma)^{1+p} F(1+p, p; 2; \sigma) \\ &+ \frac{p(1+p)(\eta+\xi)(\eta_0^2+\xi_0^2-2\eta\xi)}{[(\eta^2-\xi_0^2)(\eta_0^2-\xi^2)]^{1-p}} F(1+p, p; 2; \sigma). \end{aligned}$$

The Lemma 1 is proved.

Let's return to the formula (14). First, we transform I_2 by representing it as the sum of two integrals $I_2 = i_1 + i_2$, where

$$i_1 = \frac{2p(1+p)}{(\eta_0^2 - \xi_0^2)^{2p-1}} \int_{\xi_0}^{\eta_0 - \varepsilon} \left[\frac{(\eta + \xi) \eta \xi F(1-p, -p; 2; \sigma) u(\xi, \eta)}{[(\eta^2 - \xi_0^2)(\eta_0^2 - \xi^2)]^{-p} [\eta(\eta_0^2 - \xi^2) + \xi(\eta^2 - \xi_0^2)]^2} \right]_{\eta=\xi+\varepsilon} d\xi,$$

$$i_2 = \frac{p(1+p)}{2(\eta_0^2 - \xi_0^2)^{2p-1}} \int_{\xi_0}^{\eta_0 - \varepsilon} \left[\frac{(\eta + \xi) [\eta(\eta_0^2 - \xi^2) - \xi(\eta^2 - \xi_0^2)]^2 F(1-p, -p; 2; \sigma) u}{[(\eta^2 - \xi_0^2)(\eta_0^2 - \xi^2)]^{1-p} [\eta(\eta_0^2 - \xi^2) + \xi(\eta^2 - \xi_0^2)]^2} \right]_{\eta=\xi+\varepsilon} d\xi,$$

one of which (i_1) becomes convergent, and the second (i_2), taking into account the Lemma 1 (see, Eq. (15)), is transformed to the form:

$$i_2 = i_{21} + i_{22},$$

where

$$i_{21} = \frac{1}{4} (\eta_0^2 - \xi_0^2)^{-2p} \int_{\xi_0}^{\eta_0 - \varepsilon} [\varphi(\xi, \eta) u(\xi, \eta) d\psi(\xi, \eta)]_{\eta=\xi+\varepsilon} d\xi,$$

$$i_{22} = \frac{p^2}{2} (\eta_0^2 - \xi_0^2)^{1+2p} \times$$

$$\times \int_{\xi_0}^{\eta_0 - \varepsilon} \left[\frac{(\eta^2 - \xi^2)^{1+2p} [\eta(\eta_0^2 - \xi^2) - \xi(\eta^2 - \xi_0^2)] F(1+p, 1+p; 2; \sigma) u(\xi, \eta)}{[(\eta^2 - \xi_0^2)(\eta_0^2 - \xi^2)]^{1+p} [\eta(\eta_0^2 - \xi^2) + \xi(\eta^2 - \xi_0^2)]^2} \right]_{\eta=\xi+\varepsilon} d\xi,$$

$$\varphi(\xi, \eta) = \frac{(\eta + \xi) [\eta(\eta_0^2 - \xi^2) - \xi(\eta^2 - \xi_0^2)] (\eta_0^2 - \xi_0^2)}{[\eta(\eta_0^2 - \xi^2) + \xi(\eta^2 - \xi_0^2)]^2},$$

an auxiliary function $\psi(\xi, \eta)$ is defined in (16).

Now we integrate i_{21} by parts:

$$i_{21} = \frac{1}{4} (\eta_0^2 - \xi_0^2)^{-2p} \varphi(\xi, \eta) \psi(\xi, \eta) u(\xi, \eta) \Big|_{P_\varepsilon}^{Q_\varepsilon}$$

$$- \frac{1}{4} (\eta_0^2 - \xi_0^2)^{-2p} \int_{\xi_0}^{\eta_0 - \varepsilon} \{ \psi(\xi, \eta) d[\varphi(\xi, \eta) u(\xi, \eta)] \}_{\eta=\xi+\varepsilon} d\xi.$$

Let's pass to the limit in (14) at $\varepsilon \rightarrow 0$. Since

$$\varphi(Q) \rightarrow -2, \quad \varphi(P) \rightarrow 2, \quad \psi(Q) \rightarrow R(Q), \quad \psi(P) \rightarrow R(P),$$

then

$$\lim_{\varepsilon \rightarrow 0} \left[\frac{(uv)_P + (uv)_Q}{2} + \frac{1}{4} \varphi(\xi, \eta) \psi(\xi, \eta) u(\xi, \eta) \Big|_P^Q \right] = 0.$$

Further, one can show that

$$\lim_{\varepsilon \rightarrow 0} \left\{ -\frac{1}{4} \int_{P_\varepsilon Q_\varepsilon} \{ \psi(\xi, \eta) d[\varphi(\xi, \eta) u(\xi, \eta)] \}_{\eta=\xi+\varepsilon} d\xi \right\}$$

$$\begin{aligned}
 &= 2 (\eta_0^2 - \xi_0^2)^{-2p-1} \int_{\xi_0}^{\eta_0} \frac{F(-p, -p; 1; 1) \tilde{\tau}(\xi) \xi}{[(\xi^2 - \xi_0^2) (\eta_0^2 - \xi^2)]^{-p}} d\xi \\
 &- \frac{1}{2} (\eta_0^2 - \xi_0^2)^{-2p-1} \int_{\xi_0}^{\eta_0} \frac{(\eta_0^2 + \xi_0^2 - 2\xi^2) F(-p, -p; 1; 1) \tilde{\tau}'(\xi)}{[(\xi^2 - \xi_0^2) (\eta_0^2 - \xi^2)]^{-p}} d\xi, \\
 \lim_{\varepsilon \rightarrow 0} i_{11} &= 4p(1+p) \int_{\xi_0}^{\eta_0} \frac{(\eta_0^2 - \xi_0^2)^{-2p-1} F(1-p, -p; 2; 1) \tilde{\tau}(\xi) \xi}{[(\xi^2 - \xi_0^2) (\eta_0^2 - \xi^2)]^{-p}} d\xi, \\
 \lim_{\varepsilon \rightarrow 0} I_1 &= 0, \quad \lim_{\varepsilon \rightarrow 0} i_{22} = 0, \\
 \lim_{\varepsilon \rightarrow 0} I_3 &= -\frac{1-2p}{2^{1-2p}} \int_{\xi_0}^{\eta_0} \frac{(2\xi)^{-2p} F(p, p; 1; 1) \tilde{\nu}(\xi)}{[(\xi^2 - \xi_0^2) (\eta_0^2 - \xi^2)]^p} d\xi.
 \end{aligned}$$

Hence, applying the summation formula for the Gauss hypergeometric function [17, Ch.2, Eq. 2.8(46)]

$$F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}, \quad \text{Re}(c-a-b) > 0,$$

substituting the calculated limits into (10) and passing to the old variables x and y , after performing some transformations, we obtain the solution of the Cauchy problem for the equation (3) in the quarter-plane $x > 0, y > 0$ with initial data (4) and (5) as follows

$$\begin{aligned}
 U(x, y) &= k_1 \int_0^1 \frac{\tau(s^{1-2p}) dt}{t^{-p}(1-t)^{-p}} - \frac{k_1}{(1+2p)} (xy)^{(m+2)/2} \int_0^1 \frac{\tau'(s^{1-2p})(1-2t)dt}{st^{-p}(1-t)^{-p}} \\
 &+ k_2 xy \int_0^1 \frac{t^{-p}(1-t)^{-p} \nu(s^{1-2p}) dt}{s^{1-2p}}, \quad 1-2p = \frac{2}{m+2},
 \end{aligned} \tag{17}$$

where

$$k_1 = \frac{\Gamma(2+2p)}{\Gamma^2(1+p)}, \quad k_2 = \frac{\Gamma(2-2p)}{\Gamma^2(1-p)}, \quad s^2 = x^{m+2} + y^{m+2} + 2(2t-1)(xy)^{(m+2)/2}, \quad -1 < 2p < 0.$$

Theorem. If $\tau(x) \in C^3[0, 1]$ and $\nu(x) \in C^2[0, 1]$, then the function $U(x, y)$, defined by the formula (17) is a twice continuously differentiable, moreover, unique, solution of the Cauchy problem for the equation (3) with initial data (4) and (5) in D .

Proof. The uniqueness of the solution of the stated problem follows from the very method of obtaining the formula (17). The validity of the other assertions of the theorem can be verified by direct calculation.

Remark 1. In the same way, one can obtain a solution to the Cauchy problem for the Karol's equation (1) in the domain D . The solution obtained in this way coincides with the solution obtained by I.L. Karol [8] from the general solution of the Euler-Poisson-Darboux equation.

Remark 2. Since the equation (7) by replacing $\xi^2 = t, \eta^2 = s$ reduces to the Euler-Poisson-Darboux equation, the solution to the problem could also be obtained from the general solution. We considered it more interesting to show the applicability of the Riemann method in the case when the Riemann function has a singularity on the line that is the carrier of the data of the Cauchy problem.

3 Conclusion

For the future, it would be interesting to solve the Tricomi problem for the degenerate hyperbolic equation with the identically order of degeneracy.

Consider a degenerating equation of the mixed (an elliptic-hyperbolic) type

$$\operatorname{sgn} y |y|^m U_{xx} + x^m U_{yy} = 0, \quad -1 < m < 0. \quad (18)$$

Let Ω be a singly connected finite mixed domain in the plane of the variables x, y , bounded by the simple Jordan curve σ with its endpoints at $B(1, 0)$, $D(0, 1)$, lying in the upper-right quart-plane $x > 0$, $y > 0$, by the characteristics AC and BC of equation (18) at $y < 0$ and by the segment AD , where A is an origin; C is an intersection point of the characteristics AC and BC .

The problem of Tricomi consists of finding the function $u(x, y)$ which is a solution of equation (18) in domain Ω ; continuous in the closed domain Ω and assumes prescribed (continuous) values on σ and on AC .

In solving the Tricomi problem, the formula (17) will play an important role in determining the relationship between $\tau(x)$ and $\nu(x)$, brought from the hyperbolic part of the mixed region.

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