IRSTI 27.39.21 DOI: https://doi.org/10.26577/JMMCS2025125106

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SPECTRUM OF THE GENERALIZED CESÀRO OPERATOR ON LORENTZ SPACES

The aim of this paper is to investigate the boundedness and spectrum of generalized Cesàro operators defined on Lorentz spaces over a finite interval and the positive half-line. When $\beta = 1$, these operators coincide with the classical Cesàro operator. In this paper, we extend the results obtained for Sobolev spaces in [5] to Lorentz spaces. The primary tools employed in this work are C_0-groups, C_0-semigroups, and their generators. C_0-groups and C_0-semigroups are used to demonstrate the boundedness of the generalized Cesàro operator. Since the spectrum of the bounded linear operators is non-empty, we investigate the spectrum of the generalized Cesàro operator. The generators of these C_0-groups and C_0-semigroups are utilized to analyze the spectral properties of the generalized Cesàro operator. We study the spectra of the generators and determine the spectra of the generalized Cesàro operators using the spectral mapping theorem. Additionally, we provide results on the point spectrum of generalized Cesàro operators defined on Lorentz spaces over a finite interval.

Key words: Generalized Cesàro operator, spectrum, Lorentz $L_{p,q}$ spaces, C_0-group, C_0-semigroup.

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Лоренц кеңістіктерінде анықталған жалпыланған Чезаро операторының спектрі

Бұл мақалада ақырлы аралықта және оң жарты өсінде анықталған Лоренц кеңістігіндегі жалпыланған Чезаро операторларының шенелгендігі мен спектрі зерттеледі. $\beta = 1$ болған жағдайда, бұл операторлар классикалық Чезаро операторына сәйкес келеді. Бұл зерттеуде біз [5]-дағы Соболев кеңістіктеріне арналған нәтижелерді Лоренц кеңістігіне кеңейтеміз. Бұл жұмыста негізгі қолданылатын құралдар C_0-топтар, C_0-жартылай топтар және олардың туындатушы операторлары болып табылады. C_0-топтар мен C_0-жартылай топтар жалпыланған Чезаро операторының шенелгендігін дәлелдеуде қолданылады. Шенелген сызықтық операторлардың спектрі бос емес болғандықтан, біз жалпыланған Чезаро операторының спектрін зерттейміз. Осы C_0-топтар мен C_0-жартылай топтардың туындатушы операторлары жалпыланған Чезаро операторының спектрлік қасиеттерін талдауда пайдаланылады. Біз туындатушы операторлардың спектрлерін зерттеп, спектрлік бейнелеу теоремасы арқылы жалпыланған Чезаро операторларының спектрін анықтаймыз. Сонымен қатар, біз ақырлы аралықта анықталған Лоренц кеңістігінде жалпыланған Чезаро операторларының нүктелік спектрі бойынша нәтижелерді ұсынамыз.

Түйін сөздер: Жалпыланған Чезаро оператор, спектр, Лоренц $L_{p,q}$ кеңістіктер, C_0-топ, C_0-жартылай топ.

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Спектр обобщенного оператора Чезаро на пространствах Лоренца

В данной работе исследуются ограниченность и спектр обобщенных операторов Чезаро, определенных на пространствах Лоренца на конечном интервале и положительной полуоси. В случае, когда $\beta = 1$, эти операторы совпадают с классическим оператором Чезаро. В данном исследовании мы расширяем результаты, полученные для пространств Соболева в [5], на пространства Лоренца. Основными инструментами, используемыми в данной работе, являются C_0-группы, C_0-полугруппы и их порождающие операторы. C_0-группы и C_0-полугруппы используются для демонстрации ограниченности обобщенных операторов Чезаро. Поскольку спектр ограниченных линейных операторов не является пустым, мы изучаем спектр обобщенного оператора Чезаро. Порождающие операторы этих C_0-групп и C_0-полугрупп применяются для анализа спектральных свойств обобщенного оператора Чезаро. Мы изучаем спектры порождающих операторов и определяем спектры обобщенных операторов Чезаро с помощью теоремы спектрального отображения. Кроме того, мы представляем результаты по точечному спектру обобщенных операторов Чезаро, определенных на пространствах Лоренца на конечном интервале.

Ключевые слова: Обобщенный оператор Чезаро, спектр, пространства Лоренца $L_{p,q}$, C_0-группа, C_0-полугруппа.

1 Introduction

Let $\mathbb{R}_+ = (0, \infty)$. For $\beta > 0$, $1 < p < \infty$, and $1 \le q \le \infty$, the generalized Cesàro operators C^1_β and C^∞_β are defined on $L_{p,q}(0,1)$ and $L_{p,q}(\mathbb{R}_+)$, respectively, with the same formulas

$$(C^1_\beta f)(t) = \frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1} f(s)ds,\ \ t \in (0,1) \tag{1}$$

and

$$(C^\infty_\beta f)(t) = \frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1} f(s)ds,\ \ t \in \mathbb{R}_+. \tag{2}$$

The generalized Cesàro operator C^∞_β was first studied in [5] on Sobolev spaces which are contained in the Lebesgue spaces $L_p(\mathbb{R}_+)$. This work demonstrated the boundedness and spectral properties of the generalized Cesàro operator. Boundedness of the generalized Cesàro operator in L_p spaces confirmed by the following generalized Hardy inequality in [11]:

$$\left(\int_0^\infty \left|\frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1} f(s)ds\right|^p dt\right)^{\frac{1}{p}} \le \frac{\Gamma(\beta+1)\Gamma(\frac{1}{p})}{\Gamma(\beta+\frac{1}{p})}\|f\|_{L_p}, \tag{3}$$

for $1 < p < \infty$, where Γ denotes the Gamma function. The discrete version of this operator was studied in [10]. In the special case when $\beta = 1$, this operator coincides with the classical Cesàro operator. For the boundedness and other properties such as spectrum of the Cesàro operator in different spaces, we refer the reader to [8], [9], [12], [13], [14], [15], [17], [18], [19].

The aim of this paper is to study boundedness and the spectrum of the generalized Cesàro operators C^∞_β and C^1_β on Lorentz spaces $L_{p,q}(\mathbb{R}_+)$ and $L_{p,q}(0,1)$, respectively. The main tools are so-called the C_0-group and C_0-semigroup, which are denoted by $\{T(t)\}_{t\in\mathbb{R}}$ and $\{S(t)\}_{t\in\mathbb{R}_+}$, and given by

$$(T(t)f)(s) = e^{-\frac{p}{t}} f\left(e^{-t}s\right),\ \ t \in \mathbb{R} \tag{4}$$

and

$$(S(t)f)(s) = e^{-\frac{p}{t}} f(e^{-t}s), \ t \in \mathbb{R}_+. \tag{5}$$

The idea comes from the papers [5], [10], where authors studied similar problems in Sobolev spaces.

The outline of the paper is as follows. In Section 2, we introduce a notion on the spectrum of linear bounded operators in (quasi-)Banach spaces and give definitions of Lorentz spaces as well as definitions of general C_0-group and C_0-semigroup with generators. In Section 3, we study the spectrum of generators of the C_0-group and C_0 -semigroup. Finally, in Section 4, we present the main results that include the boundedness and the spectrum of the generalized Cesàro operators on Lorentz spaces $L_{p,q}(\mathbb{R}_+)$ and $L_{p,q}(0,1)$, respectively.

2 Preliminaries

In this section, we give main definitions and properties of the spectral theory of linear operators, Lorentz spaces and operator semigroups. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ be the set of positive real numbers. As usual, $\mathbb{C}$ is the set of complex numbers, $\mathbb{C}_+$ and $\mathbb{C}_-$ are sets of complex numbers with positive and negative real parts, respectively. Throughout this paper, the closure of a set Ω is indicated by $\overline{\Omega}$.

2.1 Spectrum of linear operators and Lorentz spaces

Let X be a Banach space and $B(X)$ be the algebra of all bounded linear operators on X.

Definition 1 *[1] Let $A \in B(X)$. The resolvent set of A, denoted by $\rho(A)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ has a bounded linear inverse. For each $\lambda \in \rho(A)$, the resolvent operator is defined as*

$$R(\lambda, A) = (\lambda I - A)^{-1}.$$

The spectrum of A, denoted by $\sigma(A)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ does not have a bounded linear inverse.

One can define the different parts of the spectrum as follows:

Definition 2 *[1] Let $A \in L(X)$ a linear operator. The point spectrum, continuous and residual spectrum are defined as*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda I - A \text{ is not injective}\},$$
$$\sigma_c(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda I - A \text{ is injective}, \overline{\operatorname{Im}(\lambda I - A)} = X, \text{ but} \operatorname{Im}(\lambda I - A) \neq X\},$$
$$\sigma_r(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda I - A \text{ is injective}, \overline{\operatorname{Im}(\lambda I - A)} \neq X\}.$$

Clearly, $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$ are disjoint, and

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

In order to define the Lorentz spaces, we need the following notions of the distribution function and the decreasing rearrangement of a given measurable function. Let f be a Lebesgue measurable function defined on Ω with the Lebesgue measure ν, where Ω is either $\mathbb{R}_+$ or $(0,1)$.

Definition 3 *The function $\mu_f : [0;\infty) \to [0;\infty]$ defined by*

$$\mu_f(\lambda) = \nu\left\{t \in \mathbb{R} : |f(t)| > \lambda\right\}, \quad \lambda \geq 0$$

is called the distribution function of f.

Definition 4 *The function $f^* : [0;\infty) \to [0;\infty]$ defined by*

$$f^*(t) = \inf\left\{\lambda \geq 0 : \mu_f(\lambda) \leq t\right\}, \quad t \geq 0$$

is called the decreasing rearrangement of f.

Definition 5 *The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as*

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(s)ds.$$

We now present the main objective of this paper, the Lorentz spaces $L_{p,q}(\Omega)$.

Definition 6 *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. The Lorentz space $L_{p,q}(\Omega)$ is the set of all Lebesgue measurable functions f such that the functional $\|f\|_{L_{p,q}(\Omega)} < \infty$, where*

$$\|f\|_{L_{p,q}(\Omega)} = \begin{cases} \left(\int\limits_0^{\nu(\Omega)} \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \textit{if } 1 \leq p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t), & \textit{if } 1 \leq p \leq \infty,\ q = \infty. \end{cases}$$

Note that for finite q the space $L_{\infty,q}(\Omega)$ is trivial. Furthermore, the Lorentz space $L_{p,q}(\Omega)$ is the generalization of the Lebesgue space $L_p(\Omega)$, which is quasi-Banach in general, and Banach for $1 \leq q \leq p < \infty$ or $p = q = \infty$, see for example, [2, IV.Theorem 4.2]. If $p = q$, then $L_{p,q}(\Omega)$ coincides with $L_p(\Omega)$ and

$$\|f\|_{L_{p,p}} = \|f\|_{L_p}, \quad f \in L_p(\Omega).$$

Definition 7 *For any $f \in L_{p,q}(\Omega)$ the norm $\|\cdot\|^*_{L_{p,q}(\Omega)}$ is defined by*

$$\|f\|^*_{L_{p,q}(\Omega)} = \begin{cases} \left(\int\limits_0^{\nu(\Omega)} \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \textit{if } 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^{**}(t), & \textit{if } 0 < p \leq \infty,\ q = \infty. \end{cases}$$

According to [2], if $1 < p < \infty$, $1 \leq q \leq \infty$ or $p = q = \infty$, then $\| \cdot \|^*_{L_{p,q}(\Omega)}$ is a norm on $L_{p,q}(\Omega)$. This means that $(L_{p,q}(\Omega), \| \cdot \|^*_{L_{p,q}(\Omega)})$ is a Banach space. Moreover, the quasi-norms $\| \cdot \|_{L_{p,q}}$ and $\| \cdot \|^*_{L_{p,q}(\Omega)}$ are equivalent, as shown by the Hardy inequality:

$$\|f\|_{L_{p,q}} \leq \|f\|^*_{L_{p,q}} \leq \frac{p}{p-1}\|f\|_{L_{p,q}}.$$

If $p = \infty$, then we understand that $p/(p-1) = 1$.

A measurable locally bounded function $\omega : \mathbb{R} \to \mathbb{R}$ is a weight if it satisfies $\omega(t) \geq 1$ and $\omega(t+s) \leq \omega(t)\omega(s)$ for all $t, s \in \mathbb{R}$. The weight $\omega(t)$ is called non-quasianalytic if

$$\int\limits_{\mathbb{R}} \frac{\ln \omega(t)}{1+t^2} dt < \infty.$$

Definition 8 *Let ω be a non-quasianalytic weight function on $\mathbb{R}$. The Beurling algebra $L^1_\omega(\mathbb{R})$ is the space of all integrable functions $f \in L_1(\mathbb{R})$ satisfying*

$$\|f\|_\omega = \int\limits_{\mathbb{R}} |f(t)|\omega(t) dt < \infty.$$

2.2 Strongly continuous semigroup

Definition 9 *A family $T = \{T(t)\}_{t \in \mathbb{R}_+}$ in $B(X)$ is called a C_0-semigroup (or strongly continuous semigroup) if the following properties are satisfied:*

(i) $T(0) = I$, *where I is the identity operator on X;*

(ii) $T(t+s) = T(t)T(s)$, *for every $t, s \in \mathbb{R}_+$;*

(iii) $\lim\limits_{t \to 0} \|T(t)x - x\| \to 0$, *for all $x \in X$.*

If $s, t \in \mathbb{R}$ then $T = \{T(t)\}_{t \in \mathbb{R}}$ is called a C_0-group (or strongly continuous group).

The generator of $T = \{T(t)\}_{t \in \mathbb{R}_+}$ (or $\{T(t)\}_{t \in \mathbb{R}}$) is the linear operator A defined by

$$\mathcal{A}x = \lim_{h \to 0} \frac{T(h)x - x}{h}, x \in D(A),$$

where $D(A)$ is domain of operator $\mathcal{A}$.

Definition 10 *Let A be the generator of T. The spectral bound of A is defined by*

$$s(A) = \sup\{Re\lambda : \lambda \in \sigma(A)\}.$$

Definition 11 *Let $T = \{T(t)\}_{t \in \mathbb{R}_+} (\{T(t)\}_{t \in \mathbb{R}})$ be the strongly continuous semigroup(group), then*

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ such that } \|T(t)\| \leq M_\omega e^{\omega t} \ \forall t \in \mathbb{R}_+ \, (\mathbb{R})\}$$

is called the growth bound of T.

The growth bound of the semigroup can also be determined by following formula

$$\omega_0(T) = \lim_{t\to\infty} \frac{\log \|T(t)\|}{t}.$$

Definition 12 *The open sector of angle ω is defined as*

$$S_\omega := \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \omega\},\ 0 < \omega \leq \pi,$$
$$S_0 := (0,\infty),\ \omega = 0.$$

Definition 13 *Let $0 \leq \omega < \pi$, the operator A on a Banach space X is called sectorial of angle ω if $\sigma(A) \subset S_\omega$ and*

$$\sup\left\{\|\lambda(\lambda - A)^{-1}\| : \lambda \notin \overline{S_{\omega'}}\right\} < \infty,$$

for all $\omega < \omega' < \pi$.

The following lemma is a key role in the calculations in the study of the spectrum of generators.

Lemma 1 *If $1 < p < \infty, 1 \leq q \leq \infty$, then*

(i) $t^\gamma \notin L_{p,q}(\mathbb{R}_+)$ *for* $\gamma \in \mathbb{C}$,

(ii) $(\alpha + t)^{-\gamma} \in L_{p,q}(\mathbb{R}_+)$ *for* $\mathrm{Re}\gamma > \frac{1}{p}$ *and* $\alpha > 0$,

(iii) $t^\gamma \in L_{p,q}(0,1)$ *for* $\mathrm{Re}\gamma > -\frac{1}{p}$.

Proof. First, we prove (i). By using the property of decreasing rearrangement, we have

$$((t^\gamma)^*)^q = (|t^\gamma|^q)^* = (t^{q\mathrm{Re}\gamma})^*.$$

Since $q \geq 1$, there are two possible cases for $\mathrm{Re}\gamma$, when $\mathrm{Re}\gamma \leq 0$ and $\mathrm{Re}\gamma > 0$. Let us consider each situation separately. First, consider the case $\mathrm{Re}\gamma \leq 0$. In this case, the function $t^{q\mathrm{Re}\gamma}$ is non-increasing. Its decreasing rearrangement is given by $(t^{q\mathrm{Re}\gamma})^* = t^{q\mathrm{Re}\gamma}$. Then

$$\|f\|^q_{L_{p,q}(\mathbb{R}_+)} = \int_0^\infty (t^{\frac{1}{p}}(t^{\mathrm{Re}\gamma})^*)^q \frac{dt}{t} = \int_0^\infty t^{\frac{q}{p}-1} t^{q\mathrm{Re}\gamma} dt = \int_0^\infty t^{\frac{q}{p}-1+q\mathrm{Re}\gamma} dt = \infty.$$

It means that $t^{\mathrm{Re}\gamma} \notin L_{p,q}(\mathbb{R}_+)$ with $\mathrm{Re}\gamma \leq 0$.

If $\mathrm{Re}\gamma > 0$, then $t^{\mathrm{Re}\gamma}$ is increasing and $\mu_f(\lambda) = \infty$ for all $\lambda \geq 0$. Therefore, $t^{\mathrm{Re}\gamma} \notin L_{p,q}(\mathbb{R}_+)$.

Now, let us prove (ii). Here we also consider two cases when $\mathrm{Re}\gamma \geq 0$ and $\mathrm{Re}\gamma < 0$. First, let $\mathrm{Re}\gamma \geq 0$, then the function $(a+t)^{-q\mathrm{Re}\gamma}$ is non-increasing and

$$(((a+t)^{-\beta})^*)^q = (|(a+t)^{-\gamma}|^q)^* = ((a+t)^{-q\mathrm{Re}\gamma})^* = (a+t)^{-q\mathrm{Re}\gamma}.$$

Moreover, one has

$$\|f\|^q_{L_{p,q}(\mathbb{R}_+)} = \int\limits_0^\infty t^{\frac{q}{p}-1}(a+t)^{-q\mathrm{Re}\gamma}dt$$
$$= \int\limits_0^1 t^{\frac{q}{p}-1}(a+t)^{-q\mathrm{Re}\gamma}dt + \int\limits_0^\infty t^{\frac{q}{p}-1}(a+t)^{-q\mathrm{Re}\gamma}dt.$$

The integral converges under certain conditions on the parameters p, q and $\mathrm{Re}\gamma$. To determine when it converges, let us to analyse the behaviour of integral at the endpoints.

First, let $t \to 0^+$, then near $t = 0$, the term $(a+t)^{-\mathrm{Re}\gamma}$ approaches to $a^{-\mathrm{Re}\gamma}$, so the integral behaves as $\int\limits_0^\infty t^{\frac{q}{p}-1}dt$, and this integral converges if $\frac{q}{p} > 0$. Second, let $t \to \infty$, then for large t, the term $(a+t)^{-q\mathrm{Re}\gamma}$ behaves like $t^{-q\mathrm{Re}\gamma}$, so the integral behaves as $\int\limits_1^\infty t^{\frac{q}{p}-1-q\mathrm{Re}\gamma}$. This integral converges if $\frac{q}{p} - 1 - q\mathrm{Re}\gamma < -1$, that is, $\mathrm{Re}\gamma > \frac{1}{p}$.

In the case when $\mathrm{Re}\gamma < 0$, the function $(a+t)^{-q\mathrm{Re}\gamma}$ is increasing and $\mu_f(\lambda) = \infty$ for all $\lambda \geq 0$. Then $(a+t)^{-\mathrm{Re}\gamma} \notin L_{p,q}(\mathbb{R}_+)$.

Finally, we need to prove (iii). We consider two cases: $\mathrm{Re}\gamma \leq 0$ and $\mathrm{Re}\gamma > 0$.

First, let again $\mathrm{Re}\gamma \leq 0$. Here, the function $t^{\mathrm{Re}\gamma}$ is non-increasing, so $(t^{\mathrm{Re}\gamma})^* = t^{\mathrm{Re}\gamma}$ and the integral

$$\int\limits_0^1 t^{\frac{q}{p}-1}t^{\mathrm{Re}\gamma}dt = \int\limits_0^1 t^{\frac{q}{p}-1+q\mathrm{Re}\gamma}dt = \frac{t^{\frac{q}{p}+q\mathrm{Re}\gamma}}{q(\frac{1}{p}+\mathrm{Re}\gamma)}\Bigg|_0^1 < \infty, \text{ when } \mathrm{Re}\gamma > -\frac{1}{p}.$$

Now, let $\mathrm{Re}\gamma > 0$. In this case the decreasing rearrangement of $t^{\mathrm{Re}\gamma}$ is $f^*(t) = (1-t)^{\mathrm{Re}\gamma}, t \in (0,1)$. Then holds

$$\int\limits_0^1 t^{\frac{q}{p}-1}((t^{\mathrm{Re}\gamma})^*)^q dt = \int\limits_0^1 t^{\frac{q}{p}-1}(1-t)^{\mathrm{Re}\gamma}dt < \infty.$$

Thus, the function t^γ belongs to $L_{p,q}(0,1)$ if and only if $\mathrm{Re}\gamma > -\frac{1}{p}$.

3 Spectrum of generators of the C_0-group and C_0 -semigroup

The main result of this section is the following theorem.

Theorem 1 *For $1 < p < \infty$ and $1 \leq q \leq \infty$, the family of operators $T = \{T(t)\}_{t\in\mathbb{R}}$ is C_0-group of isometries on the space $L_{p,q}(\mathbb{R}_+)$. The generator $\mathcal{A}$ of this C_0-group is given by the following form:*

$$\mathcal{A}f(s) = -sf'(s) - \frac{1}{p}f(s),$$

with the domain $D(\mathcal{A}) = \{f \in L_{p,q}(\mathbb{R}_+) : tf' \in L_{p,q}(\mathbb{R}_+)\}$.

Similarly, the family $S = \{S(t)\}_{t\in\mathbb{R}_+}$ *is* C_0*- semigroup on the space* $L_{p,q}(0,1)$*. The generator* $\mathcal{B}$ *of this* C_0*-semigroup is given by*

$$\mathcal{B}f(s) = -sf'(s) - \frac{1}{p}f(s),$$

with domain $D(\mathcal{B}) = \{f \in L_{p,q}(0,1) : tf' \in L_{p,q}(0,1)\}$.

Proof. We begin with the checking that the operators $\{T(t)\}_{t\in\mathbb{R}}$ are isometries. The following equation holds:

$$\|T(t)f\|_{L_{p,q}(\mathbb{R}_+)} = \left(\int_0^\infty (s^{\frac{1}{p}}(T(t)f)^*(s))^q \frac{ds}{s}\right)^{\frac{1}{q}} = \left(\int_0^\infty (s^{\frac{1}{p}}e^{-\frac{t}{p}}f^*(e^{-t}s))^q \frac{ds}{s}\right)^{\frac{1}{q}}$$
$$= \left(\int_0^\infty ((e^s u)^{\frac{1}{p}}e^{-\frac{s}{p}}f^*(u))^q \frac{du}{u}\right)^{\frac{1}{q}} = \|f\|_{L_{p,q}(\mathbb{R}_+)}.$$

In the case, when $q = \infty$, we have

$$\|(T(s)f)(t)\|_{L_{p,\infty}(\mathbb{R}_+)} = \sup_{t>0} t^{\frac{1}{p}}((T(s)f)(t))^* = \sup_{t>0} t^{\frac{1}{p}}e^{-\frac{s}{p}}f^*(e^{-s}t)$$
$$= \sup_{u>0} u^{\frac{1}{p}}f^*(u) = \|f\|_{L_{p,\infty}(\mathbb{R}_+)}.$$

To show that the family of operators $\{T(t)\}_{t\in\mathbb{R}}$ forms a C_0-group, we have to show that

$$\lim_{t\to 0} \|T(t)f - f\|_{L_{p,q}(\mathbb{R}_+)} = 0.$$

for each $f \in L_{p,q}(\mathbb{R}_+)$.

We first verify this property for $C_c^\infty(\mathbb{R}_+)$ by proving the next

$$\lim_{t\to 0} \|T(t)f - f\|_\infty = \lim_{t\to 0}\sup_{x>0} |(T(t)f)(x) - f(x)|$$
$$= \lim_{t\to 0}\sup_{x>0} |e^{-\frac{t}{p}}f(e^{-t}x) - f(x)| = 0.$$

Since $C_c^\infty(\mathbb{R}_+)$ is dense in $L_{p,q}(\mathbb{R}_+)$ [4, Theorem 3.3], it follows that

$$\lim_{t\to 0} \|T(t)f - f\|_{L_{p,q}(\mathbb{R}_+)} = 0.$$

By definition of generator of the C_0-group for every $f \in D(A)$ we get following

$$\mathcal{A}f(s) = \lim_{t\to 0} \frac{T(t)f(s) - f(s)}{t} = \lim_{t\to 0} \frac{e^{-\frac{t}{p}}f(e^{-t}s) - f(s)}{t}$$
$$= -sf'(s) - \frac{1}{p}f(s),$$

with $D(\mathcal{A}) = \{f \in L_{p,q}(\mathbb{R}_+) : tf' \in L_{p,q}(\mathbb{R}_+)\}$.

Next, we will prove that the family of operators $\{S(t)\}_{t\in\mathbb{R}_+}$ is C_0-semigroup. First, we show that it is bounded for each t in the following

$$\|S(t)f\|_{L_{p,q}(0,1)} = \left(\int_0^1 (s^{\frac{1}{p}}(S(t)f)^*(s))^q \frac{ds}{s}\right)^{\frac{1}{q}} = \left(\int_0^1 (s^{\frac{1}{p}} e^{-\frac{t}{p}} f^*(e^{-t}s))^q \frac{ds}{s}\right)^{\frac{1}{q}}$$
$$= \left(\int_0^{e^{-t}} ((e^s u)^{\frac{1}{p}} e^{-\frac{s}{p}} f^*(u))^q \frac{du}{u}\right)^{\frac{1}{q}} \leq \|f\|_{L_{p,q}(0,1)}.$$

Using a similar argument as above, it follows that this semigroup is a C_0-semigroup, and its generator is given by

$$\mathcal{B}f(s) = -sf'(s) - \frac{1}{p}f(s),$$

with domain $D(\mathcal{B}) = \{f \in L_{p,q}(0,1) : tf' \in L_{p,q}(0,1)\}$.

In the following proposition, we find the spectrum of the generators of the C_0-group and C_0-semigroup.

Proposition 1 *Let $1 < p < \infty$ and $1 \leq q \leq \infty$, then*

i) $\sigma_p(\mathcal{A}) = \emptyset, \quad \sigma(\mathcal{A}) = i\mathbb{R}.$

ii) $\sigma_p(\mathcal{B}) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}, \quad \sigma(\mathcal{B}) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\}.$

Proof. i) Let $\lambda \in \mathbb{C}$. The equation $\mathcal{A}f = \lambda f$ is equivalent to the differential equation

$$tf'(t) + (\lambda + \frac{1}{p})f(t) = 0.$$

Its non-zero solutions are given by $f(t) = ct^{-(\lambda+\frac{1}{p})}$ with $c \neq 0$. According to Lemma 1, these solutions do not belong to $L_{p,q}(\mathbb{R}_+)$. Therefore, $\mathcal{A}$ has no eigenvalues, and the point spectrum is empty: $\sigma_p(\mathcal{A}) = \emptyset$.

Since each $T(s)$ is an invertible isometry, its spectrum is confined to the unit circle:

$$\sigma(T(t)) \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

By the spectral mapping theorem for C_0-group (see [6, IV. Theorem 3.6]), the relation $e^{t\sigma(\mathcal{A})} \subseteq \sigma(T(t))$ holds. Hence, if $\eta \in \sigma(\mathcal{A})$, it follows that $e^{t\eta} \in \{z \in \mathbb{C} : |z| = 1\}$, implying $\sigma(\mathcal{A}) \subseteq i\mathbb{R}$.

Assume $\xi \in i\mathbb{R}$ and that $\xi \in \rho(\mathcal{A})$. Let $\eta = \xi + \frac{1}{p}$. According to Lemma 1, the function $f(t) = (1+t)^{-\eta-1}$ lies in $L_{p,q}(\mathbb{R}_+)$. Since the resolvent operator $R(\xi, \mathcal{A})$ is bounded, the function $g(t) = R(\xi, \mathcal{A})f(t)$ also belongs to $L_{p,q}(\mathbb{R}_+)$. This implies that $g(t)$ satisfies the differential equation

$$\eta g(t) + tg'(t) = f(t).$$

Solving this equation yields the general solution

$$\widetilde{g}(t) = ct^{-\eta} + \frac{1}{\eta}(1+t)^{-\eta},$$

where c is a constant. However, as in Lemma 1, it can be confirmed that $\widetilde{g}(t) \notin L_{p,q}(\mathbb{R}_+)$. Thus $\xi \in \sigma(\mathcal{A})$.

ii) In this case, we first examine the equation $\mathcal{B}f = \lambda f$. The solution to this equation is given by $f(s) = s^{-(\lambda - \frac{1}{p})}$. We can see by Lemma 1 that the function $f(s) = s^{-(\lambda + \frac{1}{p})}$ belongs to $L_{p,q}(0,1)$ if and only if $\mathrm{Re}\lambda < 0$.

We know from [6, Corollary 1.13] that

$$-\infty \leq s(\mathcal{B}) \leq \omega_0(S) < \infty.$$

By definition,

$$\omega_0(S) = \lim_{t\to\infty} \frac{\log \|S(t)\|}{t} = 0.$$

Considering these facts, the spectrum is given by

$$\sigma(\mathcal{B}) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\}.$$

4 Spectrum of Generalized Cesàro operators on Lorentz spaces

In this section, we establish the spectrum of the generalized Cesàro operator on Lorentz spaces.

4.1 The case $\mathbb{R}_+$

The following result shows the boundedness of the generalized Cesàro operator C_β^∞ on $L_{p,q}(\mathbb{R}_+)$ spaces.

Theorem 2 *Let $\beta > 0$, $1 < p < \infty$ and $1 \leq q \leq \infty$, then the operator C_β^∞ is bounded on $L_{p,q}(\mathbb{R}_+)$.*

If $f \in L_{p,q}(\mathbb{R}_+)$, then

$$C_\beta^\infty f(t) = \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})}(s) f(t) ds. \tag{6}$$

Proof. Let us first demonstrate the equality (6). By changing the variable $\tau = te^{-s}$, we obtain the following

$$C_\beta^\infty f(t) = \frac{\beta}{t^\beta} \int_0^t (t-\tau)^{\beta-1} f(\tau) ds = \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} T(s) f(t) ds.$$

Considering the density of simple functions in the $L_{p,q}(\mathbb{R}_+)$ space and utilizing the properties of the Bochner integrable functions, we can observe that the operator C_β^∞ is well-defined and bounded on $L_{p,q}(\mathbb{R}_+)$. When $1 \le q \le p < \infty$, $p \neq 1$, then, we have

$$\|C_\beta^\infty f\|_{L_{p,q}(\mathbb{R}_+)} \le \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} \|T(s)f\| ds$$
$$= \beta \|f\|_{L_{p,q}(\mathbb{R}_+)} \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} ds = \beta \|f\|_{L_{p,q}(\mathbb{R}_+)} \int_0^1 (1-u)^{\beta-1} u^{1-\frac{1}{p}} \frac{du}{u}$$
$$= \beta \|f\|_{L_{p,q}(\mathbb{R}_+)} \int_0^1 (1-u)^{\beta-1} u^{1-\frac{1}{p}-1} du = \|f\|_{L_{p,q}(\mathbb{R}_+)} \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p})}{\Gamma(\beta+1-\frac{1}{p})}.$$

Here, the Beta function is applied to evaluate the integral. In general case, when $1 < p < q \le \infty$ by [18], we get that C_β^∞ is bounded on $L_{p,q}(\mathbb{R}_+)$ with respect to $\|\cdot\|^*_{L_{p,q}}$. Therefore, we have

$$\|C_\beta^\infty f\|_{L_{p,q}(\mathbb{R}_+)} \le c_{\beta,p} \|f\|_{L_{p,q}(\mathbb{R}_+)},$$

where $c_{\beta,p} > 0$ is a constant depending only on β and p.

The first main result is the following theorem.

Theorem 3 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $\beta > 0$. For the operator C_β^∞ on $L_{p,q}(\mathbb{R}_+)$ we have*

$$\sigma(C_\beta^\infty) = \overline{\left\{ \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)} : t \in \mathbb{R} \right\}}.$$

Proof. In the previous theorem, we demonstrated that the operator C_β^∞ can be expressed in terms of the semigroup $T(t)$, i.e.,

$$C_\beta^\infty f(t) = \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} T(s) f(t) ds = \int_{-\infty}^\infty g_{\beta,p}(s) T(s) f(t) ds,$$

where $g_{\beta,p}(s) = \chi_{[0,\infty)}(s) \beta (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})}$ for $s \in \mathbb{R}$. According to [7], if the function $g_{\beta,p}$ belongs to the space $L^1_\omega(\mathbb{R})$, then it follows that

$$\sigma(C_\beta^\infty) = \overline{\widehat{g_{\beta,p}}(\sigma(i\mathcal{A}))},$$

where $\widehat{g_{\beta,p}}$ is the Fourier transform of the function $g_{\beta,p}$. In our case, the non-quasianalytic weight is equal to 1. Therefore, it is straightforward to verify that $g_{\beta,p} \in L^1_1(\mathbb{R})$ due to the properties of the Beta function.

Then, for $t \in \sigma(i\mathcal{A}) = \mathbb{R}$ (see Proposition 1) we have

$$\widehat{g_{\beta,p}}(\lambda) = \beta \int_0^\infty e^{-i\lambda s}(1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}ds = \beta \int_0^1 (1-u)^{\beta-1}u^{1-\frac{1}{p}+it-1}du$$
$$= \beta B(\beta, 1-\frac{1}{p}+it) = \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)}.$$

4.2 The case $(0,1)$

Let $L_{p,q}(0,1)$. In contrast to the previous subsection, we now describe the spectrum of the generalized Cesàro operator on $L_{p,q}(0,1)$. The main result of this section is given in the following theorem.

Theorem 4 *Let $\beta > 0$, $1 < p < \infty$ and $1 \leq q \leq \infty$, then the operator C_β^1 is bounded on $L_{p,q}(0,1)$.*

If $f \in L_{p,q}(0,1)$, then

$$C_\beta^1 f(t) = \beta \int_0^\infty (1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}S(s)f(t)ds. \tag{7}$$

Proof. We apply the change of variable $\tau = te^{-s}$ to obtain the following

$$C_\beta^1 f(t) = \frac{\beta}{t^\beta}\int_0^t (t-\tau)^{\beta-1}f(\tau)ds = \beta \int_0^\infty (1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}S(s)f(t)ds.$$

It proves the equality (7).

Note that, due to this equality, C_β^1 is well-defined and acts as a bounded operator on $L_{p,q}(0,1)$ for $1 \leq q \leq p < \infty$, $p \neq 1$, then

$$\|C_\beta^1 f\|_{L_{p,q}(0,1)} \leq \beta \int_0^\infty (1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}\|S(s)f\|_{L_{p,q}(0,1)}ds$$
$$\leq \beta\|f\|_{L_{p,q}(0,1)} \int_0^\infty (1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}ds = \beta\|f\|_{L_{p,q}(0,1)} \int_0^1 (1-u)^{\beta-1}u^{1-\frac{1}{p}}\frac{du}{u}$$
$$= \beta\|f\|_{L_{p,q}(0,1)} \int_0^1 (1-u)^{\beta-1}u^{1-\frac{1}{p}-1}du = \|f\|_{L_{p,q}(0,1)}\frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p})}{\Gamma(\beta+1-\frac{1}{p})}.$$

As in Theorem 4, in general case when $1 < p < q \leq \infty$, we have

$$\|C_\beta^1 f\|_{L_{p,q}} \leq c_{\beta,p}\|f\|_{L_{p,q}},$$

where $c_{\beta,p} > 0$ is a constant depending only on β and p.

Theorem 5 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $\beta > 0$. For the operator C_β^1 on $L_{p,q}(0,1)$ we have*

$$\sigma_p(C_\beta^1) = \left\{ \frac{\Gamma(\beta+1)\Gamma(\lambda+1-\frac{1}{p})}{\Gamma(\beta+\lambda+1-\frac{1}{p})} : \lambda \in \mathbb{C}_+ \right\}$$

and

$$\sigma(C_\beta^1) = \overline{\left\{ \frac{\Gamma(\beta+1)\Gamma(\lambda+1-\frac{1}{p})}{\Gamma(\beta+\lambda+1-\frac{1}{p})} : \lambda \in \mathbb{C}_+ \cup i\mathbb{R} \right\}}.$$

Proof. Define the function

$$h_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \ \ \gamma \in \mathbb{C}.$$

The functions h_γ are eigenfunctions of the operator C_β^1, it means that

$$(C_\beta^1 h_\gamma)(t) = \frac{\beta}{\Gamma(\gamma)t^\gamma} \int_0^t (t-s)^{\beta-1} s^{\gamma-1} ds = \frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} h_\gamma(t).$$

According to Lemma 1, the function h_γ belongs to $L_{p,q}(0,1)$ if and only if $\mathrm{Re}\gamma - 1 > -\dfrac{1}{p}$. It follows that the point spectrum of the operator C_β^1 in $L_{p,q}(0,1)$ is the set

$$\sigma_p(C_\beta^1) = \left\{ \frac{\Gamma(\beta+1)\Gamma(\lambda+1-\frac{1}{p})}{\Gamma(\beta+\lambda+1-\frac{1}{p})} : \lambda \in \mathbb{C}_+ \right\}.$$

Next, we consider the Hille-Phillips functional calculus for the generator $\mathcal{B}$ of the semigroup $S = \{S(t)\}_{t\ge 0}$. According to Theorem 4, we can write $C_\beta^1 = \mathcal{L}(g)(-\mathcal{B})$ that is

$$C_\beta^1 f = \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} S(s) f(t) ds = \int_0^\infty g_{\beta,p}(t) S(t) f dt = \mathcal{L}(g_{\beta,p})(-\mathcal{B}) f,$$

where $g_{\beta,p}(t) = \beta(1-e^{-t})^{\beta-1} e^{-t(1-\frac{1}{p})}$ and $\mathcal{L}$ is the Laplace transform.

$$\begin{aligned}\mathcal{L}(g_{\beta,p})(z) &= \beta \int_0^\infty e^{-zs}(1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} ds \\ &= \frac{\Gamma(\beta+1)\Gamma(z+1-\frac{1}{p})}{\Gamma(\beta+z+1-\frac{1}{p})} = h_{\beta,p}(z), \ \ z \in \overline{\mathbb{C}_+}.\end{aligned}$$

By [10, p. 1458], the function $h_{\beta,p}$ satisfies Spectral Mapping Theorem [16, Theorem 2.7.8]. Since $-\mathcal{B}$ is a sectorial operator of angle $\dfrac{\pi}{2}$ and $\mathcal{B}$ is injective ($0 \notin \sigma_p(\mathcal{B})$), we have

$$\sigma(C_\beta^1) = \sigma(h_{\beta,p}(-\mathcal{B})) = \overline{h_{\beta,p}(\sigma(-\mathcal{B}))} = \overline{\left\{ \frac{\Gamma(\beta+1)\Gamma(\lambda+1-\frac{1}{p})}{\Gamma(\beta+\lambda+1-\frac{1}{p})} : \ \lambda \in \mathbb{C}_+ \cup i\mathbb{R} \right\}}.$$

4.3 Conclusion

In this paper, we studied the boundedness and spectral properties of the generalized Cesàro operators C^1_β and C^∞_β defined on the Lorentz spaces $L_{p,q}(0,1)$ and $L_{p,q}(\mathbb{R}_+)$, respectively. Using tools such as the C_0-group $\{T(t)\}_{t\in\mathbb{R}}$ and the C_0-semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$, we analyzed the boundedness and spectrum of these operators. The spectral properties of the generators of these groups and semigroups were studied, which played a central role in determining the spectrum of the generalized Cesàro operators. The main results demonstrated that the generalized Cesàro operators are bounded on Lorentz spaces and provided a detailed characterization of their spectra.

5 Acknowledgements

The work was partially supported by the grant No. AP23483532 of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

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Received: January 28, 2025
Accepted: February 24, 2025