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ON A SPECTRAL PROBLEM FOR A FOURTH-ORDER DIFFERENTIAL OPERATOR

This paper considers a generalized spectral problem for a fourth-order differential operator. The primary goal of the research is to analyze the spectral properties of the operator arising in boundary value problems for the Stokes and Navier-Stokes equations, as well as to utilize the obtained eigenfunctions to construct a fundamental system in the space of solenoidal functions. The work combines theoretical analysis with practical applications, making it relevant for numerical modeling of hydrodynamic processes. The main methodology is based on the method of separation of variables and the use of curl operators for different domain dimensions. In particular, the paper proposes approaches to introducing curl operators for the three- and four-dimensional cases, which generalize the problem formulation. The key results include proving the existence and distribution of eigenvalues, as well as constructing an orthonormal basis in functional spaces. This study contributes to the development of spectral analysis of high-order operators and can be useful for developing efficient algorithms for solving hydrodynamic problems. The practical significance of the results lies in their application to numerical modeling of fluid flows in various fields of science and engineering.
Key words: spectral problem, curl operator, eigenvalues, eigenfunction.

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Төрінші ретті дифференциалдық оператор үшін бір спектралды есеп туралы

Бұл жұмыста төртінші ретті дифференциалдық оператор үшін жалпыланған спектрлік есеп қарастырылады. Зерттеудің негізгі мақсаты – Стокс және Навье-Стокс теңдеулері үшін шекаралық есептерді шешу барысында туындайтын оператордың спектрлік қасиеттерін талдау, сондай-ақ алынған меншікті функцияларды соленоидалды функциялар кеңістігінде іргелі жүйені құру үшін пайдалану. Жұмыс теориялық талдауды практикалық қолданумен үйлестіретіндіктен, бұл оны гидродинамикалық үдерістердің сандық модельдеуі үшін өзекті етеді. Негізгі әдіс айнымалыларды бөлу әдісіне және әртүрлі өлшемдегі облыстар үшін ротор операторларын қолдануға негізделген. Атап айтқанда, үш және төрт өлшемді жағдайлар үшін ротор операторларын енгізу тәсілдері ұсынылып, бұл өз кезегінде есептің қойылымын жалпылауға мүмкіндік береді. Негізгі нәтижелерге меншікті мәндердің бар екендігі мен орналасуын дәлелдеу, сондай-ақ функционалдық кеңістіктерде ортонормаланған базисті құру жатады. Бұл зерттеу жоғары ретті операторлардың спектрлік талдауының дамуына үлес қосып, гидродинамикалық есептерді шешудің тиімді алгоритмдерін әзірлеуге пайдалы болуы мүмкін. Жұмыстың практикалық маңызы – алынған нәтижелердің әртүрлі ғылыми және инженерлік салалардағы сұйықтық ағындарын сандық модельдеуде қолданылуында.
Түйін сөздер: спектрлік есеп, ротор операторы, меншікті мән, меншікті функция.

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Об одной спектральной задаче для дифференциального оператора четвертого порядка

В данной работе рассматривается обобщенная спектральная задача для одного дифференциального оператора четвертого порядка. Основной целью исследования является анализ спектральных свойств оператора, возникающего при решении краевых задач для уравнений Стокса и Навье-Стокса, а также использование полученных собственных функций для построения фундаментальной системы в пространстве соленоидальных функций. Работа сочетает теоретический анализ с практическим применением, что делает её актуальной для численного моделирования гидродинамических процессов. Основная методология основана на методе разделения переменных и использовании роторных операторов для различных размерностей области. В частности, предлагаются способы введения операторов ротор для трех- и четырехмерного случаев, что позволяет обобщить постановку задачи. Основными результатами являются доказательство существования и расположения собственных значений, а также построение ортонормированного базиса в функциональных пространствах. Данное исследование вносит вклад в развитие спектрального анализа операторов высокого порядка и может быть полезно для разработки эффективных алгоритмов решения гидродинамических задач. Практическая значимость результатов заключается в их применении в численном моделировании потоков жидкости в различных областях науки и техники.

Ключевые слова: спектральная задача, оператор ротор, собственные значения, собственные функции.

Introduction

In this paper, we consider a generalized spectral problem for a fourth-order differential operator.

By introducing a scalar or vector stream function, the spectral problem for the two-, three-, and four-dimensional Stokes operators can be reduced to a generalized spectral problem for the biharmonic operator.

Let us provide the mathematical formulations of the latter statement.

First, let us formulate the spectral problem for the d-dimensional Stokes operator. Let $x = (x_1, ..., x_d) \in \Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded simply connected domain with boundary $\partial\Omega$. The goal is to find nontrivial solutions $\{\vec{w}_k(x),\ p_k(x),\ x \in \Omega,\ \ k \in \mathbb{N}\}$ and the corresponding values of the parameter $\{\lambda_k^2,\ k \in \mathbb{N}\}$ for the following boundary value problem ([1], Chapter II, § 4; [2], Chapter I, § 6, Corollary 6.1; [3], Chapter I, § 2, Subsection 2.6):

$$\begin{cases} -\Delta\vec{w}(x) + \nabla p(x) &= \lambda^2 w(x), \quad x \in \Omega, \\ \operatorname{div}\{\vec{w}(x)\} &= 0, \quad x \in \Omega, \\ \vec{w}(x) &= 0, \quad x \in \partial\Omega. \end{cases} \tag{1}$$

Let $\dim\{\Omega\} = 2$, and consider the two-dimensional curl operator **curl** defined as follows:

$$\{w_1, w_2\} = \mathbf{curl}\{0, 0, U(x)\} = \{\partial_{x_2} U, -\partial_{x_1} U\}, \tag{2}$$

where $U(x)$ is a scalar function known as the stream function. From equation (1) using the formulas in (2), we can proceed as follows: first, by substituting the vector function $\vec{w}$ in (1) with $\operatorname{curl} U$; second, by applying the operator curl, to the resulting expressions; and third, by summing the results obtained after the second step. As a result, we obtain:

$$\begin{cases} (-\Delta)^2 U(x) &= \lambda^2(-\Delta)U(x), \quad x \in \Omega, \\ U(x) &= 0, \quad x \in \partial\Omega, \\ \partial_{\vec{n}} U(x) &= 0, \quad x \in \partial\Omega, \end{cases} \tag{3}$$

where $\vec{n}$ is the outward normal to the boundary $\partial\Omega$.

Since the differential equation in (3) contains the operator $-\Delta$ on the right-hand side, we will refer to problem (3) as a generalized spectral problem for the biharmonic operator $(-\Delta)^2$. It is evident that the key role in transforming problem (1) into the spectral problem (3) is played by the **curl** operator given in (2).

Let $\dim\{\Omega\} = 3$, and consider the three-dimensional **curl** operator defined as follows:

$$\operatorname{curl}\vec{U}(x_1,x_2,x_3) = \vec{w}(x_1,x_2,x_3), \ \operatorname{div}\vec{w}(x_1,x_2,x_3) = 0, \ (x_1,x_2,x_3) \in \Omega, \tag{4}$$

where $\vec{U} = \{U_1, U_2, U_3\}$, $\vec{w} = \{w_1, w_2, w_3\}$ are three-dimensional vector functions,

$$\vec{w} = \operatorname{curl}\vec{U} = \{\partial_{x_2}U_3 - \partial_{x_3}U_2,\ \partial_{x_3}U_1 - \partial_{x_1}U_3,\ \partial_{x_1}U_2 - \partial_{x_2}U_1\}. \tag{5}$$

If we assume that all three components of the vector $\vec{U}$ are equal, i.e., $U_1 = U_2 = U_3 = U(x_1,x_2,x_3)$ in Ω, then, similarly to the two-dimensional case, using equations (4)–(5), we can derive from (1) the following:

$$\begin{cases} -\Delta(-\Delta + S)U(x) &= \lambda^2(-\Delta + S)U(x), & x \in \Omega, \\ U(x) &= 0, & x \in \partial\Omega, \\ \partial_{\vec{n}}U(x) &= 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

where $S = \partial^2_{x_1x_2} + \partial^2_{x_2x_3} + \partial^2_{x_3x_1}$. If we temporarily remove the operator S from the differential equation in (6), we once again obtain a spectral problem of the form (3), but now in the three-dimensional case.

Let $\dim\{\Omega\} = 4$, and consider the four-dimensional **curl** operator defined as follows:

$$\operatorname{curl}\vec{U}(x_1,x_2,x_3,x_4) = \vec{w}(x_1,x_2,x_3,x_4), \ \operatorname{div}\vec{w}(x_1,x_2,x_3,x_4) = 0, \ (x_1,x_2,x_3,x_4) \in \Omega, \tag{7}$$

where $\vec{U} = \{U_1, U_2, U_3, U_4, U_5, U_6\}$, $\vec{w} = \{w_1, w_2, w_3, w_4\}$,

$$\vec{w} = \operatorname{curl}\vec{U} = \begin{pmatrix} \partial_{x_4}U_1 + \partial_{x_3}U_5 - \partial_{x_2}U_6 \\ \partial_{x_4}U_2 + \partial_{x_1}U_6 - \partial_{x_3}U_4 \\ \partial_{x_4}U_3 + \partial_{x_3}U_4 - \partial_{x_1}U_5 \\ -\partial_{x_1}U_1 - \partial_{x_2}U_2 - \partial_{x_3}U_3 \end{pmatrix}, \ \operatorname{div}\operatorname{curl}\vec{U} = 0. \tag{8}$$

Remark 1 *The* curl *operator in equations* (7)–(8) *acts on a six-dimensional vector function* $\vec{U}$*, which, in particular, corresponds to the following vector composed of the electric* $\vec{E}$ *and magnetic* $\vec{H}$ *field intensity vectors:* $\vec{E} = \{E^1, E^2, E^3\}$, $\vec{H} = \{H^1, H^2, H^3\}$ *([4], Chapter V, § 1, Chapter VII, § 1; [5], Chapter III, § 8 and § 9; [6], Chapter I, § 5), namely,*

$$\vec{U} = \{E^1, E^2, E^3, H^1, H^2, H^3\}.$$

From equation (1), using formulas (7)–(8), we can derive the following:

$$\begin{cases} (-\Delta)^2U(x) &= 3\lambda^2(-\Delta)U(x), & x \in \Omega, \\ U(x) &= 0, & x \in \partial\Omega, \\ \partial_{\vec{n}}U(x) &= 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

If we disregard the factor of 3 in front of the spectral parameter λ^2, the spectral problem (9) fully coincides with problem (3), but now in the four-dimensional case, i.e., $\dim\Omega = 4$.

Once again, it is evident that the key role in transforming problem (1) into the spectral problem (9) is played by the **curl** operator, which is defined by formulas (7)–(8).

The aim of this work is to construct a fundamental system in the space of solenoidal functions. If we were able to solve spectral problems for the biharmonic operator (3) in domains of various dimensions $\dim\{\Omega\} = d,\ d \geq 2$, we would succeed in constructing such a fundamental system, which is important not only from a theoretical point of view but also for the development of computationally efficient algorithms for the approximate solution of boundary value problems for the Stokes and Navier–Stokes systems [7]. In this work, we will limit ourselves to solving a certain generalized spectral problem for a fourth-order differential operator.

It is worth noting that spectral problems for the Stokes operator (but with periodic boundary conditions) in a cubic domain have also been considered in the works [8], [9], and [10].

In [8], the spectra of the curl and Stokes operators in a cube are studied for functions satisfying periodic boundary conditions. The Cauchy problem for the 3D Navier-Stokes equations with periodic conditions in the spatial variable was investigated in [10].

Since our approach actively utilizes the properties of the **curl** operator, which is closely related to vortex theory, we refer to the foundational works on vortex theory [11], [12], [13], [14], [15], [16], and others. Some ideas from these works have been used in establishing our statements.

Let us introduce the main function spaces that will be used in this work. Let $x = (x_1, ..., x_d) \in \Omega \subset \mathbb{R}^d$ where $d \geq 2$, be an open bounded simply connected domain with a sufficiently smooth boundary $\partial\Omega$, and let $m \geq 0$ be an integer,

$$W_2^m(\Omega) = \left\{v|\ \partial_x^{|\alpha|} v \in L^2(\Omega),\ |\alpha| \leq m\right\}, \quad \text{where} \ \ \partial_x^{|\alpha|} = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d},\ |\alpha| = \sum_{j=1}^{d} \alpha_j,\ \partial_{x_j} = \frac{\partial}{\partial_{x_j}},$$

$$\overset{\circ}{W}{}_2^m(\Omega) = \left\{v|\ v \in W_2^m(\Omega),\ \partial_{\vec{n}}^j v = 0,\ j = 0, 1, 2..., m-1,\ \vec{n} \text{ is the outward normal to } \ \partial\Omega\right\}.$$

For the notation of function spaces, we will follow the monographs [17], [18], [19], and [20].

1 Formulation of the Spectral Problem

Let us consider the following spectral problem for a fourth-order differential operator.

Problem 1

$$\sum_{k=1}^{d} \partial_{x_k}^4 u(x) = \lambda^2 (-\Delta) u(x), \ \ x \in \Omega, \tag{10}$$

$$u(x) = \partial_{\vec{n}} u(x) = 0, \ \ x \in \partial\Omega, \tag{11}$$

where $\vec{n}$ is the outward normal to $\partial\Omega$.

Let us introduce the following spaces:

Definition 1 *Let us denote by $V_1(\Omega)$ and $V_2(\Omega)$ the Hilbert spaces with the corresponding inner products*

$$(\nabla u, \nabla v)_{L^2(\Omega)}, \qquad \forall\, u, v \in \overset{\circ}{W}{}_2^1(\Omega), \tag{12}$$

$$((u, v)) \overset{def}{=} \sum_{k=1}^{d} \left(\partial_{x_k}^2 u, \partial_{x_k}^2 v\right)_{L^2(\Omega)}, \qquad \forall\, u, v \in \overset{\circ}{W}{}_2^2(\Omega), \tag{13}$$

and norms

$$\|u\|_{V_1(\Omega)} = \sqrt{\|\nabla u\|_{L^2(\Omega)}^2}, \;\; \|u\|_{V_2(\Omega)} = \sqrt{\sum_{k=1}^{d} \|\partial_{x_k}^2 u\|_{L^2(\Omega)}^2}. \tag{14}$$

It is obvious that the norms (14), induced by the inner products (12)–(13), define equivalent norms in the spaces $\overset{\circ}{W}{}_2^1(\Omega)$ and $\overset{\circ}{W}{}_2^2(\Omega)$, respectively..

Assumption 1 *In the spectral problem* (10)–(11)*, the fourth-order operator is elliptic and possesses the properties of symmetry and positive definiteness in the space $V_2(\Omega)$. Therefore, the eigenvalues $\{\lambda_n^2,\ n \in \mathbb{N}\}$ of this problem are real and located on the positive semi-axis. Moreover, the smallest eigenvalue is bounded away from zero, i.e., $\lambda_1 \geq \delta > 0$.*

The following statement holds true.

Assumption 2 *The spectral problem* (10)–(11) *possesses a set of "generalized eigenfunctions" $\{u_n(x),\ n \in \mathbb{N}\}$, which belong to the space $V_2(\Omega)$ and form an orthonormal basis in the space $V_1(\Omega)$.*

Let us formulate the main result of this work.

Theorem 1 (Main result) *The spectral problem* (10)–(11) *has the following solution*

$$u_n(x) = X_{1n}(x_1) X_{2n}(x_2) ... X_{dn}(x_d), \qquad \lambda_n^2, \qquad n \in \mathbb{N}, \tag{15}$$

where $X_{1n}(x_1) = \Phi_n(y)_{|y=x_1},\ X_{2n}(x_2) = \Phi_n(y)_{|y=x_2}, ..., X_{dn}(x_d) = \Phi_n(y)_{|y=x_d}$*:*

$$\begin{cases} \Phi_{2n-1}(y) = \sin^2 \frac{\lambda_{2n-1} y}{2},\ \lambda_{2n-1}^2 = \left(\frac{2\pi n}{l}\right)^2,\ \ n \in \mathbb{N}, \\ \Phi_{2n}(y) = \left[\lambda_{2n} l - \sin \lambda_{2n} l\right] \sin^2 \frac{\lambda_{2n} y}{2} - \sin^2 \frac{\lambda_{2n} l}{2} \left[\lambda_{2n} y - \sin \lambda_{2n} y\right], \\ \qquad \lambda_{2n}^2 = \left(\frac{2\nu_n}{l}\right)^2,\ \ n \in \mathbb{N}, \end{cases} \tag{16}$$

and $\{\nu_n,\ n \in \mathbb{N}\}$ are the positive roots of the equation $\tan \nu = \nu,\ n \in \mathbb{N}$.

The arrangement of the eigenvalues on the positive semi-axis is shown in Figure 1.1 (here $l = 2$).

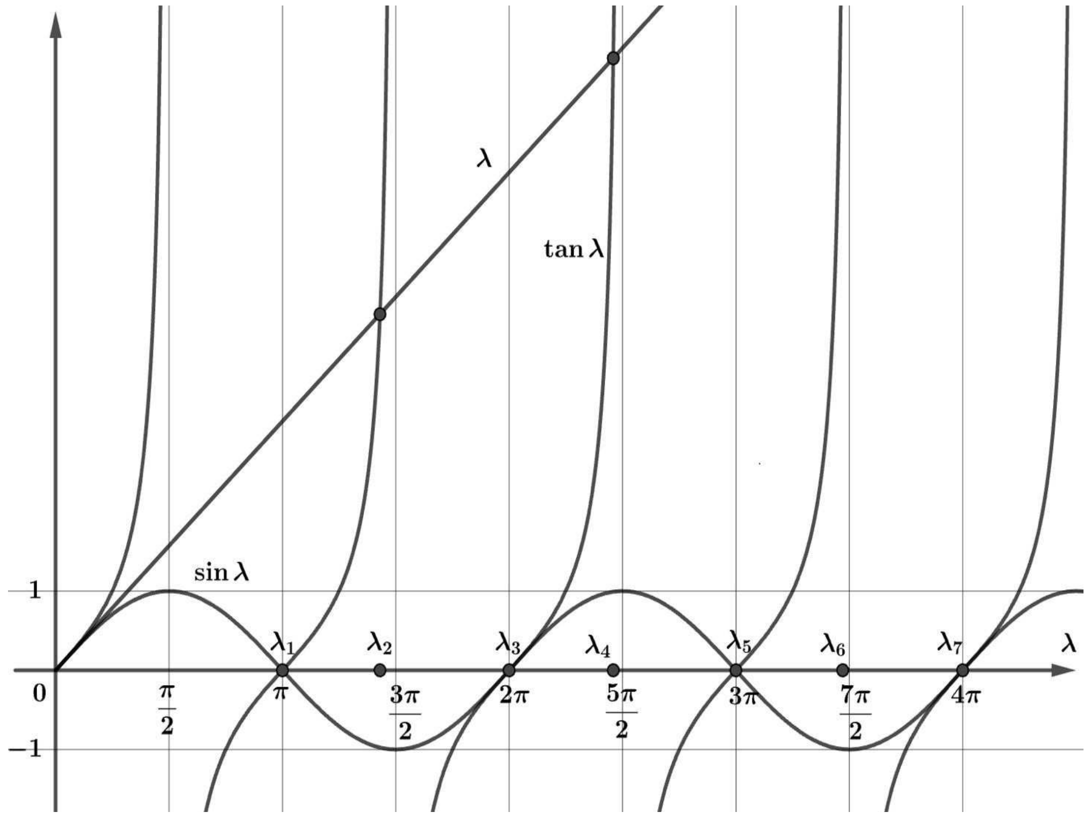

Figure 1.1. The positive roots of the equations (for $l = 2$):
$\tan \nu_n = \nu_n, \;\; \nu_n = \frac{\lambda_n l}{2} = \lambda_n; \;\; \sin \lambda_n = 0, \;\; n \in \mathbb{N}$.

From Figure 1.1 we have:

$$0 < \lambda_1 = \pi < \lambda_2 = \frac{3\pi}{2} - \varepsilon_1 < \lambda_3 = 2\pi < \lambda_4 = \frac{5\pi}{2} - \varepsilon_2$$

$$< \lambda_5 = 3\pi < \lambda_6 = \frac{7\pi}{2} - \varepsilon_3 < \lambda_7 = 4\pi < ...$$

Next, from Theorem 1, we obtain:

Corollary 1 *The eigenvalues* $\{\lambda_{2n}, \; n \in \mathbb{N}\}$ *are ordered as follows:*

$$0 < \lambda_{2n} = \frac{2\nu_n}{l} < \frac{(2n+1)\pi}{2}, \;\; \forall\, n \in \mathbb{N},$$

$$\lambda_{2n} = \frac{2\nu_n}{l} \to \frac{(2n+1)\pi}{2}, \;\; n \to \infty,$$

where $\{\nu_n, \; n \in \mathbb{N}\}$ *are the positive roots of the equation* $\tan \nu = \nu$.

2 Proof of Theorem 1

We will use the method of separation of variables. Substituting the expression $u_n(x) = X_{1n}(x_1)X_{2n}(x_2)...X_{dn}(x_d)$ into the relations (10)–(11) for each $n \in \mathbb{N}$, we obtain:

$$\begin{cases} X_{kn}^{IV}(x_k) + \lambda_n^2 X_{kn}^{II}(x_k) - \alpha_{kn}\mu_n X_{kn}(x_k) = 0, \;\; x_k \in (0, l), \\ X_{1n}(0) = X_{1n}(l) = X_{1n}^{I}(0) = X_{1n}^{I}(l) = 0, \end{cases} \tag{17}$$

where $k=1,..,d$, and $\{\alpha_{kn},\ k=1,...d\}$ are arbitrarily chosen numbers for each $n\in\mathbb{N}$ from the set $\{\alpha_{kn}\in\mathbb{R}^1\setminus\{0\},\ \sum\limits_{k=1}^{d}\alpha_{kn}=0\}$; moreover, $\mu_n\in\mathbb{C},\ n\in\mathbb{N}$, are (in the general case) unknown complex numbers.

Firstly, note that due to the positivity of the numbers λ_n^2 (as shown earlier in Proposition 1), the parameter μ_n can only take real values. Let us separately consider the following cases: (a) $\mu_n\neq 0$, (b) $\mu_n=0$.

(a) $\mu_n\neq 0$. The general solutions of the equations from (17) have the form

$$X_{kn}(x_k)=A_{kn}\sinh\theta_{(2k-1)n}x_k+B_{kn}\cosh\theta_{(2k-1)n}x_k+C_{kn}\sin\theta_{2kn}x_k+D_{kn}\cos\theta_{2kn}x_k, \quad (18)$$

where $\{A_{kn},B_{kn},C_{kn},D_{kn},\ k=1,...,d\}$ are constant values, and the constants $\{\theta_{kn},\ k=1,...,d\}$ must satisfy the equations:

$$2\theta_{(2k-1)n}\theta_{2kn}\left[1-\cosh\theta_{(2k-1)n}l\cdot\cos\theta_{2kn}l\right]=\left(\theta_{2kn}^2-\theta_{(2k-1)n}^2\right)\sinh\theta_{(2k-1)n}l\cdot\sin\theta_{2kn}l, \quad (19)$$

where $k=1,...,d$, and they ensure the fulfillment of the boundary conditions from (17).

In terms of the original constants λ_n^2 and $\sigma_{kn}=\alpha_{kn}\mu_n,\ k=1,...,d$, the equations (19) take the following form:

$$\pm 4i\sqrt{\sigma_{kn}}\left[1-\cosh\left(l\sqrt{\frac{-\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}\right)\cdot\cos\left(l\sqrt{\frac{\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}\right)\right]$$

$$=\lambda_n^2\sinh\left(l\sqrt{\frac{-\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}\right)\cdot\sin\left(l\sqrt{\frac{\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}\right),\quad k=1,...,d, \quad (20)$$

where

$$\theta_{(2k-1)n}^2\theta_{2kn}^2=\sigma_{kn},\ \ \theta_{2kn}^2-\theta_{(2k-1)n}^2=\lambda_n^2,\ \ k=1,...,d.$$

(a1). Let $\sigma_{kn}>0$ for some fixed index k. If $\mu_n\neq 0$, then such an index k always exists! In this case, the relation (20) is equivalent to the equation:

$$\pm i4\sqrt{\sigma_{kn}}\left[1-\cosh\xi_{kn}\cos\eta_{kn}\right]=\lambda_n^2\sinh\xi_{kn}\sin\eta_{kn},\ \ \xi_{kn}\neq\eta_{kn},\ \ \xi_{kn},\ \eta_{kn}\in\mathbb{R}_+^1,$$

which cannot be satisfied, where the following notations are introduced:

$$\xi_{kn}=l\sqrt{\frac{-\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}},\ \ \eta_{kn}=l\sqrt{\frac{\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}.$$

Thus, the remaining case is when $\mu_n=0$, i.e. $\sigma_{kn}=0,\ k=1,...,d$.

(b). Let $\mu_n=0$. In this case, the boundary value problems (17) take the following form:

$$\begin{cases} X_{kn}^{IV}(x_k)+\lambda_n^2X_{kn}^{II}(x_k)=0,\ \ x_k\in(0,l), \\ X_{kn}(0)=X_{kn}(l)=X_{kn}^{I}(0)=X_{kn}^{I}(l)=0, \end{cases}\quad k=1,...,d. \quad (21)$$

The general solutions of the equations from (21) are the following functions:

$$X_{kn}(x_k) = A_{kn} + B_{kn}x_k + C_{kn}\sin\lambda_n x_k + D_{kn}\cos\lambda_n x_k, \tag{22}$$

where the roots of the characteristic equations for (22) are respectively given by:

$$\theta_{kn1} = 0,\ \theta_{kn2} = 0,\ \theta_{kn3} = i\lambda_n,\ \theta_{kn4} = -i\lambda_n,\ k = 1, ..., d.$$

Moreover, the constant λ_n is a solution of the equation:

$$\lambda_n\left\{4\sin^4\frac{\lambda_n l}{2} - [\lambda_n l - \sin\lambda_n l]\sin\lambda_n l\right\} = 0. \tag{23}$$

The equation (23) is equivalent to the following relations:

$$\lambda_n \neq 0, \quad \begin{cases} \sin\dfrac{\lambda_{2n-1}l}{2} = 0, & \lambda_{2n-1}^2 = \left(\dfrac{2\pi n}{l}\right)^2, \\ \tan\dfrac{\lambda_{2n}l}{2} = \dfrac{\lambda_{2n}l}{2}, & \lambda_{2n}^2 = \left(\dfrac{2\nu_n}{l}\right)^2, \end{cases} \quad n \in \mathbb{N}, \tag{24}$$

where $\{\nu_n,\ n \in \mathbb{N}\}$ are the positive roots of the equation $\tan\nu = \nu$.

By ensuring the fulfillment of the boundary conditions from (21) for the solutions (22) with the constants $A_{kn}, B_{kn}, C_{kn}, D_{kn},\ k = 1, ..., d$, we establish the statement of Theorem 1.

Conclusion

The paper solves the generalized spectral problem for a fourth-order differential operator in a domain Ω, which has dimension $\dim\{\Omega\} = d \geq 2$. In the future, it is assumed that the eigenfunctions of the generalized spectral problem will be used to construct a fundamental system in the space of solenoidal functions. It is worth noting that in the works [23] and [24], a solution to the spectral problem (3) for the biharmonic operator in the domain Ω, represented by a 3-D sphere, was found.

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