IRSTI 22.07.01 DOI: https://doi.org/10.26577/JMMCS2025125103

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ON THE LAPLACE-BELTRAMI OPERATOR IN STRATIFIED SETS COMPOSED OF PUNCTURED CIRCLES AND SEGMENTS

This paper discusses the introduction of local coordinates on the circle S^1 and the analysis of various classes of functions defined on it. It is proved that every smooth function on the circle corresponds to a smooth 2π -periodic function on the real axis. The Laplace-Beltrami operator on S^1 is introduced using the apparatus of exterior differential forms and the Hodge operator. Its explicit expression in local coordinates is calculated, and it is shown that it can be reduced to the double differentiation operator. Then, the spectral analysis of the Laplace-Beltrami operator is performed, its eigenvalues and the corresponding eigenfunctions expressed in terms of the Chebyshev polynomials of the first and second kind are found. Well-solved problems for the Laplace-Beltrami operator on a punctured circle are written out. In the final paragraph of the article "On the Laplace-Beltrami operator on stratified sets composed of punctured circles and segments"the eigenvalues and systems of eigenfunctions on one stratified set composed of two punctured circles and a finite interval are written out.
Key words: Laplace-Beltrami operator, one-dimensional punctured sphere, well-posed problems.

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Ойылған шеңберлер мен кесінділерден тұратын қабатты жиындардағы Лаплас-Бельтрами операторы туралы

Бұл мақалада S^1 шеңберіне локальді координаттарды енгізу және онда анықталған функциялардың әртүрлі кластарын талдау қарастырылады. Шеңбердегі әрбір тегіс функция нақты осьтегі тегіс 2π -периодтық функцияға сәйкес келетіні дәлелденді. S^1 бойындағы Лаплас-Бельтрами операторы сыртқы дифференциалдық формалар аппараты мен Ходж операторы арқылы еңгізілген. Оның локальді координаттардағы айқын өрнегі есептеліп, оны екі еселі дифференциалдау операторына келтіруге болатыны көрсетілген. Одан әрі Лаплас-Бельтрами операторының спектрлік талдауы жүргізіліп, оның меншікті мәндері мен бірінші және екінші Чебышев көпмүшелері арқылы өрнектелген сәйкес меншікті функциялар табылады. Ойылған шеңбер бойынша Лаплас-Бельтрами операторына қисынды шешілетін есептер жазылған. "Ойылған шеңберлер мен кесінділерден тұратын қабатты жиындардағы Лаплас-Бельтрами операторы туралы"мақаланың соңғы бөлімінде екі ойылған шеңберден және ақырлы интервалдан тұратын бір қабатты жиындағы меншікті мәндер мен меншікті функциялардың жүйелері жазылған.
Түйін сөздер: Лаплас-Бельтрами операторы, бір өлшемді ауытқыған сфера, қисынды шешілетін есеп.

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Об операторе Лапласа-Бельтрами на стратифицированных множествах, составленных из проколотых окружностей и отрезков

В данной работе рассматривается введение локальных координат на окружности S^1 и анализ различных классов функций, определенных на ней. Доказывается, что каждая гладкая функция на окружности соответствует гладкой 2π -периодической функции на числовой оси. Вводится оператор Лапласа-Бельтрами на S^1 с использованием аппарата внешних дифференциальных форм оператора Ходжа. Вычисляется его явное выражение в локальных координатах, показывается, что он сводится к оператору двухкратного дифференцирования. Далее проводится спектральный анализ оператора Лапласа-Бельтрами, находятся его собственные значения и соответствующие собственные функции, выраженные через полиномы Чебышева первого и второго родов. Выписаны корректно разрешимые задачи для оператора Лапласа-Бельтрами на проколотой окружности. В заключительном параграфе статьи "Об операторе Лапласа-Бельтрами на стратифицированных множествах, составленных из проколотых окружностей и отрезков"выписаны собственные значения и системы собственных функции на одном стратифицированном множестве, составленных из двух проколотых окружностях и конечного интервала.
Ключевые слова: Оператор Лапласа-Бельтрами, возмущенная одномерная сфера, корректно разрешимые задачи.

1 Introduction

The circle S^1 is one of the simplest examples of manifolds studied in differential geometry and analysis. Despite its simplicity, it plays a key role in many areas of mathematics and physics, including spectral theory, harmonic analysis, and quantum mechanics.

One of the fundamental objects of study on manifolds is the Laplace-Beltrami operator, which generalizes the classical Laplace operator on Euclidean spaces. In the case of a circle, it is closely related to the theory of trigonometric series and the analysis of periodic functions.

In this paper, we consider local coordinates on S^1 and classes of functions defined on it. We define the Laplace-Beltrami operator and study its spectral structure.

The eigenfunctions of this operator form an orthonormal basis in the space of square integrable functions, which makes them an important tool for expanding functions in Fourier series. This fact has wide applications, from solving equations of mathematical physics to signal analysis in applied sciences.

Additional interest in the spectral properties of the Laplace-Beltrami operator on the circle is due to their connection with quantum mechanics and statistical physics. In particular, similar spectral problems arise in the study of string vibrations, heat conduction, and wave propagation. In addition, the circle serves as a model object for studying more complex manifolds with symmetries.

The goal of this paper is to conduct a detailed study of the circle S^1 as a differential manifold, describe its local coordinates, consider the main classes of functions defined on it, and study the spectral properties of the Laplace-Beltrami operator. The results obtained will allow a better understanding of the role of the circle in spectral geometry and its connections with various sections of analysis and mathematical physics.

2 Circle as a manifold of dimension one

In the two-dimensional space $R^2_{x^1x^2}$, consider the circle

$$S^1 = \left\{(x^1, x^2) \mid (x^1)^2 + (x^2)^2 = 1\right\}.$$

We introduce the local coordinates of the circle. It is not possible to introduce universal coordinates in the entire circle S^1, so the circle S^1 is represented as a union of two maps V_1 and V_2. Each map can define its own individual coordinates. For example, $V_1 = S^1 \backslash \{(1,0)\}$ is a punctured circle with coordinate t :

$$x^1 = \cos t, x^2 = \sin t,$$

where t runs through the interval $(0, 2\pi)$. On the map $V_2 = S^1 \backslash \{(-1,0)\}$ we enter the coordinate τ :

$$x^1 = \cos \tau, x^2 = \sin \tau,$$

where τ runs through the interval $(\pi, 3\pi)$. Note that the maps V_1 and V_2 intersect and their intersection

$$V_1 \cap V_2 = S^1_+ \cup S^1_-,$$

where S^1_+ and S^1_- are semicircles without intersections.

In S^1_+ the transition from t to τ is carried out by formula $\tau = t + 2\pi$, and in S^1_- the coordinates of t and τ coincide. Therefore, the maps V_1 and V_2 are consistent maps and define an atlas on S^1.

3 Function classes on S^1

Now we define the function classes $C^\infty(S^1)$. If the function $f \in C(S^1)$ is given, then $f = f(x^1, x^2)$. Then we define the restriction of f to S^1_+, which we denote by $f|_{S^1_+} = f_1(x^1, x^2)$. We can define the restriction of f to S^1_- in exactly the same way, that is, $f|_{S^1_-} = f_2(x^1, x^2)$. We denote by $f_1(\cos t, \sin t) = \hat{f}_1(t) \quad 0 < t < \pi$. The function $f_1(x^1, x^2)$ on S^1_+ can be represented as a function of t:

$$f_1(\cos t, \sin t) = \hat{f}_1(t), \quad 0 < t < \pi.$$

The same function $f_1(x^1, x^2)$ in S^1_+ can be written in terms of τ coordinates:

$$f_1(\cos \tau, \sin \tau) = \hat{\hat{f}}_1(\tau), 2\pi < \tau < 3\pi.$$

It is clear that

$$\hat{\hat{f}}_1(t + 2\pi) = \hat{f}_1(t).$$

If the restriction of $f(x^1, x^2)$ to S^1_- is denoted by $\hat{f}_2(t) = \hat{\hat{f}}_2(\tau)$, where $\tau = t$. Since $f(x^1, x^2) \in C(S^1)$, then the function $f(x^1, x^2)$ is continuous at the point $(1, 0)$. That is,

$$\lim_{(x^1,x^2)\to(1,0)} f(x^1, x^2) = f(1,0) \text{ or } \lim_{t\to 0} \hat{f}_1(t) = f(1,0)$$

or $\lim_{\tau\to 2\pi}\hat{f}_1(\tau) = f(1,0) = \hat{\hat{f}}(2\pi) \Rightarrow$

$$\Rightarrow \lim_{t\to 0}\hat{f}_1(t) = \hat{f}_1(2\pi).$$

Therefore, $\hat{f}_1(0) := \hat{f}_1(2\pi)$. Similarly, $f(x^1,x^2)$ is continuous at $(-1,0)$. That is, $\lim_{(x^1,x^2)\to(-1,0)} f(x^1,x^2) = f(-1,0)$ or

$$\lim_{t\to\pi}\hat{f}_1(t) = f(-1,0) = \lim_{\tau\to 3\pi}\hat{f}_1(\tau) = \hat{f}_1(\pi).$$

Therefore, $\hat{f}_1(3\pi) = \hat{f}_1(\pi)$. In the same way, we can extend $\hat{\hat{f}}_1(\tau)$ to the point $\tau = \pi$. For example,

$$\lim_{\substack{\tau\to\pi\\ \tau>\pi}}\hat{f}_1(\tau) = f(-1,0) = \hat{f}_1(\pi) \quad \text{or}$$
$$\hat{\hat{f}}_1(\pi+0) = \hat{f}_1(\pi) = \hat{f}_1(3\pi-0)$$

That is, $\hat{f}_1(\tau)$ can be defined at the points $\tau = \pi$ and $\tau = 3\pi$. In other words, $\hat{\hat{f}}_1(t)$ is defined by $\tau \in [\pi, 3\pi]$, and $\hat{\hat{f}}_1(\pi+0) = \hat{\hat{f}}_1(3\pi-0)$. Thus, the function $\hat{\hat{f}}_1(\tau)$ has a continuous 2π-periodic extension to the entire axis. In the same way, $\hat{f}_1(t)$ has a 2π-periodic continuous extension to the entire number axis. More of these extensions give the same periodic function

$$\hat{f}_1(t) = \hat{\hat{f}}_1(t), \quad \forall t \in R$$

Remark 1: if $f \in \mathbb{C}^\infty(S^1)$, then restriction $f|_{V_1} = \hat{f}(t)$ can be extended to the entire number line and the extension has the following properties:
1) 2π is periodic;
2) infinitely many times continuously differentiable.

Now consider an arbitrary continuous 2π - periodic function on the entire number line.

$$\hat{f}(t) = \hat{f}(t+2\pi), \forall t \in R$$

Let $(x^1,x^2) \in S^1\backslash\{1,0\} = V_1$. Find a unique $t \in (0,2\pi)$ such that $x = \cos t, x^2 = \sin t$. Since $\hat{f} - 2\pi$ is a continuous periodic function, then $\hat{f}(t) = \frac{a_0}{2} + \sum(a_k\cos kt + b_k \sin kt) =$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(k\,\mathrm{arcsos}\,x^1) + b_k\sin(k\arccos x^1)\right)$$
$$= \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_kT_K(x^1) + b_kx^2U_k(x^1)\right) = f(x^1,x^2)$$

where $T_k(.)$ and $U_k(.)$ are Chebyshev polynomials of the first and second kind.

Remark 2: Thus, for each smooth $2\pi-$ periodic function $\hat{f}(t)$, we can uniquely construct a smooth function $f(x^1,x^2)$, defined on the circle S^1.

4 Laplace-Beltrami operator on the circle

We denote by $C^{\infty}\left(S^{1}\right)$ the set of infinitely continuously differentiable functions on S^1. By $\Lambda^1(S^1)$ we denote the exterior differential forms of the first order in the circle S^1, that is

$$\omega^1(x)=\alpha_1(x)dx^1+\alpha_2(x)dx^2, \quad x=\left(x^1,x^2\right)\in S^1,$$

where $\alpha_1,\alpha_2\in C^{\infty}\left(S^1\right)$.

By $\Lambda^2\left(S^1\right)$ we denote the exterior differential forms of the second order in the circle S^1, that is

$$w^2(x)=\alpha(x)dx^1\wedge dx^2,$$

where $\alpha\in C^{\infty}\left(S^1\right)$. Let us recall how the Hodge operator acts on the basis elements:

$$*dx^1=dx^2, *dx^2=-dx^1, *dx^1\wedge dx^2=1.$$

Take a scalar function $h\in C^{\infty}(S^1)$. Calculate its differential $dh(x)=\frac{\partial h}{\partial x^1}dx^1+\frac{\partial h}{\partial x^2}dx^2$. Now apply the Hodge operator

$$*dh(x)=\frac{\partial h}{\partial x^1}dx^2-\frac{\partial h}{\partial x^2}dx^1.$$

Let's calculate the differential of 1-form

$$d*dh(x)=\frac{\partial}{\partial x^1}\left(\frac{\partial h}{\partial x^1}dx^1+\frac{\partial h}{\partial x^2}dx^2\right)\wedge dx^2-\frac{\partial}{\partial x^2}\left(\frac{\partial h}{\partial x^1}dx^1+\frac{\partial h}{\partial x^2}dx^2\right)\wedge dx^1=$$

$$=\left(\frac{\partial^2 h}{\left(\partial x^1\right)^2}+\frac{\partial^2 h}{\left(\partial x^2\right)^2}\right)dx^1\wedge dx^2$$

It remains to apply the Hodge operator, as a result we obtain the Laplace-Beltrami operator on the circle

$$\Delta h\equiv *d*dh(x)=\frac{\partial^2 h}{\left(\partial x^1\right)^2}+\frac{\partial^2 h}{\left(\partial x^2\right)^2}$$

The Laplace-Beltrami operator has physical and geometric meanings, therefore, this definition of the Laplace-Beltrami operator has an invariant description. Indeed, the Laplace-Beltrami operator is defined through exterior differential forms and operations on forms that are invariant with respect to the choice of local coordinates.

Now we calculate the Laplace-Beltrami operator in local coordinates. Let $x=(x^1,x^2)$ belong to the map V_1, that is, $x^1=\cos t, x^2=\sin t, t\in(0,2\pi)$. According to the results of point 2, the scalar function $h(x)\in C^{\infty}\left(S^1\right)$ in local coordinates has the form $h(x)=\hat{h}(\cos t,\sin t)=\hat{h}(t)$ for $t\in(0,2\pi)$. Moreover, $\hat{h}(t)-2\pi$-periodically extends to the entire real axis, and the extension $\hat{h}(t)$ is an infinitely differentiable function. Calculate the differential

$$dh(x)=d\hat{h}(t)=\frac{d\hat{h}(t)}{dt}dt$$

Apply the Hodge operator, then $*dh(x)=*d\hat{h}(t)=\frac{d\hat{h}}{dt}$. Hence $\Delta h=*d*dh(x)=*d*d\hat{h}(t)=\frac{d^2\hat{h}(t)}{dt^2}$. Therefore, the Laplace-Beltrami operator on the real axis represents the double differentiation operator.

5 Spectral analysis of the Laplace-Beltrami operator on the circle

In this section, we calculate the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on the circle. In the previous paragraph, the Laplace-Beltrami operator is defined by the formula

$$\Delta = *d * d$$

where d is the exterior differentiation operator, and * is the Hodge operator.

Since the Laplace-Beltrami operator is defined in invariant form, its eigenvalues do not depend on the choice of local coordinates on the circle.

The eigenvalues of the Laplace-Beltrami operator are determined from the equation.

$$-\Delta u(x) = \lambda u(x), x \in S^1. \qquad (1)$$

In this case, the complex number λ will be an eigenvalue of the Laplace-Beltrami operator if equation (1) has a non-zero solution for the corresponding λ.

To find the eigenvalues λ of the Laplace-Beltrami operator, equation (1) can be considered in local coordinates, since the eigenvalues are invariant with respect to local coordinates. Therefore, we write equation (1) on the local map V_1.

The role of the local coordinate in V_1 is played by the variable t, which was introduced in point 1. Then, according to the results of point 3, equation (1) takes the form

$$-\hat{u}''(t) = \lambda \hat{u}(t), \quad t \in \mathbb{R}, \qquad (2)$$

where $\hat{u}(t) - 2\pi$ is a periodic function on $\mathbb{R}$.

Thus, we need to find λ, for which equation (2) has non-trivial 2π periodic solutions.

The solution to this problem is known [1]:

the numbers $\lambda = 0, 1, 4, 9, \dots$ are eigenvalues, and the corresponding eigenfunctions take the form

$$\widehat{u}_0(t) \equiv 1, \hat{u}_{+\sqrt{\lambda}}(t) = \cos\sqrt{\lambda}t, \widehat{u}_{-\sqrt{\lambda}}(t) = \sin\sqrt{\lambda}t$$

Thus, $\lambda = 0$ is a simple eigenvalue, and all non-zero eigenvalues have multiplicity equal to two.

Now we rewrite the eigenfunctions $\widehat{u}_{\pm\sqrt{\lambda}}(t)$ in the variables $x = (x^1, x^2) \in S^1$. To do this, we need to use Chebyshev polynomials.

$$u_{+\sqrt{\lambda}}(x) = T_{\sqrt{\lambda}}(x^1), \quad u_{-\sqrt{\lambda}}(x) = x^2 \cdot U_{\sqrt{\lambda}}(x^1))$$

where $T_{\sqrt{\lambda}}(x^1)$ and $U_{\sqrt{\lambda}}(x^1)$ are Chebyshev polynomials of genus 1 and 2.

Thus, the system of eigenfunctions of the Laplace-Beltrami operator on the circle has the form

$$\left\{1, T_k(x^1), x^2 U_k(x^1), k = 1, 2, 3, \dots\right\}, \quad \text{where} \quad x = (x^1, x^2) \in S^1.$$

6 Inverse operator to the Laplace-Beltrami operator on the circle

It follows from the results of section 4 that the equation $(I-\Delta)u(x)=f(x)$ for $x\in S^1$ has a unique solution for any $f\in L_2\left(S^1\right)$, and

$$\begin{aligned}\hat{u}(t)=&-\int_0^t \hat{f}(t)\,\mathrm{sh}(t-\tau)d\tau-\\ &-\frac{\mathrm{ch}\,t}{2-2\,\mathrm{ch}\,2\pi}\cdot\int_0^{2\pi}\hat{f}(\tau)\begin{vmatrix}\mathrm{sh}\,2\pi & \mathrm{sh}(2\pi-\tau)\\ \mathrm{ch}\,2\pi-1 & \mathrm{ch}(2\pi-\tau)\end{vmatrix}d\tau+\\ &+\frac{\mathrm{sh}\,t}{2-2\,\mathrm{ch}\,2\pi}\int_0^{2\pi}\hat{f}(\tau)\begin{vmatrix}\mathrm{ch}\,2\pi-1 & \mathrm{sh}(2\pi-\tau)\\ \mathrm{sh}\,2\pi & \mathrm{ch}(2\pi-\tau)\end{vmatrix}d\tau,\end{aligned}\tag{3}$$

where $\hat{f}(t)=f\left(x^1,x^2\right)|_{V_1}, \quad \hat{u}(t)=u\left(x^1,x^2\right)|_{V_1}.$

In order to write out the formula for the solution $u(x^1,x^2)$, in the last formula we need to go from the local coordinate t to $(x^1,x^2)\in V_1$. For this we need the following auxiliary statement.

Lemma 1. For any smooth 2π - periodic function $\hat{F}(t)$ the integral identity holds

$$\int_0^t \hat{F}(\tau)d\tau=\int_{\gamma_x}F(\xi^1,\xi^2)(\xi^1 d\xi^2-\xi^2 d\xi^1),$$

where γ_x is a positively oriented arc of S^1 connecting the point $(1,0)$ with the point (x^1,x^2).

Proof of Lemma 1. The identity holds

$$\int_0^t \hat{F}(\tau)d\tau=\int_0^t \hat{F}(t)(\cos^2(t)+\sin^2(t))dt=$$

$$\begin{aligned}&=\int_0^t \hat{F}(t)(\cos t d\sin t-\sin t d\cos t)=\\ &=\int_{\gamma_x}F\left(x^1,x^2\right)\left(x^1dx^2-x^2dx^1\right).\end{aligned}$$

Lemma 1 is completely proved.

We expand the function $\mathrm{sh}\,t$ and $\mathrm{ch}\,t$ on $(0,2\pi)$ into trigonometric Fourier series, that is,

$$\begin{aligned}cht&=\frac{c_0}{2}+\sum_{k=1}^{\infty}\left(c_k\cos kt+d_k\sin kt\right),\\ sht&=\frac{s_0}{2}+\sum_{k=1}^{\infty}\left(s_k\cos kt+r_k\sin kt\right),\end{aligned}$$

where $\{c_k\},\{d_k\},\{s_k\},\{r_k\}$ are the corresponding Fourier coefficients in the trigonometric system.

We introduce two functions by the formulas

$$\mathbb{C}(x) = \mathbb{C}\left(x^1, x^2\right) = \frac{c_0}{2} + \sum_{k=1}^{infty} \left(c_k T_k\left(x^1\right) + d_k x^2\left(U_k\left(x^1\right)\right)\right),$$

$$\mathbb{S}(x) = \mathbb{S}\left(x^1, x^2\right) = \frac{s_0}{2} + \sum_{k=1}^{infty} \left(s_k T_k\left(x^1\right) + r_k x^2 U_k\left(x^1\right)\right).$$

On the circle S^1 we introduce the concept of convolution of two functions $f(x)$ and $g(x)$ for $x \in S^1$. We choose a fixed point $a = (\cos t_0, \sin t_0) \in S^1$. Then the convolution of two functions f and g at point a is defined by the integral

$$(f *_a g)(a) = \int_0^{t_0} \hat{f}(t)\hat{g}(t_0 - t)dt,$$

where $\hat{f}(t) = f(\cos t, \sin t)$, $\quad \hat{g}(t) = g(\cos t, \sin t)$.

Then from representation (3) taking into account Lemma 1 we have the relation

$$u(x) = -\left(f *_x \mathbb{S}\right)(x) -$$
$$-\frac{\operatorname{sh} 2\pi}{2 - 2\operatorname{ch} 2\pi}\begin{vmatrix} \mathbb{C}(x) & \left(f *_\eta \mathbb{S}\right)(\eta) \\ \mathbb{S}(x) & \left(f *_\eta \mathbb{C}\right)(\eta) \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \mathbb{S}(x) & \left(f *_\eta \mathbb{S}\right)(\eta) \\ \mathbb{C}(x) & \left(f *_\eta \mathbb{C}\right)(\eta) \end{vmatrix},$$

where $\eta = (1, 0) \in S^1$.

Thus, the inverse operator to the Laplace-Beltrami operator has the form

$$(I - \Delta)^{-1} f(x) = -\left(f *_x \mathbb{S}\right)(x) -$$
$$-\frac{\operatorname{sh} 2\pi}{2 - 2\operatorname{ch} 2\pi}\begin{vmatrix} \mathbb{C}(x) & \left(f *_\eta \mathbb{S}\right)(\eta) \\ \mathbb{S}(x) & \left(f *_\eta \mathbb{C}\right)(\eta) \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \mathbb{S}(x) & \left(f *_\eta \mathbb{S}\right)(\eta) \\ \mathbb{C}(x) & \left(f *_\eta \mathbb{C}\right)(\eta) \end{vmatrix}.$$

From this it is clear that the inverse operator is a linear integral operator. Denote by $G(x, \xi)$ the kernel of the inverse operator to the Laplace-Beltrami operator on the circle.

7 Well-solvable restrictions of the Laplace-Beltrami operator on a punctured circle

Choose an arbitrary point $x_0 \in S^1$. Denote by S_0^1 the punctured circle $S^1 \setminus \{x_0\}$. Consider the equation

$$(I - \Delta)w(x) = f(x), x \in S_0^1 \qquad (4)$$

Note that the inhomogeneous equation (4) for any right-hand side $f \in L_2(S^1)$ has infinitely many solutions. Indeed, let $u(x)$ be a solution to the inhomogeneous equation $(I - \Delta)w(x) = f(x)$ for $\forall x \in S^1$. In the previous paragraph it was proved that such a solution exists. Let us choose an arbitrary number $\alpha \in \mathbb{R}$ and consider the expression

$$w(x) = u(x) + \alpha G(x, x_0), \quad \text{for} \quad x \in S_0^1.$$

Since $x \neq x_0$, then

$$(I - \Delta)G(x, x_0) = 0.$$

Therefore, for any $\alpha \in \mathbb{R}$ the function $W(x) = u(x) + \alpha G(x, x_0)$ for $x \in S_0^1$ satisfies the inhomogeneous equation (4).

Thus, with respect to the inhomogeneous equation (4), the following question arises:

How many and what additional conditions should be added to the inhomogeneous equation (4) so that for $\forall f \in L_2(S^1)$ equation (4) has a unique solution.

To answer this question, we introduce two linear functionals. Let $x_0 = (\cos t_0, \sin t_0)$, where t_0 is a fixed number.

$$U_0(w) = \lim_{\delta \to +0} \left(\hat{w}(t_0 + \delta) - \hat{w}(t_0 - \delta) \right). \qquad (5)$$

The functional $U_0(\cdot)$ can be rewritten in another form. We will write $x'' < x_0 < x'$, if $x'', x_0, x' \in S^1$ and they preserve positive ordering on the circle S^1. Then

$$U_0(w) = \lim_{x'' < x_0 < x' x' \to x^0 x'' \to x^0} \left[w(x'') - w(x') \right].$$

We introduce another linear functional in the same way.

$$U_1(w) = \lim_{x'' < x_0 < x' x' \to x^0 x'' \to x^0} \left[\frac{\partial}{\partial \tau} w(x'') - \frac{\partial}{\partial \tau} w(x') \right],$$

where $\frac{\partial}{\partial \tau}$ is the derivative in the tangent direction.

Now we can formulate one of the main results of this article.

Theorem 1. For any function $f \in L_2(S^1)$ and any numbers γ_0 and γ_1, the inhomogeneous equation (4) is supplemented by the conditions

$$U_0(w) = \gamma_0, \quad U_1(w) = \gamma_1 \qquad (6)$$

has a unique solution.

The proof of Theorem 1 is proved by repeating the arguments given in the works [2, 3]. Theorem 1 can also be proved by simpler arguments.

Indeed, equation (4) is equivalent to the equation

$$\hat{w}(t) - \hat{w}''(t) = \hat{f}(t), \quad 0 \le t \le 2\pi, \quad t \ne t_0, \qquad (7)$$

with the boundary conditions

$$\begin{cases} \hat{w}(t_0 + 0) - \hat{w}(t_0 - 0) = \gamma_0, \\ \hat{w}'(t_0 + 0) - \hat{w}'(t_0 - 0) = \gamma_1 \end{cases} \qquad (8)$$

and the periodicity conditions

$$\begin{cases} \hat{w}(0) = \hat{w}(2\pi), \\ \hat{w}'(0) = \hat{w}'(2\pi). \end{cases} \qquad (9)$$

Problems (7)-(9) have a unique solution, since $ch 2\pi \ne 1$. Thus, Theorem 1 is completely proven.

From Theorem 1 it follows that

Corollary 1 Let $\gamma_0(\cdot), \gamma_1(\cdot)$ be arbitrary linear continuous functionals on $L_2(S^1)$. Then for any function $f \in L_2(S^1)$ the following boundary value problem applies.

$$\begin{cases} (I-\Delta)w(x) = f(x), \quad x \in S_0^1, \\ U_0(w) = \gamma_0((I-\Delta)w), \qquad (10) \\ U_1(w) = \gamma_1((I-\Delta)w) \end{cases}$$

has a unique solution.

By choosing the linear functionals $\gamma_0(\cdot)$ and $\gamma_1(\cdot)$ in a special way, we can refine Corollary 1 as follows.

We choose two functions $\alpha_0(x)$ and $\alpha_1(x)$ for $x \in S^1$ such that $(I-\Delta)\alpha_j(x) = 0, \quad \forall x \in S^1, \quad j = 0, 1$. Let the linear functionals γ_0 and γ_1 be chosen according to the Riesz theorem in the form $\gamma_j(f) = \int_{S^1} f(x)\overline{\alpha_j(x)}dS_x^1, j = 0, 1$. Then the boundary conditions (10) can be rewritten as

$$\begin{cases} U_0(w) + \overline{\hat{\alpha}_0'}(t_0)U_0(w) - \overline{\hat{\alpha}_0}(t_0)U_1(w) = A_0 w(0) + B_0 w'(0), \\ U_1(w) + \overline{\hat{\alpha}_1'}(t_0)U_0(w) - \overline{\hat{\alpha}_1}(t_0)U_1(w) = A_1 w(0) + B_1 w'(0), \end{cases}$$

where $A_0 = \hat{\alpha_0}'(2\pi) - \hat{\alpha_0}'(0), \quad B_0 = \overline{\hat{\alpha_1}'(2\pi) - \hat{\alpha_1}'(0)}$.

Thus, the following statement is true.

Corollary 2 Let $\alpha_0(x)$ and $\alpha_1(x)$ for $x \in S^1$ be a solution of the equation $(I-\Delta)\alpha_j(x) = 0, x \in S^1, j = 0, 1$. Then for any function $f \in L_2(S^1)$ the following boundary value problem applies.

$$(I-\Delta)w(x) = f(x), \quad x \in S_0^1,$$

$$\begin{cases} U_0(w) + \overline{\hat{\alpha}_0'}(t_0)U_0(w) - \overline{\hat{\alpha}_0}(t_0)U_1(w) = A_0 w(0) + B_0 w'(0), \\ U_1(w) + \overline{\hat{\alpha}_1'}(t_0)U_0(w) - \overline{\hat{\alpha}_1}(t_0)U_1(w) = A_1 w(0) + B_1 w'(0), \end{cases}$$

has a unique solution.

Similar problems on punctured balls and punctured spheres can be found in [4]- [13].

8 Spectral analysis of the Laplace-Beltrami operator on one stratified set

In this section, we consider a stratified set consisting of two punctured circles and one segment connecting these circles. Consider the eigenvalue problem on the stratified set $\mathbb{S} = \{X^1, X^2, X^3, A, B\}$, where A and B are two points on X^1 and X^2, respectively. That is, consider the system of equations

$$\begin{cases} (I-\Delta_1)w_1(x_1) = \lambda w_1(x_1), x_1 \in X^1, x_1 = (x_1^1, x_1^2) \\ (I-\Delta_2)w_2(x_2) = \lambda w_2(x_2), x_2 \in X^2, \qquad (11) \\ w_3(x_3) - w"_3(x_3) = \lambda w_3(x_3), x_3 \in X^3 \equiv (0,1), \end{cases}$$

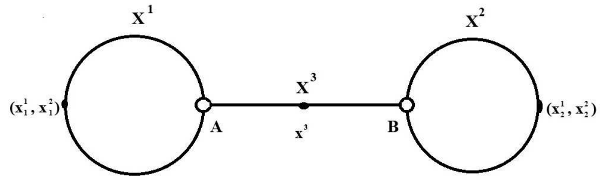

Рис. 1: Stratified set

with boundary conditions

$$\begin{cases} U_0^1(w_1) = w_3(0), \\ U_1^1(w_1) = w_3'(0), \\ U_0^2(w_2) = w_3(1), \\ U_1^2(w_2) = w_3'(1), \\ w_3(0) = 0, \\ w_3(1) = 0 \end{cases} \qquad (12)$$

Here, the functionals $U_0^j(\cdot)$ and $U_1^j(\cdot)$ for $j = 1, 2$ are defined the same way as the functionals U_0 and U_1 were defined in step 6 for the punctured circle S_0^1..

Now we calculate the eigenvalues and eigenfunctions of problem (11)-(12). The eigenvalues of problem (11)-(12) consist of eigenvalues of two types:

1. Each eigenvalue $\lambda_n(D)$ of the Dirichlet problem in the interval $(0,1) : y(t) - y"(t) = \lambda y(t), t \in (0,1), y(0) = 0, y(1) = 0$ are also eigenvalues of the original problem (11)-(12). If $\lambda_n(D)$ corresponds to an eigenfunction $y_n(t)$, then the eigenvector function $(w_{1n}(x_1), w_{2n}(x_2), w_{3n}(x_3))$ of problem (11)-(12) has the form

$$w_{3n}(x_3) = y_n(x_3), x_3 \in X^3,$$

for $j = 1, 2$ the function $w_{jn}(x_j)$ coincides with the solution of the problem

$$(I - \Delta_j)w_{jn}(x_j) = \lambda_n(D))w_{jn}(x_j), \quad U_0^j(w_{jn}) = \gamma_{0j}, \quad U_1^j(w_{jn}) = \gamma_{1j},$$

where $\gamma_{01} = y_n(0), \gamma_{21} = y_n'(0), \gamma_{02} = y_n(1), \gamma_{12} = y_n'(1)$.

2. Each eigenvalue $\lambda_m(S^1)$ problems

$$(I - \Delta_{S^1})u(x) = \lambda_m(S^1)u(x), x \in S^1$$

is also an eigenvalue of the original problem (11)-(12). If $\lambda_m(S^1)$ corresponds to an eigenfunction $u_m(x)$, then the eigenvector function $(w_{1m}(x_1), w_{2m}(x_2), w_{3m}(x_3))$ of problem (11)-(12) has the form

$$w_{1m}(x_1) = u_m(x_1),$$
$$w_{2m}(x_2) = \varepsilon u_m(x_2),$$
$$w_{3m}(x_3) \equiv 0,$$

where ε can be equal to 0 and/or 1.
The eigenvalue problem studied in this section is analogous to spectral problems on graphs [14, 15].

9 Conclusion

In this paper, a detailed characterization of the circle S^1 as a differential manifold was carried out, local coordinates and classes of functions defined on it were considered. Particular attention was paid to the Laplace-Beltrami operator, its spectral properties and connection with harmonic analysis.

The results obtained confirm the fundamental role of the circle in spectral theory and analysis of periodic functions. The study of the spectrum of the Laplace-Beltrami operator demonstrates its connection with trigonometric functions, which is the basis for many applications in mathematical physics and signal theory.

Thus, the circle S^1 remains an important object of mathematical analysis, and further study of its properties can lead to new results in related areas, such as geometric analysis and representation theory.

10 Acknowledgements

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22685565 and AP19678089).

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Received: January 31, 2025
Accepted: February 23, 2025