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LIMITING ERROR OF THE OPTIMAL COMPUTING UNIT FOR FUNCTIONS FROM THE CLASS $W_2^{r;\alpha}$

In the problem of optimal recovery of an infinite object (functions on a continuum, integrals of continuous functions, solutions of partial differential equations, derivative of functions,...) from finite numerical information about it, the problem of finding the limiting error of the optimal computing unit naturally arises, since the numerical information about the infinite object to be restored , as a rule, will not be accurate. In this article, the limiting error of the optimal computing unit is found in the problem of optimal recovery of periodic functions of many variables from the anisotropic Sobolev class $W_2^{r;\alpha}$ in a power-logarithmic scale in the space metric L^2. The actuality of this work is determined by the following factors: firstly, the found limiting error $\overline{\varepsilon}_N$ of the optimal computing unit preserves the exact order of the smallest recovery error , when replacing exact numerical information about a function $f \in W_2^{r;\alpha}$ with inaccurate information and is unimprovable in order; secondly, the problem of finding the limiting error of an optimal computing unit has not previously been studied in the class under consideration; thirdly, the anisotropic Sobolev class in the power-logarithmic scale is a finer scale of classification of periodic functions according to the rate of decrease of their trigonometric Fourier coefficients than the anisotropic Sobolev class in the power scale.
Key words: optimal recovery, optimal computing unit, linear functionals, exact order, anisotropic Sobolev class, trigonometric Fourier coefficients, limiting error

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$W_2^{r;\alpha}$ класы функциялары үшін оптималды есептеу агрегатының шектік қателігі

Ақырсыз объектіні (континуумда анықталған функцияны, үзіліссіз функциялар интегралдарын, дербес туындылы дифференциалдық теңдеулер шешімдерін, функция туындыларын,...) одан алынған саны ақырлы мәліметтер арқылы оптималды қалыптастыру есебінде табиғи түрде, қалыптастырылуға тиіс ақырсыз объекттен алынатын сандық мәліметтер әдетте дәл болмайтындықтан, оптималды есептеу агрегатының шектік қателігін табу есебі пайда болады. Бұл мақалада L^2 кеңістігі метрикасында дәреже – логарифмдік шкаладағы анизотропты Соболев $W_2^{r;\alpha}$ класына тиесілі көп айнымалылы периодты функцияларды оптималды қалыптастыру есебіндегі оптималды есептеу агрегатының шектік қателігі табылған. Осы жұмыстың өзектілігі оптималды есептеу агрегатының келесі факторлар арқылы қамтамасыз етіледі: біріншіден, оптималды есептеу агрегатының табылған $\overline{\varepsilon}_N$ шектік қателігі $f \in W_2^{r;\alpha}$ функциясынан алынған дәл сандық мәліметті дәл емес мәліметке ауыстырғанда да қалыптастырудың ең аз қателігінің дәл ретін сақтайды және реті бойынша жақсармайды; екіншіден, оптималды есептеу агрегатының шектік қателігін табу есебі осы күнге дейін қарастырып отырған класта зерттелмеген; үшіншіден, периодты функцияларды олардың тригонометриялық Фурье коэффициенттерінің кему жылдамдығы бойынша классификациялап сипаттауда логарифм-дәрежелік шкаладағы анизотропты Соболев класы дәрежелік шкаладағы анизотропты Соболев класымен салыстырғанда кең, әрі дәл сипаттама болып келеді.

Түйін сөздер: оптималды қалыптастыру, оптималды есептеу агрегаты, сызықтық функционалдар, дәл рет, анизотропты Соболев класы, тригонометриялық Фурье коэффициенттері, шектік қателік.

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Предельная погрешность оптимального вычислительного агрегата для функций из класса $W_2^{r;\alpha}$

В задаче оптимального восстановления бесконечного объекта (функции на континууме, интегралы от непрерывных функций, решения дифференциальных уравнений в частных производных, производной функций, ...) по конечной числовой информации о нем естественным образом возникает задача нахождения предельной погрешности оптимального вычислительного агрегата, поскольку числовая информация о подлежащем к восстановлению бесконечном объекте, как правило, не будет точной. В данной статье найдена предельная погрешность оптимального вычислительного агрегата в задаче оптимального восстановления периодических функций многих переменных из анизотропного класса Соболева $W_2^{r;\alpha}$ в степенно – логарифмической шкале в метрике пространства L^2. Актуальность настоящей работы обусловлена следующими факторами: во – первых, найденная предельная погрешность $\overline{\varepsilon}_N$ оптимального вычислительного агрегата сохраняет точный порядок наименьшей погрешности восстановления при замене точной числовой информации о функции $f \in W_2^{r;\alpha}$ на неточную и является неулучшаемой по порядку; во – вторых, ранее задача нахождения предельной погрешности оптимального вычислительного агрегата не изучалась на рассматриваемом классе; в – третьих, анизотропный класс Соболева в степенно – логарифмической шкале является более тонкой шкалой классификаций периодических функций по скорости убывания их тригонометрических коэффициентов Фурье, чем анизотропный класс Соболева в степенной шкале.

Ключевые слова: Оптимальное восстановление, оптимальный вычислительный агрегат, линейные функционалы, точный порядок, анизотропный класс Соболева, тригонометрические коэффициенты Фурье, предельная погрешность

1 Introduction

Using the notations of the articles [1] and [2], we present definitions of the computing unit, the exact order of error of the optimal recovery, the optimal computing unit and its limiting error. Let a natural number N, normalized spaces X and Y of numerical functions defined on sets Ω and Ω_1 respectively, a functional class $F \subset X$, an operator $T : F \mapsto Y$, a function

$$\varphi_N \equiv \varphi_N(z_1, \ldots, z_N; y) : \mathbb{C}^N \times \Omega_1 \to \mathbb{C},$$

which for each fixed $(z_1, \ldots, z_N)$ as a function of a variable y belongs to the space Y, are given. Further, the symbol $l^{(N)}$ will be used to denote a $N-$ dimensional vector $\left(l_N^{(1)}, \ldots, l_N^{(N)}\right)$ with functionals $l_N^{(1)} : F \to \mathbb{C}, \ldots, l_N^{(N)} : F \to \mathbb{C}$.

Definition 1. *For a given pair* $\left(l^{(N)}, \varphi_N\right)$, *a numerical function*

$$\varphi_N\left(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); y\right)$$

of a variable y is called a computing unit.

Every below, we will use $C(\alpha, \beta, \ldots)$ to denote positive quantities that depend only on the parameters indicated in brackets. For positive sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ the notation $a_n \underset{\alpha,\beta,\ldots}{\ll} b_n$ will mean the existence of some quantity $C(\alpha, \beta, \ldots) > 0$ such that

$a_n \leq C(\alpha, \beta, \ldots) b_n$ for all $n \in \mathbb{N}$. It should be taken into account that the values of $C(\alpha, \beta, \ldots) > 0$ in different expressions may be different. And the simultaneous fulfillment of the relations $a_n \underset{\alpha,\beta,\ldots}{\ll} b_n$ and $b_n \underset{\alpha,\beta,\ldots}{\ll} a_n$ is written as $a_n \underset{\alpha,\beta,\ldots}{\succ\prec} b_n$.

Further, for given F, Y, D_N and $T : F \mapsto Y$ we determine the quantity

$$\delta_N(D_N, T, F)_Y = \inf_{(l^{(N)}, \varphi_N) \in D_N} \delta_N((l^{(N)}, \varphi_N), T, F)_Y, \tag{1}$$

where D_N is a subset of the set of all pairs $(l^{(N)}, \varphi_N)$,

$$\delta_N((l^{(N)}, \varphi_N), T, F)_Y = \sup_{f \in F} \left\| (Tf)(\cdot) - \varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); \cdot) \right\|_Y.$$

Definition 2. *A positive sequence* $\{\psi_N\}_{N \geq 1}$ *such that*

$$\delta_N(D_N, T, F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \psi_N \tag{2}$$

is called the exact order of error of the optimal recovery of the operator $T : F \to Y$ *by computing units from* D_N *in the metric of the space* Y.

Definition 3.*A computing unit* $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right) \equiv \widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); \cdot\right)$ *such that*

$$\delta_N(D_N, T, F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N), T, F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \psi_N \tag{3}$$

is called optimal.

Thus optimal computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ realizes the exact order ψ_N.

Here we note that in the relations (2) and (3), instead of the parameters $\alpha, \beta, \ldots$ the parameters of the class F and the space Y are taken.

Calculations for each function f from class F the value $l_N^{(1)}(f), \ldots, l_N^{(N)}(f)$ of functionals $l_N^{(1)} : F \to \mathbb{C}, \ldots, l_N^{(N)} : F \to \mathbb{C}$, with rare exceptions, cannot be exact. Therefore, for the optimal computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$, the problem arises of finding the error $\widetilde{\varepsilon}_N$ in calculating the values $\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f)$ of the functionals $\tilde{l}_N^{(1)} : F \to \mathbb{C}, \ldots, \tilde{l}_N^{(N)} : F \to \mathbb{C}$, which preserves the optimality of $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ and is the limiting in order. In [1] the error $\widetilde{\varepsilon}_N$ was called the limiting error of the optimal computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$. Now we present definition of $\widetilde{\varepsilon}_N$, formulated in [2].

Definition 4. *A sequence* $\widetilde{\varepsilon}_N > 0$ *is called the limiting error of an optimal computing unit* $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$, *if*

$$\triangle_N\left(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), T, F\right)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \delta_N(D_N, T, F)_Y \quad \text{and} \tag{4}$$

$$\overline{\lim_{N\to\infty}} \frac{\triangle_N(\eta_N \widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), T, F)_Y}{\delta_N(D_N, T, F)_Y} = +\infty \tag{5}$$

for any positive sequence $\{\eta_N\}_{N\geq 1}$ *increasing arbitrarily slowly to* $+\infty$, *where*

$$\triangle_N(\varepsilon_N,(\tilde{l}^{(N)},\widetilde{\varphi}_N),T,F)_Y =$$

$$= \sup_{f\in F}\sup_{z_1,\ldots,z_N}\left\{\left\|(Tf)(\cdot)-\widetilde{\varphi}_N(z_1,\ldots,z_N;\cdot)\right\|_Y : \left|z_i-\tilde{l}_N^{(i)}(f)\right|\leq\varepsilon_N, i=1,\ldots,N\right\}\equiv$$

$$\equiv \sup_{f\in F}\sup_{|\gamma_N^{(1)}|\leq 1,\ldots,|\gamma_N^{(N)}|\leq 1}\left\|(Tf)(\cdot)-\widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f)+\gamma_N^{(1)}\varepsilon_N,\ldots,\tilde{l}_N^{(N)}(f)+\gamma_N^{(N)}\varepsilon_N;\cdot\right)\right\|_Y$$

for any positive sequence ε_N.

The relation (4) means that when calculating the values of the optimal computing unit $\widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f),\ldots,\tilde{l}_N^{(N)}(f);\cdot\right)$ each number $\tilde{l}_N^{(\tau)}(f)(\tau=1,\ldots,N=N(K))$ can be replaced with error $\widetilde{\varepsilon}_N$ by a number z_τ such that $|z_\tau-\tilde{l}_N^{(\tau)}(f)|\leq\varepsilon_N(\tau=1,\ldots,N=N(K))$, preserving the exact order of error of the optimal recovery.

According to equality (5), we can state that the error $\widetilde{\varepsilon}_N$ is of limiting error, because an arbitrarily slow infinite increase in the value of $\widetilde{\varepsilon}_N$ (i.e., replacement of $\widetilde{\varepsilon}_N$ by $\eta_N\widetilde{\varepsilon}_N$) violates the exact order of error of optimal recovery.

Many mathematicians have been and continue to be concerned with the problems of establishing the relation (1) and constructing optimal computing units for various F, Y, D_N and $T : F \mapsto Y$ (see, for example, [3-6] and the bibliography therein). The problem of finding limiting errors is a relatively new problem in approximation theory, computational mathematics and numerical analysis. Results on this problem can be found in the works [1], [2], [7] and [8]. In this article, when

$$Tf=f, F=W_2^{r;\alpha}[0,1]^s, Y=L^2[0,1]^s, D_N=L_N,$$

where $W_2^{r;\alpha}\equiv W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ is the anisotropic Sobolev class on a power –logarithmic scale (the definition of the class is given below), L_N is the set of computing units $(l^{(N)},\varphi_N)$ with linear functionals $l_N^{(1)}: W_2^{r;\alpha}\to\mathbb{C},\ldots,l_N^{(N)}: W_2^{r;\alpha}\to\mathbb{C}$, the limiting error of the optimal computing unit $\left(\bar{l}^{(N)},\overline{\varphi}_N\right)$ from [5] is found.

The importance of studying this work lies in the following: firstly, the anisotropic Sobolev class $W_2^{r;\alpha}\equiv W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ in the power-logarithmic scale is a finer scale of classifications of periodic functions in terms of the rate of decrease of their trigonometric Fourier coefficients than the usual anisotropic Sobolev class $W_2^{r_1,\ldots,r_s}[0,1]^s$ in the power scale; secondly, the recovery of functions from the class $W_2^{r;\alpha}[0,1]^s$ is carried out by computing units from a fairly wide set containing all partial sums of Fourier series over all possible orthonormal systems, all possible finite convolutions $\sum_{i=1}^{N} f(\xi_i)K_N(x-\xi_i)$ with special kernels K_N, and all finite sums of approximation used in orthowidths, linear widths, and greedy algorithms; thirdly, previously, the problem of finding the limiting error of the optimal computing unit was not considered on a multidimensional functional class $W_2^{r;\alpha}$; fourthly, in the problem of finding the limiting error of the optimal computing unit on optimal recovery of functions from the class $W_2^{r;\alpha}$, unlike the classes Sobolev SW_2^r with a dominant mixed derivative, Korobov E_s^r and Sobolev W_p^r, and the exact order and limiting error does not depend on the number variable functions $f(x)=f(x_1,\ldots,x_s)$ (see, for example, [9] and [10]).

2 Main result

First, let's agree on the notation used. Everywhere below, the symbols $[a]$ and $|E|$ will denote the integer part of the number a and the amount elements of the finite set E. For each vector $r = (r_1, \ldots, r_s)$ with positive components, we assume $\lambda = 1/(1/r_1 + \ldots + 1/r_s)$. Instead of symbols

$$\|\cdot\|_{L^2}, \underset{s,r_1,\ldots,r_s,\alpha_1,\ldots,\alpha_s}{\ll}, \underset{s,r_1,\ldots,r_s,\alpha_1,\ldots,\alpha_s}{\gg} \quad \text{and} \quad \underset{s,r_1,\ldots,r_s,\alpha_1,\ldots,\alpha_s}{\succ\prec}$$

we will use the symbols $\|\cdot\|_2, \ll, \gg$ and $\succ\prec$ respectively. The symbol $\square$ will mean the end of the proofs.

Now we give a definition of the anisotropic Sobolev class $W_2^{r;\alpha}$ on a power – logarithmic scale. Let an integer number $s \geq 2$, vectors $\mathrm{r} = (r_1, \ldots, r_s)$ and $\alpha = (\alpha_1, \ldots, \alpha_s)$ be given such that $r_i > 0$ and $\alpha_i \in \mathbb{R}$ for each $i = 2, 3, \ldots, s$. The class $W_2^{r;\alpha} \equiv W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ consist of all functions $f(x) = f(x_1, \ldots, x_s)$ that are summable on $[0,1]^s$ and 1 – periodic on each variable and whose trigonometric Fourier – Lebesgue coefficients $\hat{f}(m) = \int\limits_{[0,1]^s} f(x)e^{-2\pi i(m,x)}dx, m \in Z^s$ satisfy the condition

$$\sum_{m \in Z^s} |\hat{f}(m)|^2 (\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \ldots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1)) \leq 1,$$

where $\overline{m}_j = \max\{1; |m_j|\}$ for each $j = 1, \ldots, s$.

The main result of this article is the following

Theorem. *Let an integer number $s \geq 2$, vectors $\mathrm{r} = (r_1, \ldots, r_s), r_1 > 0, \ldots, r_s > 0$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathrm{R}^s$ be given such that $r_i + \alpha_i > 0$ for each $i = 2, 3, \ldots, s$, and let the inequality*

$$\left(\frac{1}{\min\{r_1, r_1 + \alpha_1\}} + \ldots + \frac{1}{\min\{r_s, r_s + \alpha_s\}}\right)^{-1} > \frac{1}{2} \tag{6}$$

hold. Then the quantity

$$\overline{\varepsilon}_N = \frac{1}{N^{\lambda+1/2}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}}$$

is limiting error of the optimal computing unit

$$\left(\overline{l}^{(N)}, \overline{\varphi}_N\right) \equiv \overline{\varphi}_N\left(\overline{l}_N^{(1)}(f), \ldots, \overline{l}_N^{(N)}(f); x\right) = \sum_{\tau=1}^{N} \hat{f}(\overline{m}^{(\tau)}) e^{2\pi i(\overline{m}^{(\tau)}, x)},$$

where $N \equiv N(K) = \prod\limits_{i=1}^{s}(2N_i + 1)$,
$N_i \equiv N_i(K) = [K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}], K \geq 2$ for each $i \in \{1, \ldots, s\}, \{\overline{m}^{(1)}, \overline{m}^{(2)}, \ldots, \overline{m}^{(N)}\}$ is some ordering of the set $A_K = \{m \in Z^s : |m_1| \leq N_1, \ldots, |m_s| \leq N_s\}$, in the problem of optimal recovery of functions from the class $W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ in the metric of the space $L^2[0,1]^s$.

In the case $\alpha_1 = \alpha_2 = \ldots = \alpha_s = 0$ from this theorem we obtain the following statement.

Corollary. *Let an integer number $s \geq 2$, vector $\mathrm{r} = (r_1, \ldots, r_s)$ be given such that $r_i > 0$ for each $i = 2, 3, \ldots, s$, and let the inequality $\left(\frac{1}{r_1} + \ldots + \frac{1}{r_s}\right)^{-1} > \frac{1}{2}$ hold. Then the quantity $\overline{\varepsilon}_N = \frac{1}{N^{\lambda+1/2}}$ is limiting error of the optimal computing unit*

$$\left(\bar{l}^{(N)}, \overline{\varphi}_N\right) \equiv \overline{\varphi}_N\left(\bar{l}_N^{(1)}(f), \ldots, \bar{l}_N^{(N)}(f); x\right) = \sum_{\tau=1}^{N} \hat{f}(\overline{m}^{(\tau)}) e^{2\pi i (\overline{m}^{(\tau)}, x)},$$

where $N \equiv N(K) = \prod_{i=1}^{s}(2N_i + 1), N_i \equiv N_i(K) = [K^{\lambda/r_i}], K \geq 2$ for each $i \in \{1, \ldots, s\}, \{\overline{m}^{(1)}, \overline{m}^{(2)}, \ldots, \overline{m}^{(N)}\}$ is some ordering of the set $A_K = \{m \in Z^s : |m_1| \leq N_1, \ldots, |m_s| \leq N_s\}$, in the problem of optimal recovery of functions from the class $W_2^{r_1,\ldots,r_s}[0,1]^s$ in the metric of the space $L^2[0,1]^s$.

3 Auxiliary statements

Lemma 1. *Let sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be given such that $\lim\limits_{n\to\infty} x_n = +\infty$ and $\lim\limits_{n\to\infty} y_n = +\infty$. Then for the sequence $z_n = \min\{x_n, y_n\}$ the equality $\lim\limits_{n\to\infty} z_n = +\infty$ holds.*
Proof. According to the equalities $\lim\limits_{n\to\infty} x_n = +\infty$ and $\lim\limits_{n\to\infty} y_n = +\infty$, for any positive number $\varepsilon > 0$ there is a number N_ε such that for all natural numbers $n \geq N_\varepsilon$ the inequalities $x_n > \varepsilon$ and $y_n > \varepsilon$ are satisfied. From these inequalities follows the inequality $\min\{x_n, y_n\} > \varepsilon$, which is true for each $n \geq N_\varepsilon$. Therefore, $\lim\limits_{n\to\infty} z_n = +\infty$. $\square$
Lemma 2. *For each $\gamma \in \mathrm{R}$ there exists a quantity $C_1(\gamma) \geq 2$ such that for all integers $K \geq C_1(\gamma)$ the relation*

$$\ln(K \ln^\gamma K) \underset{\gamma}{\succ\!\prec} \ln K \tag{7}$$

holds.
Proof. In case $\gamma \geq 0$ for all integers $K \geq 2$ the inequalities

$$\begin{gathered} K \leq K \ln^\gamma K \leq K^{\gamma+1} \Leftrightarrow \\ \Leftrightarrow \ln K \leq \ln(K \ln^\gamma K) \leq (\gamma+1) \ln K \end{gathered} \tag{8}$$

are satisfied.

Let $\gamma < 0$. Since $\lim\limits_{K\to\infty} \sqrt{K} \ln^\gamma K = +\infty$, there exists a number $K^{(0)} = K^{(0)}(\gamma)$ such that for all integers $K \geq K^{(0)}$ the inequalities

$$\frac{1}{2} \ln K \leq \ln(K \ln^\gamma K) \leq \ln K \tag{9}$$

hold. Therefore, taking $C_1(\gamma) = \begin{cases} 2, & \text{if} \quad \gamma \geq 0; \\ K^{(0)} + 1, & \text{if} \quad \gamma < 0, \end{cases}$ by virtue of the inequalities (8) and (9) for all integers $K \geq C_1(\gamma)$ we obtain (7). $\square$
Lemma 3. *For any function $f \in L^2$ the inequality*

$$\max\left\{\|f(\cdot) - \varphi_N(0, \ldots, 0; \cdot)\|_2, \|(-f)(\cdot) - \varphi_N(0, \ldots, 0; \cdot)\|_2\right\} \geq \|f\|_2$$

is satisfied.

Proof. Let us introduce the following notations:

$$a = \|f(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2 \quad \text{and} \quad b = \|(-f)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2.$$

Then, according to the inequalities $\max\{a,b\} \geq (a+b)/2$ and $\|x\|_2 + \|y\|_2 \geq \|x-y\|_2$, we have

$$\max\Big\{\|f(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2, \|(-f)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2\Big\}$$
$$\geq \frac{\|f(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2 + \|(-f)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|}{2} \geq$$
$$\geq \frac{(\|f(\cdot) - \varphi_N(0,\ldots,0;\cdot)) - ((-f)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2}{2} \geq \|f\|_2.\square$$

4 Proof of the main result

We will begin the proof by checking the validity of the relations

$$\triangle_N(\overline{\varepsilon}_N, (\overline{l}^{(N)}, \overline{\varphi}_N), Tf = f, W_2^{r;\alpha})_{L^2} \succ\prec \delta_N(L_N, Tf = f, W_2^{r;\alpha})_{L^2}. \tag{10}$$

For arbitrarily given numbers $\gamma_N^{(\tau)}$ such that $|\gamma_N^{(\tau)}| \leq 1 (\tau = 1,\ldots,N)$ there is an inequality

$$\left\|f(\cdot) - \overline{\varphi}_N(\overline{l}_N^{(1)}(f) + \gamma_N^{(1)}\overline{\varepsilon}_N, \ldots, \overline{l}_N^{(N)}(f) + \gamma_N^{(N)}\overline{\varepsilon}_N;\cdot)\right\|_2 \leq$$
$$\leq \left\|f(\cdot) - \overline{\varphi}_N(\overline{l}_N^{(1)}(f), \ldots, \overline{l}_N^{(N)}(f);\cdot)\right\|_2 + \left\|\sum_{\tau=1}^{N}(-\gamma_N^{(\tau)})\overline{\varepsilon}_N e^{2\pi i(\overline{m}^{(\tau)},\cdot)}\right\|_2. \tag{11}$$

According to the theorem from [5], the relations

$$\delta_N(L_N, Tf = f, W_{L^2}^{r;\alpha})_{L^2} \succ\prec \delta_N\left((\overline{l}^{(N)}, \overline{\varphi}_N), Tf = f, W_2^{r;\alpha}\right)_{L^2} \succ\prec$$
$$\succ\prec \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \ldots + \alpha_s/r_s)}} \tag{12}$$

are valid. Using the Parseval equality, we find

$$\left\|\sum_{\tau=1}^{N}(-\gamma_N^{(\tau)})\overline{\varepsilon}_N e^{2\pi i(\overline{m}^{(\tau)},\cdot)}\right\|_2 \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \ldots + \alpha_s/r_s)}}.$$

Further, due to inequalities (11) and (12), we obtain

$$\left\|f(\cdot) - \overline{\varphi}_N\left(\overline{l}_N^{(1)}(f) + \gamma_N^{(1)}\overline{\varepsilon}_N, \ldots, \overline{l}_N^{(N)}(f) + \gamma_N^{(N)}\overline{\varepsilon}_N;\cdot\right)\right\|_2 \ll$$

$$\ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \ldots + \alpha_s/r_s)}}.$$

From where, by virtue of the arbitrariness of the numbers $\gamma_N^{(\tau)} (\tau = 1,\ldots,N)$ and the function f , we have

$$\triangle_N\left(\overline{\varepsilon}_N,(\overline{l}^{(N)},\overline{\varphi}_N),Tf=f,W_2^{r;\alpha}\right)_{L^2}\ll\frac{1}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}}. \quad (13)$$

Since

$$\delta_N(L_N,Tf=f,W_2^{r;\alpha})_{L^2}\le\delta_N\left((\overline{l}^{(N)},\overline{\varphi}_N),Tf=f,W_2^{r;\alpha}\right)_{L^2}\le$$
$$\le\triangle_N\left(\overline{\varepsilon}_N,(\overline{l}^{(N)},\overline{\varphi}_N),Tf=f,W_2^{r;\alpha}\right)_{L^2},$$

then taking into account (12) and (13) we obtain (10).

Let the set of pairs $\left(l^{(N)},\varphi_N\right)$ with functionals

$$l_N^{(1)}(f)=\hat{f}(m^{(1)}),\ldots,l_N^{(N)}(f)=\hat{f}(m^{(N)})$$

be denoted by Φ_N. Now let us verify that for all $(l^{(N)},\varphi_N)\in\Phi_N$ and any arbitrarily slowly increasing to $+\infty$ sequence $\{\eta_{N(K)}\}_{K\ge1}$ the equality

$$\overline{\lim_{N\to\infty}}\frac{\triangle_N(\eta_N\overline{\varepsilon}_N,(\overline{l}^{(N)},\overline{\varphi}_N),Tf=f,W_2^{r;\alpha})_{L^2}}{\delta_N(L_N,Tf=f,W_2^{r;\alpha})_{L^2}}=+\infty. \quad (14)$$

holds. Next, for each integer $K>C(r,\alpha,s)$ we define the set

$$H_K^*=\{m\in Z^s:[M_1^*]\le|m_1|\le2\cdot[M_1^*],\ldots,[M_s^*]\le|m_s|\le2\cdot[M_s^*]\},$$

where $M_i^*=N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}(\ln N)^{-\alpha_i/r_i}\beta_K^{-1/r_i}$ for all $i\in\{1,2,\ldots,s\},N=N(K),\beta_K=\min\{\eta_N,\ln N\}$.

Since $\lim\limits_{K\to+\infty}\beta_K=+\infty$ (see Lemma 1), there exists a number $K_0\ge C(r,\alpha,s)$ such that for all integers $K\ge K_0$ the inequality $\beta_K\ge1$ holds.

Now for any component $m_i(i=1,2,\ldots,s)$ of the vector $m\in H_K^*$ we will prove the inequality

$$\ln^{\alpha_i}(2\overline{m}_i)\ll\ln^{\alpha_i}N,\alpha_i\in\mathbb{R}. \quad (15)$$

If $\alpha_i\ge0$, then by virtue of inequalities $(\ln N)^{-\alpha_i/r_i}\le1$ and $\beta_K^{-1/r_i}\le1$ and Lemma 2, we have

$$\ln^{\alpha_i}(2\overline{m}_i)\ll\ln^{\alpha_i}\left(2N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}(\ln N)^{-\alpha_i/r_i}\beta_K^{-1/r_i}\right)\ll$$
$$\ll\ln^{\alpha_i}\left(2N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}\right)\ll\ln^{\alpha_i}N.$$

Comparing the beginning and the end of this chain of inequalities, we obtain (15). Let $\alpha_i<0$. Since $\beta_K=\min\{\eta_N,\ln N\}$, then $\beta_K^{-1/r_i}\ge\ln^{-1/r_i}N$.

Therefore,

$$2\overline{m}_i\gg2N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}(\ln N)^{-\alpha_i/r_i}\beta_K^{-1/r_i}\gg$$
$$\gg N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i-\alpha_i/r_i-1/r_i},$$

whence by virtue of Lemma 2 and the inequality $\alpha_i<0$, we again obtain (15).

From (15) follows the inequality

$$\overline{m}_i^{r_i}\ln^{\alpha_i}(2\overline{m}_i) \ll N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}\beta_K^{-1}. \tag{16}$$

Consider the function $h_K(x) = \beta_K\overline{\varepsilon}_N \sum\limits_{m\in H_K^*} e^{2\pi i(m,x)}$. Then, using the relation

$$|H_K^*| \succ\prec N\cdot\beta_K^{-1/\lambda} \tag{17}$$

and the inequalities (16) and $\beta_K \geq 1 (K \geq K_0)$, we have

$$\sum_{m\in Z^s}|\hat{h}_K(m)|^2\left(\overline{m}_1^{2r_1}\ln^{2\alpha_1}(\overline{m}_1+1)+\ldots+\overline{m}_s^{2r_s}\ln^{2\alpha_s}(\overline{m}_s+1)\right)=$$

$$=\sum_{m\in H_K^*}|\hat{h}_k(m)|^2\left(\overline{m}_1^{2r_1}\ln^{2\alpha_1}(\overline{m}_1+1)+\ldots+\overline{m}_s^{2r_s}\ln^{2\alpha_s}(\overline{m}_s+1)\right)\ll$$

$$\ll\sum_{m\in H_K^*}|\hat{h}_K(m)|^2\left(\overline{m}_1^{2r_1}\ln^{2\alpha_1}(2\overline{m}_1)+\ldots+\overline{m}_s^{2r_s}\ln^{2\alpha_s}(2\overline{m}_s)\right)\ll$$

$$\ll\beta_K^2\overline{\varepsilon}_N^2\sum_{m\in H_K^*}N^{2\lambda}(\ln N)^{2\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}\beta_K^{-2}\ll$$

$$\ll\frac{1}{N}\sum_{m\in H_K^*}1\ll\frac{1}{\beta_K^{1/\lambda}}\ll 1.$$

Therefore, for some $C(r,\alpha,s)>0$ the function $t_K(x)=C(r,\alpha,s)\cdot h_K(x)$ belongs to the class $W_2^{r;\alpha}$. By virtue of Parseval's equality and relation (17) we have

$$\|t_K\|_2 \gg \frac{\beta_K^{1-1/(2\lambda)}}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}}. \tag{18}$$

Further, having fixed the values of $K\geq K_0$ for any $\tau=1,\ldots,N\equiv N(K)$ we put

$$\tilde{\gamma}_N^{(\tau)}=-\frac{\hat{t}_K(m^{(\tau)})}{\overline{\varepsilon}_N\eta_N}\quad\text{and}\quad\widetilde{\omega}_N^{(\tau)}=-\frac{(-\hat{t}_K)(m^{(\tau)})}{\overline{\varepsilon}_N\eta_N}.$$

Since for each $\tau=1,\ldots,N$ inequalities $|\tilde{\gamma}_N^{(\tau)}|\leq 1, |\widetilde{\omega}_N^{(\tau)}|\leq 1$ and equalities

$$\hat{t}_K(m^{(\tau)})+\eta_N\tilde{\gamma}_N^{(\tau)}\overline{\varepsilon}_N=0, (-\hat{t}_K)(m^{(\tau)})+\eta_N\widetilde{\omega}_N^{(\tau)}\overline{\varepsilon}_N=0$$

are true, then by virtue of Lemma 3 for each pair $(l^{(N)},\varphi_N)\in\Phi_N$ we have

$$\sup_{f\in W_2^{r;\alpha}}\sup_{|\gamma_N^{(1)}|\leq 1,\ldots,|\gamma_N^{(N)}|\leq 1}\Big\|f(\cdot)-\varphi_N\Big(\hat{f}(m^{(1)})+\gamma_N^{(1)}\eta_N\overline{\varepsilon}_N,\ldots,$$

$$\hat{f}(m^{(N)})+\gamma_N^{(N)}\eta_N\overline{\varepsilon}_N;\cdot\Big)\Big\|_2\geq$$

$$\geq\max\Big\{\Big\|t_K(\cdot)-\varphi_N\Big(\hat{t}_K(m^{(1)})+\tilde{\gamma}_N^{(1)}\eta_N\overline{\varepsilon}_N,\ldots,\hat{t}_K(m^{(N)})+\tilde{\gamma}_N\eta_N\overline{\varepsilon}_N;\cdot\Big)\Big\|_2,$$

$$\left\|(-t_K)(\cdot) - \varphi_N\Big((-\hat{t}_K)(m^{(1)}) + \widetilde{\omega}_N^{(1)}\eta_N\overline{\varepsilon}_N, \ldots, (-\hat{t}_K)(m^{(N)}) + \widetilde{\omega}_N^{(N)}\eta_N\overline{\varepsilon}_N; \cdot\Big)\right\|_2\Big\} =$$

$$= \max\left\{\left\|t_K(\cdot) - \varphi_N(0,\ldots,0;\cdot)\right\|_2, \left\|(-t_K)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\right\|_2\right\} \geq \|t_K\|_2.$$

Next, using inequality (18), we find

$$\triangle_N\left(\eta_N\overline{\varepsilon}_N, (l^{(N)}, \varphi_N), Tf = f, W_2^{r;\alpha}\right)_{L^2} \gg \delta_N\left(L_N, Tf = f, W_2^{r;\alpha}\right)_{L^2} \beta_K^{1-1/(2\lambda)}. \tag{19}$$

Since

$$\left(\frac{1}{\min\{r_1, r_1+\alpha_1\}} + \ldots + \frac{1}{\min\{r_s, r_s+\alpha_s\}}\right)^{-1} > \frac{1}{2}$$

(see condition (6)), then $2\lambda > 1$. Therefore, in view of the equality $\lim\limits_{K\to+\infty}\beta_K = +\infty$ and the inequalitiy (19) for each pair $(l^{(N)}, \varphi_N) \in \Phi_N$ and any positive sequence $\{\eta_{N(K)}\}_{K\geq 1}$ that increases arbitrarily slowly to $+\infty$, the inequality

$$\lim_{N\to\infty}\frac{\triangle_N(\eta_N\overline{\varepsilon}_N, (l^{(N)}, \varphi_N), Tf = f, W_2^{r;\alpha})_{L^2}}{\delta_N(L_N, Tf = f, W_2^{r;\alpha})_{L^2}} = +\infty \tag{20}$$

holds. It is clear that $(\overline{l}^{(N)}, \overline{\varphi}_N) \in \Phi_N$. Consequently, from (20) follows (14). $\square$

Remark. *Since the equality (20) is proved for all pairs $(l^{(N)}, \varphi_N) \in \Phi_N$, then any optimal computing unit $\varphi_N\left(\hat{f}(m^{(1)}), \ldots, \hat{f}(m^{(N)}); \cdot\right), N = N(K)$ does not have a greater (in order) limiting error than $\overline{\varepsilon}_N$.*

5 Conclusion

The theorem proved above is a new result in approximation theory, numerical analysis, and computational mathematics. This study can be continued by replacing condition (6) with a weaker condition that ensures absolute convergence of the trigonometric Fourier series $\sum\limits_{m\in Z^s}\hat{f}(m)e^{2\pi i(m,x)}$ functions $f \in W_2^{r;\alpha}$.

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