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THE MULTIPLICATIVE INTEGRAL AND THE EVOLUTION OF THE MAGNETIC FIELD IN THE MARKOV LINEAR MODEL

The paper is devoted to the probabilistic asymptotic analysis of the magnetic field and magnetic energy in a Markov linear model of an incompressible fluid. Firstly, the paper introduces a brief history of the problem under consideration and presents the main results of the previous studies, which ultimately lead to the study of the product of independent random matrices with an increasing number of multiplicands. After that, the description of the Markov linear model considered in the paper is given, the so-called Lyapunov (generally speaking, random) bases for the multiplicative (stochastic) integral contained in the integral representation of the magnetic field are constructed. In conclusion, by decomposing the multiplicative integral over the constructed Lyapunov basis and relying on the properties of the basis, the main results - theorems on the asymptotic behavior of the magnetic field and magnetic energy - have been proven.

Key words: Multiplicative integral, Markov linear model, magnetic field, Lyapunov exponent, Lyapunov basis.

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Мультипликативті интеграл және марковтық сызықтық моделдегі магнит өрісінің эволюциясы

Жұмыс сығылмайтын сұйықтық марковтың сызықтық моделіндегі магнит өрісі мен магниттік энергияның ықтималдықтық - асимптотикалық талдауына арналған. Жұмыста алдымен қарастырылып отырған есептің бұған дейін басқа авторлар қарастырған, ақыр соңында көбейткіштерінің саны өсе беретін тәуелсіз кездейсоқ матрицалардың көбейтінділерін зерттеуге келтірілетін, жұмыстардың қысқаша тарихы баяндалған. Сосын жұмыста қарастырылатын марковтың сызықтық модельдің сипаттамасы берілген, магниттік өрістің интегралдық жазылымында пайда болатын мультипликативтік (стохастикалық) интеграл үшін ляпуновтық деп аталатын (жалпы алғанда, кездейсоқ) базис құрастырылған. Ең соңында, мультипликативтік интегралды құрастырылған ляпуновтық базис арқылы жіктеп және бұл базистік қасиеттеріне сүйене отырып, негізгі нәтижелер - магнит өрісі мен магниттік энергияның асимптотикалық беталыстары туралы теоремалар дәлелденген.

Түйін сөздер: Мультипликативтік интеграл, марковтық сызықтық модель, магнит өрісі, Ляпунов көрсеткіші, ляпуновтық базистер.

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Мультипликативный интеграл и эволюция магнитного поля в марковской линейной модели

В работе сначала изложена краткая история рассматриваемой задачи, приведены основные результаты предыдущих,сводящихся в конечном итоге к изучению произведения независимых случайных матриц при возрастании числа сомножителей, работ. После дано описание рассматриваемой в работе марковский линейной модели, построены так называемые ляпуновские (вообще говоря, случайные) базисы для содержащегося в интегральном представлении магнитного поля мультипликативного (стохастического) интеграла.

В заключение, разлагая мультипликативный интеграл по построенному ляпуновскому базису и опираясь на свойства этих базисов, доказаны основные результаты - теоремы об асимптотических поведениях магнитного поля и магнитной энергии.
Ключевые слова: Мультипликативный интеграл, марковская линейная модель, магнитное поле, показатель Ляпунова, ляпуновские базисы.

1 Introduction

The problem of the evolution of a magnetic field in a random turbulent flow of a conducting fluid is one of the most important in many physical applications. First of all, astrophysical applications can be mentioned here: stars, planets, and galaxies have magnetic fields that can vary greatly in time and space. A huge number of works have been devoted to various physical and mathematical aspects of this problem (see, for example, the monographs [1], [2], and the recent work [3]). One of the central and actual issues in this area is the study of asymptotic properties (completely non-trivial from the mathematical point of view) of the solution of the Cauchy problem for the induction equation. In this paper, the problem of the evolution of the magnetic field is considered in a kinematic formulation: this means that the statistical characteristics of a given random velocity field do not change with time, although the statistical characteristics of the magnetic field, generally speaking, change. In other words, the reverse effect of the magnetic field on the velocity field is not taken into account. The kinematic formulation allows one to remain within the linear approximation (i.e., for a given fluid velocity field), while the problem of the joint evolution of the velocity field and magnetic field requires the study of a nonlinear system of equations in six dimensions. We note that the asymptotic behavior of solutions at very large Reynolds numbers is related to the famous (until now unsolved) problem of the hydromagnetic dynamo (see [4] and the bibliography cited there).

2 Literature review and problem statement.

While for a given fluid flow the process of magnetic field transfer is fundamentally clear, the very problem of describing a turbulent fluid flow is known to be extremely complex. Therefore, one or another method of modeling the motion of a fluid is usually resorted to. In [5]- [6], the question of the evolution of the magnetic field was studied in the so-called linear model with updating, and ultimately the problem under consideration was reduced to studying the product of independent random matrices with increasing number of factors. Our present work is a generalization of works [5]- [6] in the sense that we study the asymptotic behavior of the solution of the Cauchy problem for the magnetic induction equation in a more general (than the updated model) model - a given Markov linear model (for a description of the model, see below, in Section 5) at long times. We will also consider a similar question for the total magnetic energy. In this case, we will essentially use the main result of works [7]- [8] - the Ferstenberg- type theorem (the theorem that establishes the existence of a strictly positive Lyapunov exponent associated with the introduced Markov model of a multiplicative stochastic integral of a special form). It should be noted that this Ferstenberg- type theorem for the multiplicative stochastic integral is, in a certain sense, a generalization of similar results for the product of unimodular random matrices [9], in particular, the product of independent [10] or random matrices forming a Markov sequence [11] (see also the survey article [12]).

3 Purpose and objectives of the study

The purpose and objectives of our work are to study the asymptotic properties of the solution of the Cauchy problem for the equation of magnetohydrodynamics and to generalize and extend the main results of [5]- [6] to the case of a Markov linear model of a given velocity field. In this case, special attention will be paid to the problem of finding the asymptotic form of the magnetic field and total magnetic energy present in the integral representations and determined by the introduced Markov linear model of a multiplicative stochastic integral of a special kind.

4 Materials and methods.

In the work, some well-known results and methods of the theory of magnetic fields in random media, the theory of matrices and multiplicative integral, partial differential equations and stochastic analysis will be used and refined in cases necessary for our purposes.

5 Mean result.

5.1 Model of a Markov linear velocity field.

Let $b(t), t \geq 0$ is a Brownian motion on a compact Riemannian manifold K, $dimK = \nu \geq 3$, with metric form ds^2 having the form in local coordinates $x^1, ..., x^\nu$ on K $ds^2 = \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} g_{ij} dx^i dx^j, d\sigma = \sqrt{detg}dx-$Riemannian volume element. The infinitesimal operator of the process $b(s), s \geq 0$, is the Beltrami - Laplace operator $\frac{1}{2}\Delta$, where

$$\Delta = \frac{1}{\sqrt{detg}} \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \left(\frac{\partial}{\partial x^i} \left(g_{ij} \sqrt{detg} \right) \right).$$

Let $C(\cdot) : K \to SL(\nu, R)$, where $SL(\nu, R)$ is the linear space of square $\nu \times \nu$ matrices with zero trace ($TrC = 0$). Functions $g_{ij}(x)$, $c_{ij}(x)$ are functions of the class $C^\infty(K)$. Then the Markov linear model of the velocity field is the velocity field of the form

$$\vec{V}(t, x) = C(b_t)x, \tag{1}$$

where the process $b_t = b(t)$, the manifold and the matrix $C(\cdot)$ are defined and described by the above conditions.

5.2 Evolution of the magnetic field in a Markov linear model.

It is well known that the evolution of the initial distribution

$$\vec{H}(x) = (H_{01}(x), H_{02}(x), ..., H_{0\nu}(x))$$

ν-dimensional $\nu \geq 3$ magnetic field

$$\vec{H}(t,x) = (H_1(t,x), H_2(t,x), ..., H_\nu(t,x))$$

in a given speed field

$$\vec{V}(t,x) = (V_1(t,x), V_2(t,x), ..., V_\nu(t,x))$$

with constant magnetic diffusion ν_m ($\nu_m > 0$) is described by the induction equation

$$\frac{\partial}{\partial t}\vec{H} = \nu_m \Delta \vec{H} + \text{rot}\left[\vec{V} \times \vec{H}\right], \tag{2}$$

$$\vec{H}(0,x) = \vec{H}(x), \tag{3}$$

where $t \geq 0, x \in \mathbb{R}^\nu$.

If we assume that the velocity field $\vec{V}$ and the initial field $\vec{H}_0$ are incompressible, i.e. (divergences in x)

$$\text{div}\,\vec{V} = 0,\ \text{div}\,\vec{H}_0 = 0, \tag{4}$$

then problem (2)-(3), under the condition

$$\text{div}\,\vec{H} = 0, \tag{5}$$

reduces to solving the Cauchy problem (herein after, the bracket $(...,...)$ means the scalar product)

$$\frac{\partial}{\partial t}\vec{H} = \nu_m \Delta \vec{H} - \left(\vec{V}, \nabla\right)\vec{H} + \left(\vec{H}, \nabla\right)\vec{V}, \quad \vec{H}(0,x) = \vec{H}_0(x). \tag{6}$$

Note that condition (5) is a consequence of condition (4): from $\text{div}\,\vec{H}_0(x) = 0$ it follows that $\text{div}\,\vec{H}(t,x) = 0$ for all $t \geq 0$.

Indeed, taking the divergence from both parts of (2) and taking into account the relation $\text{div}\,(rot) = 0$, we obtain

$$\frac{\partial}{\partial t}\text{div}\,\vec{H} = \nu_m \Delta\left(\text{div}\,\vec{H}\right). \tag{7}$$

Therefore, by the uniqueness theorem, condition (5) will be satisfied for all $t > 0$ if it is satisfied for $t = 0$, i.e. for the initial condition $\vec{H}_0(x)$.

The initial magnetic field $\vec{H}_0(x)$ is given by the distribution of currents, and these currents are concentrated in a limited region of space. It is known that then $\vec{H}_0(x) = O\left(|x|^{-\nu}\right), x \to \infty$ and this condition ensures the solvability of system (6). Let us now write out the solution of equation (6) in the Markov linear model (1).

To do this, we first look for a particular solution in the form

$$\vec{H}(t,x) = \vec{h}(t,k) exp\left\{i(\vec{\varkappa}(t,\vec{k}), x)\right\}. \tag{8}$$

where $t \geq 0, \vec{k} \in \mathbb{R}^{\nu}$, i is imaginary unit, and $\vec{\varkappa}(\vec{h}_0, \vec{k}) = \vec{k}$. Substituting (8) into (6) and equating the real and imaginary parts of the received relation, we obtain

$$\frac{d}{dt}\vec{\varkappa}(t, \vec{k}) = -C^*(b_t)\vec{\varkappa}(t, \vec{k}), \quad \vec{\varkappa}(0, \vec{k}) = \vec{k}, \tag{9}$$

$$\frac{d}{dt}\vec{h}(t, \vec{k}) + \nu_m \vec{\varkappa}^2 \vec{h}(t, \vec{k}) = C(b_t)\vec{h}(t, \vec{k}), \quad \vec{h}(0, \vec{k}) = h_0(\vec{k}), \tag{10}$$

where $*$ is the transposition operation, the scalar square $\vec{\varkappa}^2 = \left(\vec{\varkappa}(t, \vec{k}), \vec{\varkappa}(t, \vec{k})\right)$ and the condition $\operatorname{div} \vec{H}_0(x) = 0$ is equivalent to the condition $\left(\vec{h}_0, \vec{k}\right)$. In addition, the condition $\operatorname{div} \vec{V} = 0$ means that the matrices $C(b_t), C^*(b_t)$ have zero traces:

$$trC(b_t) = trC^*(b_t) = 0,$$

those $C(b_t) \in SL(\nu, \mathbb{R})$. As is known ([13] Ch. XV, §5, §6), the unimodular $\nu \times \nu$ matrix $X(t)$, is a solution to the equation

$$\frac{d}{dt}X(t) = C(t)X(t), \quad X(0) = E,$$

where the E is identity matrix is called the multiplicative integral (in terms of [13]-matrix) and is denoted by the symbol

$$X(t) = \Omega_0^t(D) = \int_0^t (E + D(s)ds)\,.$$

Then, introducing into consideration the matrix (multiplicative integral, more precisely, multiplicative stochastic integral)

$$G_t = \int_0^t (E + C(b_s)ds)\,. \tag{11}$$

as a solution to the equation

$$\frac{d}{dt}G_t = -C(b_t)G_t, \quad G_0 = E,$$

and noting that the matrix of system (9) is $(G_t^*)^{-1}$ and the property 2^0 ([13], p. 431) we get that the solutions of systems (9) - (10) can be written, respectively, in the form

$$\vec{\varkappa}(t, \vec{k}) = (G_t^*)^{-1}\vec{k},$$

$$\vec{h}(t, \vec{k}) = G_t \vec{h}_0 exp\left\{-\nu_m \int_0^t \vec{\varkappa}^2(s, \vec{k})ds\right\},$$

Now, to find a general solution of the problem (6) in model (7), we need to expand the initial condition $\vec{H}_0(x)$ into a Fourier integral and force each component of this expansion to evolve according to systems (9), (10). In other words, if by $\widehat{\vec{H}}_0(\vec{k})$ we denote the Fourier image of the initial field $\vec{H}_0(x)$:

$$\vec{H}_0(x) = \frac{1}{(2\pi)^{\nu/2}} \int\limits_{\mathbb{R}^\nu} e^{i(\vec{k},x)} \widehat{\vec{H}}_0(\vec{k}) d\vec{k}. \tag{12}$$

then

$$\vec{H}(t,x) = \frac{1}{(2\pi)^{\nu/2}} \int\limits_{\mathbb{R}^\nu} G_t \widehat{\vec{H}}_0(\vec{k}) exp\left\{ i((G_t^*)^{-1}\vec{k}, x) \right\} \cdot exp\left\{ -\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k} \right)^2 ds \right\} d\vec{k}. \tag{13}$$

It is easy to see that for the total magnetic energy $E(t)$ we obtain the integral representation

$$\mathcal{E}(t) = \int\limits_{\mathbb{R}^\nu} \vec{H}^2(t,x)dx = \int\limits_{\mathbb{R}^\nu} \left(G_t \widehat{\vec{H}}_0(\vec{k}) \right)^2 exp\left\{ -2\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k} \right)^2 ds \right\} d\vec{k} \tag{14}$$

Thus, the solution $\vec{H}(t,x)$ of the magnetic induction equation (6) in the Markov linear model (1) by formula (13), and its magnetic energy $\mathcal{E}(t)$ by formula (14) are expressed as some functionals of the multiplicative integral (matrix) of the form (11). And this means that in order to study questions about the asymptotic behaviors of the magnetic field $\vec{H}(t,x)$ and magnetic energy $\mathcal{E}(t)$ as $t \to \infty$, it is important to know the asymptotic behavior as the multiplicative integral G_t itself and integrals (13) and (14) depending on it as $t \to \infty$.

In connection with the above, the following tasks arise:

A) Find out the asymptotic behavior of G_t as $t \to \infty$;

B) Carry out an asymptotic analysis of the magnetic field $\vec{H}(t,x)$ and magnetic energy $\mathcal{E}(t)$ at $t \to \infty$.

Other equally interesting problems are also possible (for example, those related to various moments of the magnetic field in the Markov linear model). But in this paper we will not consider such problems.

The solution of problem A) was announced in [7], and the complete solution of problem A) was given in [8] in the following setting.

Let θ_0 is an arbitrary (non-random) column vector of unit length: $\theta_0 \in S^{\nu-1}$ is the unit sphere in $\mathbb{R}^\nu$, $\nu \geq 3$. Let us act on θ_0 by the multiplicative integral G_t with the matrix $C(\cdot) : K \to SL(\nu, \mathbb{R})$ and denote by r_t the Euclidean length of the resulting vector: $r_t = \|G_t\theta_0\|$.

The Lyapunov exponent of the matrix G_t is the almost-probably (a.s.) limit

$$\gamma = \lim_{t\to\infty} \frac{1}{t} ln\, r_t.$$

Then the following is true.

Theorem 1 *(Ferstenberg type theorem) For any fixed initial phase $\theta_0 \in S^{\nu-1}$ there exists a.s.and a strictly positive limit*

$$\gamma = \lim_{t\to\infty} \frac{1}{t} ln \left\| G_t\theta_0 \right\| > 0. \tag{15}$$

In other words, it was proved in [8] that for$t \to \infty$; the asymptotic $r_t = \|G_t\theta_0\| \sim exp\{\gamma t\}, \gamma > 0$, i.e., for sufficiently large$t$, the action of the matrix G_t on a vector of unit length leads to its exponential expansion. The proposed paper is devoted to solving problem C). In this case, the above Theorem 1 from [8] will play an essential role.

5.3 Construction of Lyapunov bases for G_t

This section is devoted to obtaining such results on the multiplicative integral G_t of a Markov random matrix that would be convenient for studying the magnetic field $\vec{H}(t,x)$ and its energy $\mathcal{E}(t)$ expressed by formulas (13) and (14).

Let us now proceed to the construction of Lyapunov (generally speaking, random) bases and show that, with the appropriate use of these bases, the multiplicative integral G_t defined in model (1) by formula (11) as $t \to \infty$ almost does not differ from the degree of some constant matrix.

To do this, first of the multiplicative integral (matrix) G_t, like any matrix, we represent as a product of orthogonal (U_t) and upper triangular (K_t) matrices. Technically, this can be done as follows. We orthonormalize the columns of the matrix G_t, starting from the first one, and form a new basis from them. As U_t, we take the transition matrix from the original basis to the new one. Then the diagonal elements of the matrix K_t will have the following form: K_{11} is the length of the first column of the matrix G_t, K_{22} are the length of the component of the second column orthogonal to the first column, and so on. Let us now substitute this representation $G_t = U_tK_t$ into the equation for G_t, if the matrix $U_t^{-1}C(b_t)U_t$ is represented as the sum of antisymmetric (F_t) and upper triangular (B_t) matrices, then for U_t and K_t we obtain the equations

$$\frac{dU_t}{dt} = U_tF_t, \quad \frac{dK_t}{dt} = B_tK_t \tag{16}$$

The first of these equations is a non-linear equation closed with respect to the orthogonal matrix U_t, which determines the orientation of the matrix G_t. After solving this equation, we can find the diagonal elements of K_t:

$$K_{ii} = K_{ii}(0)exp\left\{t\gamma_i + \sqrt{t}\xi_i(t)\right\},$$

where

$$\gamma_i = \lim_{t\to\infty} \frac{1}{t}\int\limits_0^t b_{ii}(s)ds, \quad \xi_i(t) = \frac{1}{\sqrt{t}}\int\limits_0^t \left(b_{ii}(s) - \gamma_i\right)ds. \tag{17}$$

According to the central limit theorem, as $t \to \infty$, the process $\xi_i(t)$ has a normal distribution (note that we can always put $K_{ii}(0) = 1$).

Moreover, the matrix U_t is Markov and the group of orthogonal matrices is compact in $SL(\nu, \mathbb{R})$. Whence follows the existence of a stationary distribution μ of the matrices U, over which the averaging is performed in the first of formulas (17). Further, $detK_t = 1$, so $\gamma_1 + \gamma_2 + ... + \gamma_\nu = 0$. From the decomposition method of G_t into the product $U_t K_t$ it follows that $\gamma_1 \geq \gamma_2 \geq ...\gamma_{\nu-1} \geq \gamma_\nu$. Take $\theta_0 = (1, 0, ..., 0) \in S^{\nu-1}$ and make sure that $\gamma_1 = \gamma > 0$, where γ_ν is the largest Lyapunov exponent appearing in the Ferstenberg-type theorem (see (15)). After that, we act to the unit vector $\theta_0 \in S^{\nu-1}$ by the inverse matrix G_t^{-1}. Then the resulting vector, for the same reasons as above, changes as $exp|\gamma_\nu|t$, where γ_ν is the "highest exponent in the reverse course of time i.e. the lowest index and $\gamma_\nu < 0$. However, the negativity of γ_ν also follows from the fact that $\gamma_1 > 0, \gamma_1 + \gamma_2 + ... + \gamma_\nu = 0$. The signs of other $\gamma_j (j = 2, 3..., \nu - 1)$ can be arbitrary. For example, if $\nu = 3$ and the distribution of matrices also has symmetry under the change $G_t \to G_t^{-1}$, then $\gamma_2 = 0,\ \gamma_1 = -\gamma_3$. In addition, some of γ_j can be the same. However, it turns out that in our assumptions, they are all different, or rather true

Theorem 2 *(simplicity theorem for the spectrum of characteristic Lyapunov exponents). The exponents of the Lyapunov matrix G_t are different, i.e. there are strict inequalities*

$$\gamma = \gamma_1 \geq \gamma_2 \geq ...\gamma_{\nu-1} \geq \gamma_\nu. \tag{18}$$

Proof 1 *We divide the interval $[0, t]$ into n parts by points $0 = t_0 < t_1 < t_2 < ... < t_n = t$ and represent the matrix G_t as ([13], p. 433, formula (46))*

$$G_t = g_n \cdot g_{n-1} \cdot ... \cdot g_1, \tag{19}$$

where

$$g_k = \int\limits_{t_{k-1}}^{t_k} (E + C(b_s)ds)\,, \quad k = 1, 2, ..., n. \tag{20}$$

It is clear that $g_1, g_2...$ are stationary Markov processes with values in the group $SL(\nu, \mathbb{R})$ and mean value $Mln\|g_1\| < \infty$. Studying this sequence $g_1, g_2...$ from the point of view of [11]. Given that, according to the results of [8], the pair (b_t, G_t) has a smooth transition density, we see that this sequence satisfies all the requirements of the main theorem of [11] ([11], §2, p.122). Thus, according to the main theorem [11], the characteristic Lyapunov exponents are simple, i.e.

$$\gamma_1 > \gamma_2 > ... > \gamma_\nu.$$

The simplicity of the spectrum of characteristic exponents is one of the central properties that determine the asymptotic behavior of the multiplicative integral G_t. Below, using this property, we will prove a theorem that, when using an appropriate Lyapunov basis, for large t $(t \to \infty)$ G_t almost does not differ from the degree of some fixed matrix (see Theorem 3 below). For these purposes, we first prove an auxiliary lemma.

Lemma 1 *a) For large t, matrix K_t can be represented as $K_t = D_t\overline{K}_t$, where K_t is an upper triangular matrix such that there are limits $\lim\limits_{t\to\infty}\overline{K}_{ij}(t) = \overline{K}_{ij}(\infty)$, and D_t is diagonal matrix:*

$$D_t = diag\left\{exp\left\{(\gamma_1 + \alpha_{11}(t))t\right\}, ..., exp\left\{(\gamma_\nu + \alpha_{\nu\nu}(t))t\right\}\right\} \tag{21}$$

$$\alpha_{ij} = \frac{1}{t}\int\limits_0^t (b_{ij}(s) - \gamma_j)\, ds.$$

b) Let $\overline{K}(\infty)$ is a matrix with entries $\overline{K}_{ij}(\infty)$. Let us set $\overline{K}_t = \overline{\overline{K}}_t\overline{K}(\infty)$. Then for any $\alpha > 0$ there are numbers $\beta_{ij} > 0$ such that, for sufficiently large t, the inequalities hold for $j < i$ with probability 1

$$-\beta_{ij}exp\left\{(\gamma_j - \gamma_i)t - \alpha t\right\} \le \overline{\overline{K}}_{ij}(t) \le \beta_{ij}exp\left\{(\gamma_j - \gamma_i)t + \alpha t\right\} \tag{22}$$

where $\overline{\overline{K}}_{ij}(t)$ are the elements of the matrix $\overline{\overline{K}}_t$.

Proof 2 *For simplicity, we will carry out the proof for matrices K_t of order 3×3.*

a) We know the form of the diagonal elements of K_t (formula (16). Using (17) for off-diagonal elements, we obtain:

$$K_{12}(t) = K_{11}(t)\overline{K}_{12}(t),$$

$$K_{13}(t) = K_{11}(t)\overline{K}_{13}(t),$$

$$K_{22}(t) = K_{22}(t)\overline{K}_{23}(t),$$

where

$$\overline{K}_{12}(t) = 1 + \int\limits_0^t b_{12}(s)K_{22}(s)K_{11}^{-1}ds,$$

$$\overline{K}_{13}(t) = 1 + \int\limits_0^t \left(b_{12}(s)K_{23}(s) + b_{13}(s)K_{33}(s)\right)K_{11}^{-1}ds, \tag{23}$$

$$\overline{K}_{23}(t) = 1 + \int\limits_0^t b_{23}(s)K_{33}(s)K_{22}^{-1}ds,$$

whence $K_t = D_t\overline{K}_t$, where the matrix D_t is defined by proportions (21).

Further, due to the boundedness of the norm matrix $C(x)$ on the compact set K, the elements of the matrix B_t with probability 1 are bounded functions of t. Therefore, from formula (21), from the fact that inequalities $j > i$ are found for $\gamma_j - \gamma_i < 0$, means of limitation follow for $\overline{K}_{ij}(t)$ at $t \to \infty$.

b) by the definition of a matrix, $\overline{\overline{K}}_t$ we have:

$$\overline{\overline{K}}_{11}(t) = \overline{\overline{K}}_{22}(t) = \overline{\overline{K}}_{33}(t) = 1, \overline{\overline{K}}_{ij}(t) = 0,\ (j < i), \overline{\overline{K}}_{12}(t) = \overline{K}_{12}(G) - \overline{K}_{12}(\infty),$$

$$\overline{\overline{K}}_{23}(t) = \overline{K}_{23}(t) - \overline{K}_{23}(\infty),\ \overline{\overline{K}}_{13}(t) = \overline{K}_{13}(t) - \overline{K}_{13}(\infty) + \overline{\overline{K}}_{12}(t)\overline{K}_{23}(\infty).$$

Substituting now the values $\overline{K}_{ij}(t)$ from (23) into the last relations, and evaluating the integrals obtained, we obtain the inequalities we need (22).

Definition 1 *Let column vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_\nu$ form a matrix e, satisfying the condition*

$$\overline{K}(\infty)e = E,$$

where E is the identity matrix. Then the basis $\vec{e}_1, \vec{e}_2, ..., \vec{e}_\nu$ called corresponding indicators $\gamma_1, \gamma_2, ..., \gamma_\nu$ Lyapunov basis.

The name "Lyapunov bases"is justified to some extent by the following theorem.

Theorem 3 *On Lyapunov bases $\vec{e}_1, \vec{e}_2, ..., \vec{e}_\nu$ the relations are fulfilled*

$$\gamma_j = \lim_{t\to\infty} \frac{1}{t} ln \left\| G_t \vec{e}_j \right\|, \quad j = 1, 2, ..., \nu. \tag{24}$$

Proof 3 *By definition of the Lyapunov basis*

$$\overline{K}(\infty)\vec{e} = (0, ..., 0, 1, 0, ..., 0)^*,$$

where 1*(one) is in the $j-th$ place, and the $*$ is sign is the transposition operation.*

Therefore, $\|G_t\vec{e}_j\|^2$ the length is simply the square of the length of the $j-th$ column of the matrix $D_t\overline{\overline{K}}_t$, i.e.,

$$\|G_t\vec{e}_j\|^2 = \sum_{l=1}^{\nu} \overline{\overline{K}}_{ij}^2 exp\left\{2(\gamma_l + \alpha_{ll}(t))t\right\}$$

Using inequalities (22), we obtain that for any $\alpha > 0$ there are numbers $\beta_1, \beta_2 > 0$ such that

$$\beta_1 exp\left\{2\gamma_j t - \alpha t\right\} \le \|G_t\vec{e}_j\|^2 \le \beta_2 exp\left\{2\gamma_j t + \alpha t\right\}$$

Now the relations (21) we need follow from the last inequality (due to the arbitrariness of α).

Note that the Lyapunov exponents are, as averages, non-random numbers. Mean while, the Lyapunov basis corresponding to them is random; it is different for different implementations of the process b_t. However, for this implementation, the basis is the same and does not depend on time.

5.4 Asymptotic analysis of the magnetic field and total magnetic energy

Now let the vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_\nu$ constitute the Lyapunov basis, and $\vec{e} = \lambda_1\vec{e}_1 + \lambda_2\vec{e}_2 + ... + \lambda_\nu\vec{e}_\nu$. Denote by $[|G_t\vec{e}|]^2$ approximate value of the square of the Euclidean norm, calculated under the assumption of the replacement $\overline{K}(t)$ with $\overline{K}(\infty)$ and D_t on $\overline{D}_t = diag exp(\gamma_1 t), ..., exp(\gamma_\nu t)$. In other words

$$[|G_t\vec{e}|]^2 = \sum_{j=1}^{\nu} \lambda_j^2 exp(2\gamma_j t). \tag{25}$$

On the other hand, due to the infinite smallness of $\alpha_{jj}(t)(j=1,...,\nu) ast \to \infty$, for any $a>0$ there exists $T_1 = T_1(a) > 0$, such that for $t > T_1$ the inequalities hold

$$e^{-at}\left[|G_t\vec{e}|\right]^2 \leq \left\|D_t\overline{K}(\infty)\vec{e}\right\|^2 = \sum_{j=1}^{\nu}\lambda_j^2 e^{2(\gamma_j+\alpha_{jj}(t))t} \leq e^{at}\left[|G_t\vec{e}|\right]^2. \tag{26}$$

Using (6), we can write similar inequalities for the norms $\left\|D_t\overline{K}(\infty)\vec{e}\right\|^2$ and $\|G_t\vec{e}\|^2$. As a result, we come to the conclusion that the theorem is true.

Theorem 4 *For any $a>0$ there exists $T=T(a)>0$ such that for $t>T$ uniformly over all vectors $\vec{e}\in\mathbb{R}^\nu$ the inequalities hold*

$$e^{-at} \leq \frac{\left[|G_t\vec{e}|\right]^2}{\|G_t\vec{e}\|^2} \leq e^{at}. \tag{27}$$

Inequalities (27) will later play an essential role in the study of the asymptotic behavior of the magnetic field strength and its total energy. Now, before proceeding to these studies, we note the following useful information.

If $\gamma_1 > \gamma_2 > ... > \gamma_\nu$ are the Lyapunov exponents for the matrix G_t, then the Lyapunov exponents for $(G_t^*)^{-1}$ will be $(-\gamma_1) < (-\gamma_2 < ... < (-\gamma_\nu)$. The corresponding Lyapunov basis $\vec{e}'_1, \vec{e}'_2, ..., \vec{e}'_\nu$ satisfies the condition

$$\left(\overline{K}(\infty)^*\right)^{-1} e' = E$$

(e' composed of column vectors $\vec{e}'_1, \vec{e}'_2, ..., \vec{e}'_\nu$ matrix). In other words, $e^*e' = E$, i.e. the Lyapunov bases for G_t and $(G_t^*)^{-1}$ are biorthogonal.

In addition, for the elements of the matrix $(G_t^*)^{-1}$ inequalities similar to (22) hold, so inequalities (27) are also valid for $(G_t^*)^{-1}$.

Note also that in formulas (12) and (13) the quantities $(G_t^*)^{-1}\vec{k}$ and $G_t\widehat{\vec{H}}_0(\vec{k})$ increase exponentially with probability 1. Therefore, the multiplier $exp\left\{-\nu_m \int\limits_0^t \left((G_t^*)^{-1}\vec{k}\right)^2 ds\right\}$ decreases as $t\to\infty$ (and any $\nu_m > 0$) as a double exponent. But in the integral sense (due to the influence of values of $\vec{k}$ close to zero) the double exponent decreases only at the rate of the usual exponent. This circumstance determines the nontriviality of the analysis of integral expressions (12) and (13).

Further, for simplicity, we will assume that $\nu = 3$ and proceed from formulas (12) and (13). In addition, below we will additionally assume that $\gamma_2 \neq 0$ herefore, two qualitatively different cases are possible:

$$a)\gamma_1 > 0 > \gamma_2 > \gamma_3,$$

$$b)\gamma_1 > \gamma_2 > 0 > \gamma_3.$$

Let us now formulate the main results - the theorem on the exponential decrease in the magnetic field and the theorem on the exponential growth of the total magnetic energy. (Below, $|\vec{a}|$ denotes the length of the vector $\vec{a}$, and we use the sign $\|\cdot\|$ to denote the norm).

Theorem 5 *Let $|\vec{H}_0(x)| \in L^{1+\beta}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ where $0 < \beta \leq 1$. Then there is $\alpha > 0$ (which does not depend on ν_m, provided that $\nu_m > 0$) such that with probability 1 as $t \to \infty$*

$$\sup_x |\vec{H}(t,x)| = o(exp(-\alpha t)). \tag{28}$$

Theorem 6 *Let $\widehat{\vec{H}}_0(0) \neq 0$, $|\vec{H}_0(x)| \in L^1(\mathbb{R}^3)$ and the initial field $\vec{H}_0(x)$ either be nonrandom or random, but does not depend on the fluid flow, i.e. from process b_t. Then there exists a positive with probability 1 function $B(\omega)$ of the elementary ω and a constant $\alpha > 0$ (the same for all $\nu_m > 0$) such that for sufficiently large t the inequality*

$$\varepsilon(t) > B(\omega)(exp(\alpha t)). \tag{29}$$

The proofs of these theorems will essentially be based on the inequalities (27), and the methodologies of the proofs from the technical point of view are in many respects similar and rather lengthy. In this connection, here we omit the detailed proofs of these assertions and give only the proof of Theorem 5 in case b).

We will proceed from the integral representation (12) for the magnetic field strength H $\vec{H}(t,x)$. Omitting the constant before the integral, we can write

$$|\vec{H}(t,x)| \leq \int\limits_{\mathbb{R}^3} exp\left\{-\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds\right\} \left|G_t \widehat{\vec{H}}_0(\vec{k})\right| d\vec{k}. \tag{30}$$

But according to Theorem 2, for any $\alpha > 0$ there exists $T = T(a) > 0$ such that for $t > T$ we have

$$\left|G_t \widehat{\vec{H}}_0(\vec{k})\right| \leq exp\left\{(\gamma_1 + \alpha)t\right\} \left|\widehat{\vec{H}}_0(\vec{k})\right| \tag{31}$$

On the other hand, $\vec{H}_0(x) \in L^{1+\beta}(\mathbb{R}^3)$, therefore, by the Hausdorff-Young inequality, $\widehat{\vec{H}}_0(\vec{k}) \in L^q(\mathbb{R}^3)$, $\frac{1}{q} + \frac{1}{1+\beta} = 1$. Furthermore,

$$\left\|\widehat{\vec{H}}_0(\vec{k})\right\|_{L^q} \leq c_\beta \left\|\vec{H}_0(x)\right\|_{L^{1+\beta}},$$

where C_β is a constant depending on β. Therefore, applying the Holder inequality to the right-hand side of (21), we obtain

$$\left|\vec{H}(t,x)\right| \leq c_\beta \int\limits_{\mathbb{R}^3} \left(exp\left\{-(1+\beta)\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds\right\}\right)^{\frac{1}{1+\beta}} \left\|\widehat{\vec{H}}_0(\vec{k})\right\|_{L^q} exp\left\{(\gamma_1+\alpha)t\right\} d\vec{k}.$$

Let us estimate the triple integral on the right side of the last inequality. Let k_1, k_2, k_3 are the coordinates of the vector $\vec{k}$ in the Lyapunov basis for $(G_t^*)^{-1}$. Then, according to Theorem 5, for any $\alpha > 0$ and sufficiently large t, the inequalities are fulfilled

$$\int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds \geq \int\limits_0^\tau \left((G_s^*)^{-1}\vec{k}\right)^2 ds + \int\limits_{t-\tau}^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds \geq$$

$$\delta(\omega)\left(k_1^2 + k_2^2 + k_3^2\right) + \sum_{j=1}^3 f_j(t) k_j^2,$$

where $\delta(\omega) = \delta$ is a finite, positive value with probability 1 (ω is an elementary event), τ is a sufficiently small number,

$$f_j(t) = \int\limits_{t-\tau}^{t} exp\left\{-2\left(\gamma_j + \alpha\right)s\right\}ds.$$

Since $\gamma_1 > \gamma_2 > 0 > \gamma_3$ and an α is an arbitrary small number, then for large t

$$\int\limits_0^t \left((G_t^*)^{-1}\vec{k}\right)^2 ds \geq \delta(\omega)k^2 + C_1(\tau)exp\left(-2\left(\gamma_1 + \alpha\right)t\right)k_1^2 +$$
$$C_2(\tau)exp\left(-2\left(\gamma_2 + \alpha\right)t\right)k_2^2 + exp\left(-2\left(\gamma_3 + \alpha\right)t\right)k_3^2 \geq$$
$$\delta(\omega)k^2 + exp\left(-2\left(\gamma_3 + \alpha\right)t\right)k_3^2,$$

where C_j are some positive constants depending on τ.
Hence

$$\int\limits_{\mathbb{R}^3} exp\left\{-(1+\beta)\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds\right\} d\vec{k} \leq \left(\frac{\pi}{(1+\beta)\nu_m}\right)^{\frac{3}{2}} exp\left(\left(\gamma_3 + \alpha\right)t\right).$$

Substituting this into (31) and taking into account the fact that $\gamma_1 + \gamma_3 = -\gamma_2 < 0$, we verify the assertion of Theorem 5. in case b).

Remark 1 *Usually in applications the initial field $\vec{H}_0(x)$ is decreasing at infinity as $|x|^{-(3+\alpha)}, \alpha > 0$, so that in the most important practical cases the condition of Theorem 5 is satisfied.*

If the initial function is not random, then Theorem 5 can be strengthened by assuming only the finiteness of the magnetic energy, i.e. by assuming that $|\vec{H}_0(x)| \in L^2(\mathbb{R}^3)$. Namely, the following is true.

Theorem 7 *If the initial field is nonrandom or independent of the fluid flow, from the finiteness of its magnetic energy follows an exponential, with probability 1, decrease of the magnetic field as $t \to \infty$: for some $\alpha > 0$ with probability 1*

$$\sup_x |\vec{H}(t,x)| = o(exp(-\alpha t)), \quad t \to \infty. \tag{32}$$

The proof of this theorem is similar to the proof of Theorem 5.

6 Discussion of the results

The main results of the work are presented in section 5. At the same time, at the beginning, in Section 5.4, a description of the Markov linear model considered in this paper is introduced and given. Note that such a model of the velocity field in the form (1) was first defined in the previous works of the first of the authors of this article. Such a representation of the velocity field in [7]- [8] made it possible to apply the theory of degenerate elliptic-parabolic Hermander operators to prove the existence of non-degenerate joint transition densities constructed according to the multiplicative integral (matrix) G_t of diffusion processes defined by formula (11), and, ultimately, to prove theorems of Furstenberg type (Theorem 1). This theorem plays the main role later in paragraph (5.) in the construction of Lyapunov bases for G_t.

In paragraph 5.2. the magnetic induction equation is considered within the introduced Markov linear model. Explicit integral representations (in the form of some functionals of the multiplicative integral G_t) of the desired solution $\vec{H}(t, x)$ of the magnetic induction equation (6) and its total magnetic energy ε_t (formulas (13) and (14), respectively) are obtained. the main problem was reduced to studying the asymptotic behavior of the multiplicative integral G_t and integrals (14) and (15) depending on it as $t \to \infty$.

Section 5.3 is devoted to the construction of Lyapunov (generally speaking, random) bases for G_t. At the same time, using the results known from the theory of matrices (decomposition of a matrix into a product of orthogonal and upper triangular matrices, etc.) and the central limit theorem, we first find the characteristic Lyapunov exponents (formulas (17)), prove a theorem on their simplicity (Theorem 3), and the Lyapunov bases corresponding to these $t(t \to \infty)$ exponents are defined (Definition1).

In the last subsection 5.2, we present and prove the main results of this paper, Theorems 5 and 6 In proving these theorems, the results of Theorem 4 (i.e., inequalities (27)) were essentially used. Theorems 5 and 6 actually mean that in the Markov linear model for large $t(t \to \infty)$ the exponential decrease of the magnetic field occurs with probability 1, however, its total magnetic energy grows exponentially throughout space. This property of the field, which at first glance seems paradoxical, can be explained simply: in the linear model, due to the increase in velocity, the volume of space occupied by the field rapidly increases, which entails an increase in the total magnetic energy.

The results obtained in this work are similar to the results of [5]- [6], where the problem of the evolution of a magnetic field in a random linear velocity field was also studied in a kinematic setting (i.e., for a given velocity field), but the authors of these papers had to deal with with the product of independent random matrices as the number of factors increases. Our paper covers a more general situation (than papers [5]- [6]), because we are studying the magnetic induction equation in a more general (Markovian) linear model, and we had to investigate the asymptotic behavior of a more general product - a multiplicative stochastic integral.

7 Conclusion

The work was devoted to the asymptotic analysis of the solution of the Cauchy problem for the induction equation in a given Markov linear velocity model. Explicit, containing (associated with a given velocity field) some multiplicative stochastic integral G_t of a Markov random matrix, integral representations for the magnetic field and its total energy were obtained. So-called Lyapunov bases are constructed and it is shown that, with an appropriate choice of the corresponding Lyapunov bases, G_t for large $t(t \to \infty)$ almost does not differ from the degree of some constant matrix. As a result, theorems were proved on the exponential decrease in the magnetic field at each point in space and the exponential growth of the total magnetic energy (over the entire space) in a given Markov linear model.

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