IRSTI 27.31.21 DOI: https://doi.org/10.26577/JMMCS20251262011

Shakir Aidos[1,2]
[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[2]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
e-mail: ajdossakir@gmail.com

AN INVERSE PROBLEM FOR PSEUDOPARABOLIC EQUATION WITH MEMORY TERM AND DAMPING

In this paper, we study the inverse problem of determining, along with solution $u(x,t)$ of a pseudo-parabolic equation with memory (convolution term) and a damping term, also an unknown coefficient $f(t)$ determining the external effect (the free term). In the investigating inverse problem, the overdetermination condition is given in integral form, which represents the average value of a solution tested with some given function over all the domain. By reducing the considering inverse problem to an equivalent nonlocal direct problem. The applicability of the Faedo-Galerkin method to the inverse problem is analyzed. The damping term $\gamma |u|^{q-2} u$ affects as nonlinear source in the case $\gamma > 0$, and an absorption, if $\gamma < 0$. In all these cases, we establish the conditions on the range of exponent q, the dimension d, and the data of the problem for the global and local in time existence and uniqueness of a weak solution of the studing problem.

Key words: inverse problem, nonlinear pseudoparabolic equation, memory term, solvability.

Шәкір Айдос[1,2]
[1]Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан
[2]Математика және математикалық моделдеу институты, Алматы, Қазақстан
e-mail: ajdossakir@gmail.com

Жады бар және сызықты емес мүшелі псевдопараболалық теңдеу үшін кері есеп

Бұл жұмыста жады мүшесі бар (үйірткі түріндегі) және сызықты емес мүшелі псевдопараболалық теңдеудің $u(x,t)$ шешімімен қатар сыртқы әсерді сипаттайтын (бос мүше) қосылғыштың $f(t)$ коэффициентін анықтау кері есебі зерттелінген. Қарастырылып отырған кері есепте қайта анықтау қосымша шарты интегралдық түрде берілген, ал ол өз кезегінде шешімнің орташа мәні туралы ақпарат береді. Берілген кері есепті эквивалентті локалды емес тура есепке келтіру арқылы шешімнің бар болуы Фаэдо-Галеркин әдісімен дәлелденді. Теңдеудегі сызықты емес $\gamma |u|^{q-2} u$ мүшесі $\gamma > 0$ жағдайда жылу көзі, $\gamma < 0$ жағдайда абсорбция қосылғыш ретінде қатысады. Сондай-ақ, q көрсеткішіне, d кеңістік өлшеміне және бастапқы берілген функцияларға жеткілікті шарттары негізінде кері есептің әлсіз шешімінің локалды және глобалды бар болуы, сонымен қатар әлсіз шешімінің жалғыздығы дәлелденді.

Түйін сөздер: кері есеп, сызықты емес псевдопараболалық теңдеу, интегралдық мүше, шешімділік.

Шакир Айдос[1,2]
[1]Казахский национальный университет имени аль-Фараби, Алматы, Казахстан
[2]Институт математики и математического моделирования, Алматы, Казахстан
e-mail: ajdossakir@gmail.com

Обратная задача для псевдопараболического уравнения с памятью и затуханием

В данной работе изучается обратная задача определения наряду с решением $u(x,t)$ псевдопараболического уравнения с памятью (членом свертки) и затухающим членом также неизвестного коэффициента $f(t)$, определяющего внешнее воздействие (свободный член). В исследуемой обратной задаче условие переопределения задается в интегральной форме, представляющей собой среднее значение решения, проверенного с некоторой заданной функцией по всей области. Путем сведения рассматриваемой обратной задачи к эквивалентной нелокальной прямой задаче анализируется применимость метода Фаэдо-Галеркина к обратной задаче. Затухающий член $\gamma |u|^{q-2} u$ действует как нелинейный источник в случае $\gamma > 0$ и как поглощение, если $\gamma < 0$.

Во всех этих случаях устанавливаются условия на диапазон изменения показателя q, размерность d и данные задачи для глобального и локального по времени существования и единственности слабого решения изучаемой задачи.
Ключевые слова: обратная задача, нелинейное псевдопараболическое уравнение, интегральный член, разрешимость.

1 Introduction

A coefficient inverse problems for differential equations have been called problems in which, together with the solution of the corresponding differential equation, it is also necessary to determine coefficient of the equation itself or the coefficient of the right-hand side (external influence). Such problems naturally arise in the mathematical modeling of physical, biological, etc. processes occurring in environments with previously unknown characteristics, since it is the characteristics of the environment that determine coefficient of the corresponding differential equation. This work devoted to study one of these kind of problem.

The statement of problem. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with smooth boundary $\partial\Omega$, and $Q_T = \{(x,t): \ x \in \Omega, \ 0 < t \leq T\}$ is a cylinder with lateral Γ_T. Let us consider the following inverse problem of finding the pair of functions $(u(x,t), f(t))$, which satisfy the pseudoparabolic equation with memory term and damping

$$u_t - \kappa\Delta u_t - \lambda\Delta u - \int\limits_0^t K(t-s)\Delta u(x,s)ds = \gamma\,|u|^{q-2}\,u + f(t)\cdot g(x,t), \quad \text{in} \quad Q_T, \tag{1}$$

the initial condition

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{2}$$

the boundary condition

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_T, \tag{3}$$

and the integral overdetermination condition

$$\int\limits_\Omega u(x,t)\omega(x)dx = h(t), \quad t \in [0,T]. \tag{4}$$

Here, the coefficient κ, λ are given positive numbers, γ is the coefficient of the damping term might be positive $\gamma > 0$ either negative $\gamma < 0$. The functions $g(x,t), \ u_0(x), \ \omega(x)$ and $h(t)$ are given. The exponent q is given positive number such that

$$1 < q < \infty. \tag{5}$$

Pseudo-parabolic equations can be used to describe various important physical processes, such as hydrodynamics, filtration theory, continuum mechanics, the heat conduction involving two temperature systems, dispersive, viscous flow in materials with memory and so on. One of the examples is furnished by the Kelvin-Voigt (Navier-Stokes-Voigt) equations. We refer the reader to the works [1–7] and the references therein, in which these issues were discussed in detail for the model equation (1).

In the absence of the memory term ($K(t) = 0$), the equation in (1) reduces to the pseudo-parabolic equation with damping. In corresponding equation if the coefficient of the external term $f(t)$ is given, then we will obtain an initial boundary value (IBV) problem. Various IBV problems for nonlinear pseudo-parabolic equation have been extensively studied in [4, 8–14] and results concerning existence and uniqueness of solution, and asymptotic behavior like blow up have been established. In the presence of the memory term ($K(t) \neq 0$), the various IBV problems have been considered and many results were obtained, such as the existence and uniqueness of classical and weak solutions, and finite-time blow up, asymptotic behavior of solutions in [15–19] and so on.

Next, we focus on the inverse problems posed for the pseudo-parabolic equation and their different modification. Since the pioneer works of [20–23] in the field of the inverse problem brought the authors international fame. In [24], a class of abstract pseudoparabolic equations of the form

$$\begin{cases} A_0 u_t(t) - A_1 u(t) = k * A_2 u(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

for the operators $A_j, j = 0, 1, 2$, were investigated. The main focus was to pay attention to the recoverving the kernel and finding solution in the Volterra operator integral equation. Lyubanova and coauthors in [25, 26] proved existence and uniqueness, regularity results for strong solutions to the pseudoparabolic equation of the operator form

$$u_t + \eta M u_t + k(t) M u_t + g(x, t) u = f,$$

where M is a linear differential operator of the second order in the space variables. Yaman [27] discussed the coefficient inverse problem for Eq.(1) with $K(t) = 0$ and the special external source term

$$F(x, t) := f(t) \left(\omega - \Delta\omega\right), \tag{6}$$

where the test function ω replaced by $\omega - \Delta\omega$ in the overdetermination condition (4). It may restrict the statement of the problem from both of mathematical and physical view, he derived the upper bound for the blow-up time under some assumptions about the initial data. The equation consisting relation between the damping term and p-Laplacian was considered by Antontsev and et. [29] with the special right-hand side and overdetermination condition such as in [27]. The authors proved in [29] the local existence of weak solution (without the uniqueness). This work was later improved by Khompysh et al. [30] established global and local in time existence and uniqueness result. Recently, Aitzhanov and et. in [28] considered Eq. (1) with $\gamma = b(x, t)$ variable coefficients and instead of (4) assumed overdetermination condition (6). The authors showed that existence and uniqueness of weak and strong solutions under certain conditions and initial data of the corresponding inverse problem. In the present work overdetermination condition (4) cause some difficulties, thus author has to develop other techniques to overcome these difficulties. Similar problems for the equation (1) were studied earlier articles of the author [34–36].

The present paper is organized as follows. In Section 2, we introduce some auxiliary lemmas that we use in this work. In Section 3, we prove that the initial inverse problem (1)-(4) is equivalent to the direct problem (15)-(18) containing the nonlinear nonlocal operator of u. The global and local in time existence of a weak solution to the direct problem (15)-(18) is established in Section 4 in the case $\gamma > 0$, and in Section 5 in the case $\gamma \leq 0$. For that we construct Galerkin's approximations u^n and derive their priori estimates. Next using compactness arguments we realize a passage to the limit as $n \to \infty$. The Section 6 is devoted to the study the uniqueness of the weak solution to the problem (15)-(18) in both case of $\gamma > 0$ and $\gamma \leq 0$.

2 Preliminaries

In this section, we introduce some auxiliary lemmas that will be used throughout the paper. For the definitions, notations of the function spaces and for their properties, we address the reader to the monographs [31,32]. In particular, the norm in the Lebesgue spaces $L^p(\Omega)$ and $L^p(Q_T)$ are denoted as follows, respectively:

$$\|u\|_{p,\Omega} \equiv \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}, \quad \|u\|_{p,Q_T} \equiv \left(\int_0^T \int_\Omega |u(x,t)|^p dxdt\right)^{\frac{1}{p}}.$$

We use the classical and the following nonlinear Gronwall's inequality ([31]) to establish the first and second local estimates.

Lemma 1 *If $y : \mathbb{R}^+ \longrightarrow [0,\infty)$ is a continuous function such that*

$$y(t) \leq C_1 \int_0^t y^\mu(s)ds + C_2, \quad t \in \mathbb{R}^+, \quad \mu > 1$$

for some positive constants C_1 and C_2, then

$$y(t) \leq C_2 \left(1 - (\mu-1)C_1 C_2^{\mu-1} t\right)^{-\frac{1}{\mu-1}} \quad \text{for } \ 0 \leq t < t_{\max} := \frac{1}{(\mu-1)C_1C_2^{\mu-1}}.$$

The following another very important auxiliary lemma (see [33] (Lemma 2.2., p. 1809.) will be used to prove the uniqueness and passage to the limit in the Galerkin approximation.

Lemma 2 *For all $p \in (1,\infty)$ and $\delta \geq 0$, there exist constants C_1 and C_2, depending on p and d, such that for all $\xi,\ \eta \in \mathbb{R}^d,\ d \geq 1$, it*

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leq C_1|\xi-\eta|^{1-\delta}(|\xi|+|\eta|)^{p-2-\delta} \tag{7}$$

and

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi-\eta) \geq C_2|\xi-\eta|^{2+\delta}(|\xi|+|\eta|)^{p-2+\delta}. \tag{8}$$

3 Weak formulation.

Assume that the data of the problem satisfy the following conditions

$$u_0(x) \in W_0^{1,2}(\Omega) \cap L^q(\Omega), \tag{9}$$

$$|g_0(t)| := \left| \int\limits_\Omega g(x,t)\omega(x)\,dx \right| \geq l_0 > 0 \quad \text{for all} \quad t \geq 0, \tag{10}$$

$$g(x,t) \in C(0,T; L^2(\Omega)), \tag{11}$$

$$h(t) \in W^{1,2}([0,T]) \text{ and } \int\limits_\Omega u_0(x)\omega(x)dx = h(0), \tag{12}$$

$$K(t) \in L^2([0,T]), \tag{13}$$

$$\omega(x) \in W_0^{1,2}(\Omega) \cap L^q(\Omega). \tag{14}$$

Lemma 3 *Under the conditions (10) and (12)-(14), the inverse problem (1)-(4) is equivalent to the following problem for a nonlinear pseudoparabolic equation with nonlinear nonlocal operator of the function u*

$$u_t - \kappa\Delta u_t - \lambda\Delta u - \int\limits_0^t K(t-s)\Delta u(x,s)ds = \gamma\,|u|^{q-2}\,u + f(t,u)\cdot g(x,t), \quad in \quad Q_T, \tag{15}$$

$$u(x,0) = u_0(x), \; x \in \Omega, \tag{16}$$

$$u(x,t) = 0 \quad on \quad \Gamma_T, \tag{17}$$

where

$$f(t,u) = \frac{1}{g_0(t)}\left(h'(t) + \kappa\int\limits_\Omega \nabla u_t \nabla\omega dx + \lambda\int\limits_\Omega \nabla u\nabla\omega dx + \int\limits_0^t K(t-s)(\nabla u, \nabla\omega)_{2,\Omega}ds - \gamma\int\limits_\Omega |u|^{q-2}\,u\omega dx \right) \tag{18}$$

Proof 1 *1. Let the pair $(u(x,t), f(t))$ be a solution of the inverse problem (1)-(4). Multiplying both sides of (1) by ω, and integrating over Ω and applying the formula of integrating by parts, we have*

$$\int_\Omega u_t \omega dx + \kappa \int_\Omega \nabla u_t \nabla \omega dx + \lambda \int_\Omega \nabla u \nabla \omega dx + \int_0^t K(t-s) \int_\Omega \nabla u(s) \nabla \omega dx ds = \gamma \int_\Omega |u|^{q-2} u \cdot \omega dx + f(t) \int_\Omega g(x,t) \omega dx. \tag{19}$$

Using

$$\int_\Omega u_t \omega dx = h'(t) \tag{20}$$

which follows from the overdetermination condition (1), and the assumption (10), we get from (19) the equality (18).

2. Let now $u(x,t)$ be a solution to the direct problem (15)-(17) with (18). It means that the pair of functions (u, f) is satisfied the equations (1)-(3). Thus, the pair (u, f) to be a solution of the inverse problem (1)-(4) it is sufficient to prove that the function $u(x,t)$ satisfies the overdetermination condition (4). Let us assume that for contradiction, i.e. the overdetermination condition (4) doesn't hold. Suppose that

$$\int_\Omega u \omega dx = h_1(t), \quad t \geq 0 \tag{21}$$

where $h_1(t) \neq h(t)$ for all $t \geq 0$. Thus, by the conditions (4) and (12), we have $h_1(t) \in W_2^1([0,T])$ and

$$h_1(0) = \int_\Omega u_0 \omega dx = h(0) \tag{22}$$

Multiply (15) by ω and integrating by parts and using (18),we get

$$h_1'(t) + \kappa \int_\Omega \nabla u_t \cdot \nabla \omega dx + \lambda \int_\Omega \nabla u \cdot \nabla \omega dx + \int_0^t K(t-s)(\nabla u, \nabla \omega)_{2,\Omega} ds - \gamma \int_\Omega |u|^{q-2} u \cdot \omega dx = f(t,u) g_0(t), \tag{23}$$

where $f(t,u)$ is defined in (18). Plugging (18) into (23), we obtain

$$h_1'(t) + \kappa \int_\Omega \nabla u_t \, \nabla \omega dx + \lambda \int_\Omega \nabla u \, \nabla \omega dx + \int_0^t K(t-s)(\nabla u, \nabla \omega)_{2,\Omega} ds - \gamma \int_\Omega |u|^{q-2} u \, \omega dx =$$

$$h'(t) + \kappa \int_\Omega \nabla u_t \, \nabla \omega dx + \lambda \int_\Omega \nabla u \, \nabla \omega dx + \int_0^t K(t-s)(\nabla u, \nabla \omega)_{2,\Omega} ds - \gamma \int_\Omega |u|^{q-2} u \, \omega dx.$$

(24)

(24) implies that the following Cauchy problem for $H(t)=h_1(t)-h(t)$:

$$H'(t)=0,\ H(0)=h_1(0)-h(0)=0 \tag{25}$$

which yields that $h_1(t)\equiv h(t)$ for all $t>0$.

Definition 1 *A function $u(x,t)$ is a weak solution to the problem (15)-(18), if:*

1. $u\in L^\infty(0,T;W_0^{1,2}(\Omega)\cap L^q(\Omega)), u_t\in L^2(0,T;W_0^{1,2}(\Omega))$.
2. $u(0)=u_0$ *a.e. in* Ω
3. *The following identity*

$$\frac{d}{dt}\int\limits_\Omega(u\varphi+\kappa\nabla u\nabla\varphi)dx+\lambda\int\limits_\Omega\nabla u\nabla\varphi dx+\int\limits_0^t K(t-s)(\nabla u,\nabla\varphi)_{2,\Omega}ds= \gamma\int\limits_\Omega|u|^{q-2}u\varphi dx+\int\limits_\Omega f(t,u)g\varphi dx \tag{26}$$

holds for every $\varphi\in W_0^{1,2}(\Omega)\cap L^q(\Omega)$ and for a.a. $t\in[0,T]$.

4 Global and local existence: a nonlinear source case.

In this section we consider the problem (15)-(18). Let

$$1<q<\infty. \tag{27}$$

Now we present our mean result for

$$\gamma>0 \tag{28}$$

First we state the global existence theorem.

Theorem 1 *(Global existence) Let the conditions (9)-(14), (28) are fulfilled and assume, that*

$$1<q\le 2 \tag{29}$$

Also exists a positive constant m such that

$$\frac{\kappa}{l_0^2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\|\nabla\omega\|_{2,\Omega}^2\le m<2. \tag{30}$$

Then there exists a global weak solution to the problem (15)-(18) in the sense of Definition 1. Moreover, the weak solutions satisfy the following estimates

$$\sup_{t\in[0,T]}\left(\|\nabla u(t)\|_{2,\Omega}^2+\|u(t)\|_{q,\Omega}^q\right)+\|u_t\|_{2,Q_T}^2+\|\nabla u_t\|_{2,Q_T}^2\le C, \tag{31}$$

where C is positive constant depending on data of the problem.

In this theorem we establish local existence of the weak solution to the problem (15)-(18).

Theorem 2 *(Local existence) Let the conditions (9)-(14), (28), (30) are fulfilled and assume that the following condition holds*

$$2 < q \leq 2^*, 2^* = \frac{2d}{d-2} \text{ if } d > 2; 2^* = (1, \infty) \text{ if } d = 2 \tag{32}$$

Then there exists a time $T_1 \in (0, T)$ defined at (49), below such that the problem (15)-(18) has, at least, a weak solution $u(x,t)$ in the sense of Definition 1, with T_1 instead of T. Moreover, these weak solutions satisfy the estimate (31) for all $t \in (0, T_1]$ with another positive constant C depending on data of the problem.

Remark 1 *The condition (32) assures the passage to the limit as $n \to \infty$ below, see (4.4). We have assumed that the condition (32) is fulfilled, because we return to the statement of the 1 in case $q \leq 2$.*

Proof 2 *The proof of these theorems consists of the steps: construction of Galerkin's approximations; obtain energy estimates; passage to limit.*

4.1 Galerkin's approximations.

Let $\{\psi_k\}_{k \in N}$ be an orthonormal family in $L^2(\Omega)$ and their linear combinations are dense in $V := W_0^{1,2}(\Omega) \cap L^q(\Omega)$. Given $n \in N$, let us consider the n-dimentional space V^n spanned by $\psi_1, ..., \psi_n$. for each $n \in N$, we search for approximate solutions

$$u^n(x,t) = \sum_{j=1}^{n} c_j^n(t)\psi_j(x), \quad \psi_j \in V^n, \tag{33}$$

where the coefficients $c_1^n(t), ..., c_n^n(t)$ are defined as the solutions of the following n ordinary differential equations derived from

$$\begin{aligned} \int_\Omega (u_t^n \psi_k + \kappa \nabla u_t^n \nabla \psi_k)\, dx + \lambda \int_\Omega \nabla u^n \nabla \psi_k \, dx + \int_0^t K(t-s)(\nabla u^n, \nabla \psi_k)_{2,\Omega} ds = \\ \gamma \int_\Omega |u^n|^{q-2} u^n \cdot \psi_k \, dx + f(t, u^n) \int_\Omega g \psi_k dx \end{aligned} \tag{34}$$

for $k = 1,\ 2,\ \ldots,\ n$. The system (34) of ODEs is supplemented with the following Cauchy data

$$u^n(0) = u_0^n \quad \text{in} \quad \Omega \tag{35}$$

and assume that

$$u_0^n \to u_0(x) \text{ as } n \to \infty \text{ in } W_0^{1,2}(\Omega) \cap L^q(\Omega). \tag{36}$$

According to the general theory of nonlinear ODE, the problem (34)-(35) has a solution $c_j^n(t)$ in $[0, t_0]$, where $t_0 \in (0, T]$. The solution can be extended to $[0, T]$ by a priory estimate which we shall obtain below.

4.2 Global priori estimates

Let us consider the case (28). In this case we obtain the global a priori estimates.

Proof 3 *Multiplying both sides of (34) by* $\frac{dc_k^n(t)}{dt}$*, and summing on* k*, and adding* $\frac{1}{\gamma}\frac{d}{dt}\left\|u^n\right\|_{q,\Omega}^q$ *on both side, we have*

$$\frac{d}{dt}\left(\frac{\lambda}{2}\left\|\nabla u^n\right\|_{2,\Omega}^2+\frac{\gamma}{q}\left\|u^n\right\|_{q,\Omega}^q\right)+\left\|u_t^n\right\|_{2,\Omega}^2+\kappa\left\|\nabla u_t^n\right\|_{2,\Omega}^2=\frac{2\gamma}{q}\frac{d}{dt}\left\|u^n\right\|_{q,\Omega}^q+I_1+I_2 \tag{37}$$

where

$$I_1=-\int_0^t K(t-s)(\nabla u^n(s),\nabla u_t^n(t))_{2,\Omega}ds, \tag{38}$$

$$I_2=\frac{1}{g_0(t)}\left(h'(t)+\kappa\int_\Omega \nabla u_t^n\cdot\nabla\omega dx+\lambda\int_\Omega \nabla u^n\cdot\nabla\omega\ dx+\right.$$
$$\left.\int_0^t K(t-s)(\nabla u^n,\nabla\omega)_{2,\Omega}ds-\gamma\int_\Omega |u^n|^{q-2}u^n\cdot\omega dx\right)(g(t),u_t^n(t))_{2,\Omega}\,. \tag{39}$$

Now we estimate each term on the right hand side of (37) by using the Hölder and Cauchy inequalities with ε_0 *together with the assumptions in* (29)*. Thus, we have*

$$\frac{2\gamma}{q}\frac{d}{dt}\left\|u^n\right\|_{q,\Omega}^q=2\gamma\int_\Omega |u|^{q-1}u_t dx\le 2\gamma\left\|u_t^n\right\|_{2,\Omega}\left\|u^n\right\|_{2(q-1),\Omega}^{q-1}\le$$
$$\frac{\varepsilon_0}{2}\left\|u_t^n\right\|_{2,\Omega}^2+C(\varepsilon_0,\gamma,q)\left\|u^n\right\|_{2(q-1),\Omega}^{2(q-1)}\le\frac{\varepsilon_0}{2}\left\|u_t^n\right\|_{2,\Omega}^2+C(\varepsilon_0,\gamma,q,\Omega)\left(\left\|\nabla u^n\right\|_{2,\Omega}^2\right)^{q-1}. \tag{40}$$

Also the Cauchy inequality with some ε*, gives us*

$$|I_1|\le\int_0^t |K(t-s)|\left\|\nabla u^n(s)\right\|_{2,\Omega}\left\|\nabla u_t^n\right\|_{2,\Omega}ds\le\frac{\varepsilon_1}{2}\left\|\nabla u_t^n\right\|_{2,\Omega}^2+\frac{K_0^2}{2\varepsilon_1}\int_0^t\left\|\nabla u^n\right\|_{2,\Omega}^2 ds. \tag{41}$$

Exploiting the Hölder integral inequality and Cauchy inequality with $\varepsilon_2>0$*, we estimate the third term* i_2 *in (37)*

$$|I_2|\le\frac{1}{l_0}\left(|h'(t)|+\kappa\left\|\nabla u_t^n\right\|_{2,\Omega}\left\|\nabla\omega\right\|_{2,\Omega}+\lambda\left\|\nabla u^n\right\|_{2,\Omega}\left\|\nabla\omega\right\|_{2,\Omega}+\right.$$
$$\left.\int_0^t |K(t-s)|\left\|\nabla u^n(s)\right\|_{2,\Omega}\left\|\nabla\omega\right\|_{2,\Omega}ds+\gamma\left\|\omega\right\|_{q,\Omega}\left\|u^n\right\|_{q,\Omega}^{q-1}\right)\left\|g\right\|_{2,\Omega}\left\|u_t^n\right\|_{2,\Omega}dt\le$$
$$\frac{\varepsilon_2}{4}\int_0^t\left\|u_t^n\right\|_{2,\Omega}^2d\tau+\frac{\kappa^2}{l_0^2\varepsilon_2}\left\|\nabla\omega\right\|_{2,\Omega}^2\sup_{t\in[0,T]}\left\|g(t)\right\|_{2,\Omega}^2\left\|\nabla u_t^n\right\|_{2,\Omega}+\frac{\varepsilon_2}{4}\left\|u_t^n\right\|_{2,\Omega}^2+$$

$$\begin{aligned}
&\frac{1}{l_0^2\varepsilon_2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\Big(|h'(t)|^2+\lambda^2\|\nabla u^n\|_{2,\Omega}^2\|\nabla\omega\|_{2,\Omega}^2+\\
&K_0^2\|\nabla\omega\|_{2,\Omega}^2\int_0^t\|\nabla u^n(s)\|_{2,\Omega}^2\,ds+\gamma^2\|\omega\|_{q,\Omega}^2\|u^n\|_{q,\Omega}^{2(q-1)}\Big)\le\\
&\frac{\varepsilon_2}{2}\|u_t^n\|_{2,\Omega}^2+\frac{\kappa^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\|\nabla u_t^n\|_{2,\Omega}^2+\frac{1}{l_0^2\varepsilon_2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\big(|h'(t)|^2+\\
&\lambda^2\|\nabla u^n\|_{2,\Omega}^2\|\nabla\omega\|_{2,\Omega}^2+K_0^2\|\nabla\omega\|_{2,\Omega}^2\int_0^t\|\nabla u^n(s)\|_{2,\Omega}^2\,ds+C\gamma^2\|\omega\|_{q,\Omega}^2\|\nabla u^n\|_{2,\Omega}^{2(q-1)}\Big).
\end{aligned}\tag{42}$$

Plugging (40)-(42) into (37), we get

$$\begin{aligned}
&\frac{d}{dt}\left(1+\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)+\alpha\|u_t^n\|_{2,\Omega}^2+\beta\|\nabla u_t^n\|_{2,\Omega}^2\le\\
&C_1\|\nabla u^n\|_{2,\Omega}^2+C_2\left(\|\nabla u^n\|_{2,\Omega}^2\right)^{q-1}+C_3\int_0^t\|\nabla u^n(s)\|_{2,\Omega}^2\,ds+C_4\le\\
&C_1\left(1+\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)+C_2\left(1+\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)^{q-1}+\\
&C_3\int_0^t\left(1+\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)ds+C_4,
\end{aligned}\tag{43}$$

where $\alpha:=1-\frac{\varepsilon_0+\varepsilon_2}{2}$; $\quad\beta:=\kappa-\frac{\varepsilon_1}{2}-\frac{\kappa^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2$;

$$C_1:=\frac{\lambda^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2,\ C_2:=\frac{K_0^2}{2\varepsilon_1}+\frac{K_0^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2,$$

$$C_3:=C(\varepsilon_0,\gamma,q,\Omega)+\frac{C\gamma^2}{l_0^2\varepsilon_2}\|\omega\|_{q,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2,\ C_4:=\frac{1}{l_0^2\varepsilon_2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2|h'(t)|^2.$$

Now we choose $\varepsilon_i,\ i=0,1,2$ *such that* $\alpha,\beta>0$, *and* $C_j,\ j=1,2,3,4$ *to be finite. It is possible by (30).*

However $\varepsilon_1,\varepsilon_2$ *cannot be chosen such that* $m>2$, *because* $\varepsilon_2<2$ *due to* $\alpha>0$. *Thus, choosing* $\varepsilon_i,\ i=0,1,2$ *with suitable values and in case* $q-1\le 1$ *integrating (43) with respect to* τ *from* 0 *to* t *and using (36), we obtain*

$$y(t)+\int_0^t\left(\alpha\|u_\tau^n\|_{2,\Omega}^2+\beta\|\nabla u_\tau^n\|_{2,\Omega}^2\right)d\tau\le C_5\int_0^t y(\tau)d\tau+C_6,\tag{44}$$

where $y(t) := 1 + \frac{\lambda}{2} \|\nabla u^n\|_{2,\Omega}^2 + \frac{\gamma}{q} \|u^n\|_{q,\Omega}^q$,

$$C_5 := C_1 + C_2 + C_3 T; \quad C_6 := C_4 T + \frac{\lambda}{2} \|\nabla u_0\|_{2,\Omega}^2 + \frac{\gamma}{q} \|u_0\|_{q,\Omega}^q .$$

Omitting the integral terms on left hand side and applying classical Grönwall's lemma, inequality (44) implies that

$$y(t) \le C_6 e^{C_5 T}. \tag{45}$$

Thus, substituting (45) into (44) and taking suppremum, we obtain the estimate (31)

$$\sup_{t\in[0,T]} \left(\frac{\lambda}{2} \|\nabla u^n\|_{2,\Omega}^2 + \frac{\gamma}{q} \|u^n\|_{q,\Omega}^q \right) + \int_0^t \left(\alpha \|u_\tau^n\|_{2,\Omega}^2 + \beta \|\nabla u_\tau^n\|_{2,\Omega}^2 \right) ds \le M_1 < +\infty, \tag{46}$$

where $M_1 := M_1(T, C_5, C_6)$.

4.3 Local a priori estimates

Let now be $q \le 2^*$. In this case we obtain the local a priori estimates.

Proof 4 *Choosing* $\varepsilon_i, i = 0, 1, 2$ *with suitable values as we did in obtaining a priory estimates above and in case* $q - 1 > 1$ *integrating (43) with respect to* $\tau \in (0, t)$ *and using (36), we have*

$$z(t) + \int_0^t \left(\alpha \|u_\tau^n\|_{2,\Omega}^2 + \beta \|\nabla u_\tau^n\|_{2,\Omega}^2 \right) d\tau \le C_5 \int_0^t z^{q-1}(\tau) d\tau + C_6. \tag{47}$$

Omitting the second and third terms on left hand side (47) and applying Grönwall's Lemma 1, we obtain

$$z(t) \le C_6 \left[1 - (q-2) C_5 C_6^{q-2} t \right]^{-\frac{1}{q-2}} \tag{48}$$

for

$$0 \le t < T_1 := \frac{1}{(q-2) C_5 C_6^{q-2}}. \tag{49}$$

Using (48) *and maximizing* (47) *by* $t \in (0, T_1]$, *we have*

$$\sup_{t\in[0,T_1]} \left(\frac{\lambda}{2} \|\nabla u^n\|_{2,\Omega}^2 + \frac{\gamma}{q} \|u^n\|_{q,\Omega}^q \right) + \alpha \|u_t^n\|_{2,Q_{T_1}}^2 + \beta \|\nabla u_t^n\|_{2,Q_{T_1}}^2 \le M_2 < \infty, \tag{50}$$

where $M_2 := M_2(T_1, C_5, C_6)$.

4.4 Passage to the limit

By means of reflexivity and up to some subsequences, the estimate (31) implies that

$$u^n \rightharpoonup u \quad \text{weakly-}* \text{ in } L^\infty(0,T;W_0^{1,2}(\Omega)\cap L^q(\Omega)) \quad \text{as } n\to\infty, \tag{51}$$

$$u_t^n \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;W_0^{1,2}(\Omega)) \quad \text{as } n\to\infty. \tag{52}$$

On the other hand, (31) implies the existence of function R such that

$$|u^n|^{q-2}u^n \rightharpoonup R \quad \text{weakly in } L^{q'}(Q_T), \quad \text{as } n\to\infty, \tag{53}$$

where $q' = \dfrac{q}{q-1}$ is the Hölder conjugate of q. By the compact and continuous embedding

$$W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega) \hookrightarrow L^2(\Omega), \qquad \forall\, r: 2\le r<2^*$$

and by Aubin-Lions compactness lemma, (51) and (52) imply that

$$u^n \longrightarrow u \quad \text{strongly in } L^2(0,T;L^r(\Omega)), \;\; 2\le r<2^* \;\; \text{as } n\to\infty, \tag{54}$$

and in particular,

$$u^n \longrightarrow u \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \quad \text{as } n\to\infty, \tag{55}$$

where 2^* is the Sobolev conjugate of 2, i.e. $2^* = \frac{2d}{d-2}$ with $d>2$.

As a consequence of (55) and Riesz-Fischer's theorem, we have up to some subsequence,

$$u^n \longrightarrow u \quad \text{a.e. in } Q_T \quad \text{as } n\to\infty, \tag{56}$$

which together with (53) yields (see Lemma 1.3 in [32, p. 12])

$$|u^n|^{q-2}u^n \rightharpoonup |u|^{q-2}u \quad \text{weakly in } L^{q'}(Q_T), \quad \text{as } n\to\infty. \tag{57}$$

Under the assumption (32) and (52), (54), we have also that

$$u^n \longrightarrow u \quad \text{strongly in} \quad L^q(Q_T) \quad \text{as} \quad n\to\infty, \quad \text{for} \quad q<2^*$$

and consequently

$$\|u^n\|_{q,Q_t} \longrightarrow \|u\|_{q,Q_t} \quad \text{as} \quad n\to\infty. \tag{58}$$

Let $\eta(t)$ be a continuously differentiable function on $[0,T]$, where T is the maximal time such that above first and second estimates are hold. Multiplying (34) by η and integrating by $t\in[0,T]$, we obtain

$$\begin{aligned}
&\int\limits_{Q_T}(u_t^n z_k+\kappa\nabla u_t^n\nabla z_k)dxdt+\lambda\int\limits_{Q_T}\nabla u^n\nabla z_k\,dxdt+\int\limits_0^T\int\limits_0^t K(t-s)(\nabla u^n,\nabla z_k)_{2,\Omega}dsdt=\\
&\gamma\int\limits_{Q_T}|u^n|^{q-2}u^n z_k\,dxdt+\int\limits_0^T\frac{1}{g_0(t)}\Bigg(h'(t)+\kappa\int\limits_\Omega\nabla u_t^n\nabla\omega\;dx+\lambda\int\limits_\Omega\nabla u^n\nabla\omega\;dx+\\
&\int\limits_0^t K(t-s)(\nabla u^n,\nabla\omega)_{2,\Omega}ds-\gamma\int\limits_\Omega|u^n|^{q-2}u^n\omega dx\Bigg)\int\limits_\Omega g z_k\,dxdt
\end{aligned} \tag{59}$$

and using above convergence results (51), (52) and (57), we obtain

$$\int\limits_{Q_T}(u_t z_k+\kappa\nabla u_t\nabla z_k)dxdt+\lambda\int\limits_{Q_T}\nabla u\nabla z_k\,dxdt+\int\limits_0^T\int\limits_0^t K(t-s)(\nabla u,\nabla z_k)_{2,\Omega}dsdt=$$

$$\gamma\int\limits_{Q_T}|u|^{q-2}uz_k\,dxdt+\int\limits_0^T\frac{1}{g_0(t)}\left(h'(t)+\kappa\int\limits_\Omega\nabla u_t\nabla\omega\;dx+\lambda\int\limits_\Omega\nabla u\nabla\omega\;dx+\right. \tag{60}$$

$$\left.\int\limits_0^t K(t-s)(\nabla u,\nabla\omega)_{2,\Omega}ds-\gamma\int\limits_\Omega|u|^{q-2}u\omega dx\right)\int\limits_\Omega gz_k\,dxdt$$

for all $z_k=\psi_k(x)\eta(t),\quad k\in\{1,\dots,n\}$. By linearity and by a continuity argument, the equation (60) is still true for any

$$z\in Z:=\{z=\psi\zeta:\psi\in\mathcal{V},\;\zeta\in C_0^\infty(0,T)\}.$$

5 Global and local existence: an absorption case.

In this section we present existence of weak solution to the problem (15)-(18) for

$$\gamma\le 0. \tag{61}$$

For existence of the weak solution the following theorems hold.

Theorem 3 *(Global existence) Let the conditions (9)-(14), (29), (30), (61) are fulfilled. Then there exists a global weak solution to the problem (15)-(18) in the sense of Definition 1. Moreover, the weak solutions satisfy the estimate (31) for all $t\in[0,T]$ with another positive constant C depending on data of the problem.*

Theorem 4 *(Local existence) Let the conditions (9)-(14), (30), (32), (61) are fulfilled. Then there exists $T_2\in(0,T]$ and at least one weak solution to the problem (15)-(18) in the sense of Definition 1 and satisfies the estimate (31) in Q_{T_2}, where T_2 depending on data of the problem.*

Proof 5 *The proof of Theorems 3 and 4 are similar to the Theorems 1 and 2.*

6 Uniqeness

Theorem 5 *Assume that the following conditions*

$$\nabla\omega\in L^2(\Omega), \tag{62}$$

$$2\le q\le\frac{2d}{d-2},\;d>2 \tag{63}$$

hold. Moreover, there exists a positive constant m such that

$$\frac{\kappa}{l_0^2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\|\nabla\omega\|_{2,\Omega}^2\le m<2. \tag{64}$$

If $\gamma \le 0$, assume addition to (62)-(63) that all conditions of Theorems 1 and 2 are fulfilled. If $\gamma > 0$, assume addition to (62)-(63) that all conditions of Theorem 3 and 4 are fulfilled.

Then the weak solution of (15)-(18) is unique.

Proof 6 *Let u_1 and u_2 be two weak solutions to the problem (15)-(18) in the sense of Definition 1. Using $\partial_t u := \partial_t u_1 - \partial_t u_2$ as a test function in (26), it follows, by subtracting the equation for u_2 to the equation for u_1, that*

$$\frac{\lambda}{2}\frac{d}{dt}\|\nabla u\|_{2,\Omega}^2+\|u_t\|_{2,\Omega}^2+\kappa\|\nabla u_t\|_{2,\Omega}^2=D+G+F, \tag{65}$$

where

$$D=-\int_0^t K(t-s)(\nabla u(s),\nabla u_t(t))_{2,\Omega}ds, \tag{66}$$

$$G=\gamma\int_\Omega\left(|u_1|^{q-2}u_1-|u_2|^{q-2}u_2\right)\cdot u_t\,dx, \tag{67}$$

$$\begin{aligned}F=&\frac{1}{g_0(t)}\Bigg(\kappa\int_\Omega\nabla u_t\cdot\nabla\omega\,dx+\lambda\int_\Omega\nabla u\cdot\nabla\omega\,dx+\int_0^t K(t-s)(\nabla u,\nabla\omega)_{2,\Omega}ds\\&-\gamma\int_\Omega\left(|u_1|^{q-2}u_1-|u_2|^{q-2}u_2\right)\cdot\omega\,dx\Bigg)\int_\Omega g(x,t)u\,dx.\end{aligned} \tag{68}$$

Using Hölder's and Minkovskii inequalities and (7) in Lemma 2 with $\delta = 0$, we estimate D, G and F

$$|D|\le\frac{\varepsilon_0}{2}\|\nabla u_t\|_{2,\Omega}^2+\frac{K_0^2}{2\varepsilon_0}\int_0^t\|\nabla u\|_{2,\Omega}^2ds, \tag{69}$$

$$\begin{aligned}|G|=&\left|\gamma\int_\Omega\left(|u_1|^{q-2}u_1-|u_2|^{q-2}u_2\right)u_t dx\right|\le|\gamma|\int_\Omega|u|\left(|u_1|+|u_2|\right)^{q-2}|u_t|\,dx\le\\&|\gamma|\,\|u\|_{2^*,\Omega}\,\||u_1|+|u_2|\|_{\frac{(q-2)d}{2},\Omega}^{q-2}\|u_t\|_{2^*,\Omega}\le|\gamma|C^2\left(\|\nabla u_1\|_{2,\Omega}+\|\nabla u_2\|_{2,\Omega}\right)^{q-2}\times\\&\|\nabla u\|_{2,\Omega}\|\nabla u_t\|_{2,\Omega}\le\frac{\varepsilon_2}{2}\|\nabla u_t\|_{2,\Omega}^2+\frac{\lambda}{2}b_0\|\nabla u\|_{2,\Omega}^2,\ q\le\frac{2d}{d-2}\end{aligned} \tag{70}$$

where $b_0 := \frac{2|\gamma|^2 C^2 C^{q-2}}{\lambda}$.

$$|F| \le \frac{1}{l_0} \|g\|_{2,\Omega} \|u_t\|_{2,\Omega} \Big(\kappa \|\nabla u_t\|_{2,\Omega} \|\nabla \omega\|_{2,\Omega} + \lambda \|\nabla u\|_{2,\Omega} \|\nabla \omega\|_{2,\Omega} +$$

$$\int_0^t K(t-s) \|\nabla u\|_{2,\Omega} \|\nabla \omega\|_{2,\Omega} ds + |\gamma| \int_\Omega \left(|u| \left(|u_1| + |u_2| \right)^{q-2} |\omega| \, dx \right) \Big) \le$$

$$\frac{1}{l_0} \|g\|_{2,\Omega} \|u_t\|_{2,\Omega} \Big[\kappa \|\nabla u_t\|_{2,\Omega} \|\nabla \omega\|_{2,\Omega} + \lambda \|\nabla u\|_{2,\Omega} \|\nabla \omega\|_{2,\Omega} +$$

$$\int_0^t K(t-s) \|\nabla u\|_{2,\Omega} \|\nabla \omega\|_{2,\Omega} ds + |\gamma| \|u\|_{2^*,\Omega} \left(\|u_1 + u_2\|_{\frac{(q-2)d}{2},\Omega} \right)^{q-2} \|\omega\|_{2^*,\Omega} \Big] \le$$

$$\frac{\varepsilon_1}{2} \|u_t\|_{2,\Omega}^2 + \frac{\kappa^2}{4 l_0^2 \varepsilon_1} \|\nabla \omega\|_{2,\Omega}^2 \sup_{t \in [0,T]} \|g\|_{2,\Omega}^2 \|\nabla u_t\|_{2,\Omega}^2 + \frac{1}{4 l_0^2 \varepsilon_1} \|\nabla \omega\|_{2,\Omega}^2 \sup_{t \in [0,T]} \|g\|_{2,\Omega}^2 \times \tag{71}$$

$$\left(\lambda^2 \|\nabla u\|_{2,\Omega}^2 + K_0^2 \int_0^t \|\nabla u\|_{2,\Omega}^2 ds + \gamma^2 C^2 \left(\|\nabla u_1\|_{2,\Omega} + \|\nabla u_2\|_{2,\Omega} \right)^{q-2} \|\nabla u\|_{2,\Omega}^2 \right) \le$$

$$\frac{\varepsilon_1}{2} \|u_t\|_{2,\Omega}^2 + \frac{\kappa^2}{4 l_0^2 \varepsilon_1} \|\nabla \omega\|_{2,\Omega}^2 \sup_{t \in [0,T]} \|g\|_{2,\Omega}^2 \|\nabla u_t\|_{2,\Omega}^2 + \frac{\lambda}{2} b_1 \|\nabla u\|_{2,\Omega}^2 + \frac{\lambda}{2} b_2 \int_0^t \|\nabla u\|_{2,\Omega}^2 ds,$$

where

$$q \le \frac{2d}{d-2}, \frac{(q-2)d}{2} \le \frac{2d}{d-2} := 2^* \Leftrightarrow q \le \frac{2d}{d-2},$$

$$b_1 = \frac{1}{2\lambda l_0^2 \varepsilon_1} \|\nabla \omega\|_{2,\Omega}^2 \sup_{t \in [0,T]} \|g\|_{2,\Omega}^2 \left(\lambda^2 + 2\gamma^2 C^4 C^{q-2} \right),$$

$$b_2 = \frac{K_0^2}{2\lambda l_0^2 \varepsilon_1} \|\nabla \omega\|_{2,\Omega}^2 \sup_{t \in [0,T]} \|g\|_{2,\Omega}^2 .$$

Furthermore, taking into account Sobolev's inequality we derive

$$\|\nabla u\|_{2,\Omega}^2 \ge \frac{1}{C(\Omega)+1} \left(\|u\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2 \right) = \eta y(t), \tag{72}$$

where $y(t) := \|u\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2$ *and* $\eta := \frac{1}{C(\Omega)+1}$.

Using estimates for $u_i, \ i = 1, 2,$ *and choosing* $\varepsilon_i, i = 0, 1, 2$ *with suitable values as we did as obtaining a priory estimates above, we can make* α, β *to be positive and finite constants, and it is possible due to the assumption* $\frac{\kappa}{l_0^2} \sup_{t \in [0,T]} \|g(t)\|_{2,\Omega}^2 \|\nabla \omega\|_{2,\Omega}^2 \le m < 2$.

Plugging (69)–(71) into (65) and using (72), and integrating by $\tau \in (0, t)$ *we arrive to the following Cauchy problem*

$$\begin{cases} y(t) \le a \displaystyle\int_0^t y(\tau) d\tau, \\ y(0) = 0, \end{cases} \tag{73}$$

where

$$a := \frac{b_1 + b_2 T + b_0}{\eta}.$$

Due to the conditions to the Theorem 5 and then by the Gronwall's lemma, it follows from (73) that $y(t) \equiv 0$ *for all* $t \in [0, T_{max}]$*, and consequently that* $u_1 \equiv u_2$*, where* T_{max} *is a maximal time such that the weak solution to the problem (15)-(18) exists.*

7 Conclusion

In the paper, the space of a weak generalized solution of inverse problem for the pseudoparabolic equation with memory term and damping is defined. Under suitable conditions on the data of the problem, the global and local in time existence and uniqueness theorems are obtained and proved.

Acknowledgment

This research work has been funded by Grant number AP23486218 the Ministry of Science and Higher Education of the Republic of Kazakhstan.

References

[1] G. Barenblatt, I. Zheltov, I. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks. *Journal of Applied Mathematics and Mechanics*, **24** (1960), 1286–1303.

[2] T.W. Ting, Certain nonsteady flows of second-order fluids. *Archive for Rational Mechanics and Analysis*, **14** (1963), 1–26.

[3] Gurtin, M.E., Pipkin, A.C. A General Theory of Heat Conduction with Finite Wave Speed. *Archive for Rational Mechanics and Analysis.* 1968. 31, 113-126.

[4] M. O. Korpusov, A. G. Sveshnikov, *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics..* Zh. Vychisl. Mat. Mat. Fiz. 43 (2003), no. 12, 1835–1869.

[5] V.G. Zvyagin, M.V. Turbin, The study of initial-boundary value problems for mathematical models of the motion of Kelvin-Voigt fluids. *Journal of Mathematical Sciences*, **168** (2012), 157–308.

[6] A. B. Alshin, M. O. Korpusov, A. G. Sveshnikov, *Blow-up in nonlinear Sobolev type equations, Series in nonlinear analisys and applications.* De Gruyter, 2011, 648 p.

[7] S.N. Antontsev, H.B. de Oliveira, Kh. Khompysh, Generalized Kelvin-Voigt equations for nonhomogeneous and incompressible fluids. *COMMUN. MATH. SCI.*, **17(7)** (2019), 1915-1948

[8] R.E. Showalter. Existence and representation theorem for a semilinear Sobolev equation in Banach space. *SIAM J. Math. Anal.* 3 (1972) 527–543.

[9] M. Liao. Non-global existence of solutions to pseudo-parabolic equations with variable exponents and positive initial energy. *Comptes Rendus Mŭcanique* 347 (2019), no. 10, 710–715.

[10] Runzhang X., Jia Su. Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations. *Journal of Functional Analysis* Vol: 264, Issue: 12, P.: 2732-2763.

[11] Huafei Di, Yadong Shang, Xiaoming Peng. Blow-up phenomena for a pseudo-parabolic equation with variable exponents. *Applied Mathematics Letters.* Vol: 64, P.: 67-73

[12] E.D. Benedetto, M. Pierre. On the maximum principle for pseudoparabolic equations. *Indiana University Mathematics Journal.* 30 821–854.

[13] Menglan Liao, Bin Guo, Qingwei Li. On the maximum principle for pseudoparabolic equations. *Mathematical Methods in the Applied Sciences.* 43(5). 2516-2527

[14] M. Marras, S. Vernier-Piro, G. Viglialoro. Blow-up phenomena for nonlinear pseudo-parabolic equations with gradient term. *Discrete Cont. Dyn-B.* 22 (2017), 2291-2300.

[15] Huafei Di, Yadong Shang, Jiali Yu. Blow-up analysis of a nonlinear pseudo-parabolic equation with memory term. *Mathematics* 2020. 5(4). 3408-3422

[16] Gripenberg G. Global existence of solutions of Volterra integrodifferential equations of parabolic type. *Differential Equations.* 1993, 102(2): 382-390.

[17] Salim A. Messaoudi Global existence of solutions of Volterra integrodifferential equations of parabolic type. *Progress in Nonlinear Differential Equations and Their Applications* 2005, 64: 351-356.

[18] Y. D. Shang, B. L. Guo. On the problem of the existence of global solutions for a class of nonlinear convolutional integro-differential equations of pseudoparabolic type . *Acta Math. Appl. Sin.* 26 (2003), 511–524

[19] Huafei Di, Yadong Shang. Blow-Up of Solutions for a Class of Nonlinear Pseudoparabolic Equations with a Memory Term. *Abstract and Applied Analysis* Volume 2014, Article ID 507494, 7 pages.

[20] A. Asanov, E.R. Atamanov, *Nonclassical and Inverse Problems for Pseudoparabolic Equations.* De Gruyter, Berlin, 1997.

[21] V. Isakov, *Inverse Problems for Partial Differential Equations.* Springer, New-York-Berlin-Heidelberg, 1998.

[22] B.S. Ablabekov, *Inverse Problems for Pseudoparabolic Equations.* Ilim, Bishkek, 2001, (in Russian).

[23] A. Favini, A. Lorenzi, *Differential Equations, Inverse and Direct Problems.* Taylor and Francis Group, LLC, 2006.

[24] A. Lorenzi, E. Paparoni, Identification problems for pseudoparabolic integrodifferential operator equations. *Journal of Inverse and Ill-Posed Problems.*, **5** (1997), 235–253.

[25] A.Sh. Lyubanova, A. Tani, An inverse problem for pseudoparabolic equation of filtration. The existence, uniqueness and regularity. *Applicable Analysis*, **90** (2011), 1557-1568.

[26] A.Sh. Lyubanova, A.V. Velisevich, Inverse problems for the stationary and pseudoparabolic equations of diffusion, *Applicable Analysis*, **98** (2019), 1997–2010.

[27] M. Yaman, Blow-up solution and stability to an inverse problem for a pseudo-parabolic equation. *Journal of Inequalities and Applications*, **2012** (2012), 1–8.

[28] Aitzhanov S.E. Ashurova, G.R. Zhalgassova K.A. Identification of the right hand side of a quasilinear pseudoparabolic equation with memory term. *Journal of Mathematics, Mechanics and Computer Science*, **2021** 110(2), 47–63.

[29] S.N. Antontsev, S.E. Aitzhanov, and G.R. Ashurova, An inverse problem for the pseudo-parabolic equation with p-Laplacian. *Evolution equation and control theory*, **11**(2) (2022), 399–414.

[30] Kh. Khompysh, A.G. Shakir, An inverse source problem for a nonlinear pseudoparabolic equation with p-Laplacian diffusion and damping term. *Quaestiones Mathematicae*, (2022) 46(9), 1889–1914.

[31] S. N. Antontsev, J. I. Diaz, and S. Shmarev, *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics.* Progress in Nonlinear Differential Equations and their Applications 48, Birkhäuser, 2002.

[32] J.-L. Lions, *Quelques methodes de resolution des problemes aux limites non liniaires.* Paris, Dunod, 1969.

[33] J.W. Barret, W.B. Liu, Finite element approximation of the parabolic p-Laplacian. *SIAM Journal on Numerical Analysis*, **31(2)** (1994), 413–428.

[34] Shakir A. Global solvability of inverse problem for linear Kelvin-Voigt equations with memory. *Journal of Mathematics, Mechanics and Computer Science*, **2023** 118(2), 30–41.

[35] Shakir A., Kabidoldanova A., Khompysh Kh. Solvability of a nonlinear inverse problem for a pseudoparabolic equation with p-Laplacian. *Journal of Mathematics, Mechanics and Computer Science*, **2021** 110(2), 35–46.

[36] Khompysh Kh., Shakir A. The inverse problem for determining the right part of the pseudo-parabolic equation. *Journal of Mathematics, Mechanics and Computer Science*, **2020** 105(1), 87–98.

Автор туралы мәлімет:

Шәкір Айдос Ғанижанұлы (корреспондент автор) – PhD, әл-Фараби атындағы Қазақ ұлттық университеті Математика кафедрасының аға оқытушысы (Алматы, Қазақстан, электрондық пошта: ajdossakir@gmail.com);

Сведения об авторе:

Шакир Айдос Ганижанович (корреспондент автор) — PhD, старший преподаватель кафедры математики Казахского национального университета имени аль-Фараби (Алматы, Казахстан, электронная почта: ajdossakir@gmail.com);

Information about author:

Shakir Aidos Ganizhanuly (corresponding author) — PhD, Senior lecturer at the Department of Mathematics, Al-Farabi Kazakh National University, (Almaty, Kazakhstan, email: ajdossakir@gmail.com);

Received: March 5, 2025
Accepted: June 7, 2025