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SOLUTION OF MULTILAYER PROBLEMS FOR THE HEAT EQUATION BY THE FOURIER METHOD

The multilayer problems for the heat equation arise in many areas of heat and mass transfer applications. There are two main approaches to finding exact solutions to multilayer diffusion problems: separation of variables and integral transformations. The difficulty of applying the Laplace transform method is redoubled by the difficulty of finding the inverse transform. The inverse Laplace transform is often performed numerically. The most popular analytical approach to multilayer problems for the heat equation is the method of separation of variables. It is very important to obtain analytical solutions to such problems as they provide a higher level of understanding of the solution behavior and can be used for comparative analysis of numerical solutions. In this paper, the solution of the multilayer problem for the heat equation by the Fourier method is substantiated. The solution of the initial-boundary value problem for the heat equation with discontinuous coefficients by the method of separation of variables is reduced to the corresponding non-self-adjoint spectral Sturm-Liouville eigenvalue problem. Such eigenvalue problems do not belong to the ordinary type of Sturm-Liouville problems due to the discontinuity of the heat conductivity coefficients. In addition, the non-self-adjointness of the corresponding spectral problem also complicates the solution of the problem. Using the replacement, the problem is reduced to a self-adjoint spectral problem and the eigenfunctions of this problem forming an orthonormal basis are constructed. The considered problem models the process of heat propagation of the temperature field in a thin rod of finite length, consisting of several sections with different thermal-physical characteristics. In this problem, in addition to the boundary conditions of the Sturm type, the conditions of conjugation at the point of contact of different media are specified. The existence and uniqueness of the classical solution of the considered multilayer problem for the heat conduction equation are proved.

Keywords: Heat equation, Fourier method, spectral problem, orthonormal basis, classical solution.

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Жылуөткізгіштік теңдеу үшін көпқабатты есептерді Фурье әдісімен шешу

Жылуөткізгіштік теңдеуіне арналған көпқабатты есептер жылу және масса алмасудың көптеген салаларында туындайды. Көпқабатты диффузиялық есептердің дәл шешімдерін табудың екі негізгі әдісі бар: айнымалыларды ажырату және интегралдық түрлендірулер. Лаплас түрлендіруі әдісін қолданудың қиындығы кері түрлендіруді табудың күрделілігімен шиеленіседі. Көбінесе кері Лаплас түрлендіруі сандық түрде орындалады. Жылуөткізгіштік теңдеу үшін көпқабатты есептерге ең танымал аналитикалық тәсіл айнымалыларды ажырату әдісі болып табылады. Мұндай есептердің аналитикалық шешімдері өте құнды, өйткені олар шешім тәртібін түсінудің жоғары деңгейін қамтамасыз етеді және сандық шешімдерді салыстырмалы түрде талдау үшін пайдаланылуы мүмкін. Бұл ғылыми мақалада Фурье әдісі арқылы жылуөткізгіштік теңдеуінің көпқабатты есебінің шешімі негізделеді. Коэффициенттері үзілісті жылуөткізгіштік теңдеу үшін бастапқы-шекаралық есеп айнымалылар ажырату әдісі бойынша өзіне-өзі түйіндес емес спектрлік Штурм-Лиувилль меншікті мән есебіне келтіріледі. Мұндай меншікті мәндер есептері жылуөткізгіштік коэффициенттерінің үзілуіне байланысты Штурм-Лиувилль есептерінің әдеттегі түріне жатпайды.

Сонымен қатар, спектрлік есептің өзіне-өзі түйіндес емес болуы да есепті шешуді қиындатады. Алмастыру арқылы берілген есеп өзіне-өзі түйіндес спектрлік есепке келтіріледі және осы есептің ортонормалдық базисі болатын меншікті функциялары құрылады. Қарастырылып отырған мәселе әртүрлі термофизикалық сипаттамалары бар бірнеше бөліктен тұратын, ұзындықтары ақырлы жіңішке таяқшадағы температуралық өрістің жылу таралу процесін моделдейді. Штурм типіндегі шекаралық шарттарға қосымша, әртүрлі орталардың жанасу нүктесіндегі түйіндес шарттары көрсетілген. Жылуөткізгіштік теңдеу үшін қарастырылып отырған көпқабатты есептің классикалық шешімінің бар және жалғыз екендігі дәлелденді.

Түйін сөздер: Жылуөткізгіштік теңдеуі, Фурье әдісі, спектрлік есеп, ортонормалдық базис, классикалық шешім.

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Решение многослойных задач для уравнения теплопроводности методом Фурье

Проблемы многослойных задач для уравнения теплопроводности возникают во многих областях применения процессов тепло- и массообмена. Существует два основных подхода к поиску точных решений задач многослойной диффузии: разделение переменных и интегральные преобразования. Трудность применение метода преобразование Лапласа усугубляется из-за сложности нахождение обратного преобразование. Часто обратное преобразование Лапласа выполняется численно. Наиболее популярным аналитическим подходом к многослойным задачам для уравнения теплопроводности является метод разделение переменных. Аналитические решения таких задач очень ценны, поскольку они обеспечивают более высокий уровень понимания поведения решения и могут быть использованы для сравнительного анализа численных решений. В данной научной статье обосновано решение методом Фурье многослойной задачи для уравнения теплопроводности. Решения методом разделение переменных начально-краевые задачи для уравнения теплопроводности с разрывными коэффициентами сводится к соответствующей не самосопряженной спектральной задаче Штурма-Лиувилля на собственные значения. Такие задачи на собственные значения не относится к обычному типу задач Штурма-Лиувилля из-за разрыва коэффициентов теплопроводности. Кроме того не самосопряженность соответствующей спектральной задачи также усложняет решение поставленной задачи. С помощью замены поставленная задача сведена к самосопряженной спектральной задаче и построена собственные функции этой задачи, которая образует ортонормированный базис. Рассматриваемая задача моделирует процесс распространения тепла температурного поля в тонком стержне конечной длины, состоящем из нескольких участков с различными теплофизическими характеристиками. Дополнительно к граничным условиям типа Штурма задаются условия сопряжения в точке контакта различных сред. Доказано существование и единственность классического решения рассматриваемой многослойной задачи для уравнения теплопроводности.

Ключевые слова: Уравнение теплопроводности, метод Фурье, спектральная задача, ортонормированный базис, классическое решение.

1 Introduction

Parabolic equations with discontinuous coefficients with one point of discontinuity have been extensively studied [1]-[3]. In these works, the correctness of various initial-boundary value problems for parabolic equations with discontinuous coefficients has been proved by using the Green function and thermal potential methods. In [4]-[8], some boundary value problems for the heat equation with a discontinuous coefficient, with one and two points of discontinuity,

have been considered by the method of separation of variables.
The papers [9]-[13] are devoted to the solution of multilayer diffusion problems. Mathematical models of diffusion in layered materials arise in many industrial, environmental, biological and medical applications, such as thermal conductivity in composite materials, transport of polluting chemicals and gases in layered porous media, growth of brain tumors, thermal conductivity through skin, transdermal drug delivery and greenhouse gas emissions[14]-[17]. The considered problem may arise in describing the process of particle diffusion in turbulent plasma, as well as in modeling the process of heat propagation of a temperature field in a thin rod of finite length, consisting of several sections with different thermophysical characteristics. In addition to the boundary conditions, the conjugation conditions (ideal contact condition) at the contact boundary of these media with different thermophysical characteristics are specified. It is a theoretical paper, however, the obtained analytical solution can be used for numerical calculations.

2 Statement of problem

We consider the initial-boundary value problem for the heat equation with piecewise constant coefficients

$$\frac{\partial u_i}{\partial t} = k_i^2 \frac{\partial^2 u_i}{\partial x^2}, \qquad i = 1, 2, \dots, m, \tag{1}$$

in the domain

$$\Omega = \bigcup_{i=1}^{m} \Omega_i, \qquad \Omega_i = \{(x,t) :\, l_{i-1} < x < l_i,\ 0 < t < T\},$$

with the initial condition

$$u(x,0) = \varphi(x), \qquad l_0 \le x \le l_m. \tag{2}$$

The boundary conditions are of the form

$$\begin{cases} \alpha_1 \dfrac{\partial u_1}{\partial x}(l_0, t) + \beta_1 u_1(l_0, t) = 0, \\ \alpha_2 \dfrac{\partial u_m}{\partial x}(l_m, t) + \beta_2 u_m(l_m, t) = 0, \end{cases} \qquad 0 \le t \le T. \tag{3}$$

The conjugation conditions are

$$\begin{cases} u_i(l_i - 0, t) = u_{i+1}(l_i + 0, t), \\ k_i \dfrac{\partial u_i}{\partial x}(l_i - 0, t) = k_{i+1} \dfrac{\partial u_{i+1}}{\partial x}(l_i + 0, t), \end{cases} \qquad 0 \le t \le T, \quad i = 1, 2, \dots, m-1, \tag{4}$$

where the coefficients satisfy $k_i > 0$ and $\alpha_j, \beta_j \in \mathbb{R}$ for $i = 1, 2, \dots, m$ and $j = 1, 2$. In addition, $|\alpha_1| + |\beta_1| > 0$ and $|\alpha_2| + |\beta_2| > 0$.

3 Method of Solution

To solve problem (1)-(4) we employ the Fourier method and seek a separated solution

$$u_i(x,t) = X_i(x)\,T(t) \not\equiv 0.$$

Substituting into equation (1) and conditions (2)-(4), and separating the variables, we obtain the following spectral problem:

$$k_i^2\,X_i''(x) + \lambda\,X_i(x) = 0, \qquad l_{i-1} < x < l_i, \quad i = 1,2,\ldots,m. \tag{5}$$

The boundary conditions

$$\begin{cases} \alpha_1\,X_1'(l_0) + \beta_1\,X_1(l_0) = 0, \\ \alpha_2\,X_m'(l_m) + \beta_2\,X_m(l_m) = 0, \end{cases} \tag{6}$$

and the conjugation conditions are

$$\begin{cases} X_i(l_i - 0, t) = X_{i+1}(l_i + 0, t), \\ k_i\,X_i'(l_i - 0, t) = k_{i+1}\,X_{i+1}'(l_i + 0, t), \end{cases} \qquad i = 1,2,\ldots,m-1. \tag{7}$$

The function $T(t)$ satisfies the ordinary differential equation

$$T'(t) + \lambda\,T(t) = 0.$$

Lemma 1. The spectral problem (5)-(7) is non-self-adjoint in $L_2(l_0, l_m)$.
The proof is carried out by direct calculation.

After the following change of variables

$$X_i(x) = Y_i(y), \qquad i = 1,2,\ldots,m, \tag{8}$$

where

$$y = \begin{cases} \dfrac{x - l_0}{k_1}, & l_0 < x < l_1, \\ \dfrac{x - l_1}{k_2}, & l_1 < x < l_2, \\ \ldots \\ \dfrac{x - l_{m-1}}{k_m}, & l_{m-1} < x < l_m, \end{cases} \tag{9}$$

Under the change of variables (8)-(9), the spectral problem (5)-(7) takes the form

$$Y_i''(y) + \lambda\,Y_i(y) = 0, \qquad 0 < y < h_i, \quad i = 1,2,\ldots,m, \tag{10}$$

with boundary conditions

$$\begin{cases} \dfrac{\alpha_1}{k_1}\,Y_1'(0) + \beta_1\,Y_1(0) = 0, \\ \dfrac{\alpha_2}{k_m}\,Y_m'(h_m) + \beta_2\,Y_m(h_m) = 0, \end{cases} \tag{11}$$

and conjugation conditions

$$\begin{cases} Y_i(h_i-0)=Y_{i+1}(0+), \\ Y_i'(h_i-0)=Y_{i+1}'(0+), \end{cases} \qquad i=1,2,\dots,m-1, \tag{12}$$

where

$$h_i=\frac{l_i-l_{i-1}}{k_i}, \qquad i=1,2,\dots,m.$$

Lemma 2. The spectral problem (10)-(12) is self-adjoint in

$$H=L_2(0,h_1)\oplus L_2(0,h_2)\oplus\cdots\oplus L_2(0,h_m).$$

The proof is carried out by direct calculation.

Next, we determine the eigenvalues and construct the eigenfunctions of (10)-(12). The general solution of (10) has the form

$$\begin{cases} Y_1(y)=c_1\cos(\sqrt{\lambda}\,y)+c_2\sin(\sqrt{\lambda}\,y), & 0<y<h_1, \\ Y_2(y)=c_3\cos(\sqrt{\lambda}\,y)+c_4\sin(\sqrt{\lambda}\,y), & 0<y<h_2, \\ \quad\dots \\ Y_{m-1}(y)=c_{2m-3}\cos(\sqrt{\lambda}\,y)+c_{2m-2}\sin(\sqrt{\lambda}\,y), & 0<y<h_{m-1}, \\ Y_m(y)=c_{2m-1}\cos(\sqrt{\lambda}\,y)+c_{2m}\sin(\sqrt{\lambda}\,y), & 0<y<h_m, \end{cases}$$

where c_{2i-1}, c_{2i} are arbitrary constants, $i=1,2,\dots,m$.

From the boundary conditions (11) we obtain

$$\begin{cases} \dfrac{\alpha_1}{k_1}\sqrt{\lambda}\,c_2+\beta_1 c_1=0, \\ \Big(\beta_2\cos(\sqrt{\lambda}\,h_m)-\dfrac{\alpha_2}{k_m}\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,h_m)\Big)c_{2m-1}+ \\ \Big(\beta_2\sin(\sqrt{\lambda}\,h_m)+\dfrac{\alpha_2}{k_m}\sqrt{\lambda}\,\cos(\sqrt{\lambda}\,h_m)\Big)c_{2m}=0. \end{cases} \tag{13}$$

From the conjugation conditions (12) we obtain

$$\begin{cases} c_1\cos(h_1\sqrt{\lambda})+c_2\sin(h_1\sqrt{\lambda})=c_3, \\ -c_1\sin(h_1\sqrt{\lambda})+c_2\cos(h_1\sqrt{\lambda})=c_4, \\ c_3\cos(h_2\sqrt{\lambda})+c_4\sin(h_2\sqrt{\lambda})=c_5, \\ -c_3\sin(h_2\sqrt{\lambda})+c_4\cos(h_2\sqrt{\lambda})=c_6, \\ \dots \\ c_{2m-3}\cos(h_{m-1}\sqrt{\lambda})+c_{2m-2}\sin(h_{m-1}\sqrt{\lambda})=c_{2m-1}, \\ -c_{2m-3}\sin(h_{m-1}\sqrt{\lambda})+c_{2m-2}\cos(h_{m-1}\sqrt{\lambda})=c_{2m}. \end{cases} \tag{14}$$

Successively eliminating the constants c_i from (14) gives

$$c_1 \cos\big((h_1 + h_2 + \cdots + h_{m-1})\sqrt{\lambda}\big) + c_2 \sin\big((h_1 + h_2 + \cdots + h_{m-1})\sqrt{\lambda}\big) = c_{2m-1},$$
$$-c_1 \sin\big((h_1 + h_2 + \cdots + h_{m-1})\sqrt{\lambda}\big) + c_2 \cos\big((h_1 + h_2 + \cdots + h_{m-1})\sqrt{\lambda}\big) = c_{2m}.$$

Substituting the obtained c_{2m-1}, c_{2m} into system (13), we arrive at

$$\begin{cases} \beta_1 c_1 + \dfrac{\alpha_1}{k_1}\sqrt{\lambda}\, c_2 = 0, \\ \Big(\beta_2 \cos(s_m\sqrt{\lambda}) - \dfrac{\alpha_2}{k_m}\sqrt{\lambda}\, \sin(s_m\sqrt{\lambda})\Big)c_1 + \Big(\beta_2 \sin(s_m\sqrt{\lambda}) + \dfrac{\alpha_2}{k_m}\sqrt{\lambda}\, \cos(s_m\sqrt{\lambda})\Big)c_2 = 0, \end{cases}$$

where

$$s_m = \sum_{i=1}^{m} h_i = \sum_{i=1}^{m} \frac{l_i - l_{i-1}}{k_i}.$$

The characteristic determinant of the last system has the form

$$\Delta(\lambda) = \big(\alpha_1\alpha_2\lambda + \beta_1\beta_2 k_1 k_m\big)\, \sin\big(s_m\sqrt{\lambda}\big) + \big(\alpha_2\beta_1 k_1 - \alpha_1\beta_2 k_m\big)\, \sqrt{\lambda}\, \cos\big(s_m\sqrt{\lambda}\big) = 0. \quad (15)$$

We now consider all possible special cases.
1) Suppose $\alpha_1\alpha_2 \neq 0$, $\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1 = 0$, $\beta_1\beta_2 = 0$ (that is, $Y_1'(0) = 0$, $Y_m'(h_m) = 0$). Then from (15) we obtain

$$\alpha_1\alpha_2\, \lambda\, \sin\big(s_m\sqrt{\lambda}\big) = 0.$$

From $\sin\big(s_m\sqrt{\lambda}\big) = 0$ we find the eigenvalues

$$\lambda_n = \left(\frac{\pi n}{s_m}\right)^2, \qquad n \in \mathbb{Z}.$$

The corresponding eigenfunctions are

$$Y_n(y) = C \cdot \begin{cases} y_{1n} = (-1)^n \cos\big(\frac{\pi n}{s_m}\, y\big), & 0 < y < h_1, \\ y_{2n} = \cos\big(\frac{\pi n}{s_m}\, (h_2 - y + h_3 + \cdots + h_m)\big), & 0 < y < h_2, \\ y_{3n} = \cos\big(\frac{\pi n}{s_m}\, (h_3 - y + h_4 + \cdots + h_m)\big), & 0 < y < h_3, \\ \cdots \\ y_{m-1,n} = \cos\big(\frac{\pi n}{s_m}\, (h_{m-1} - y + h_m)\big), & 0 < y < h_{m-1}, \\ y_{mn} = \cos\big(\frac{\pi n}{s_m}\, (h_m - y)\big), & 0 < y < h_m, \end{cases}$$

where C is an arbitrary constant.
2) Suppose $\alpha_1\alpha_2 = 0$, $\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1 = 0$, and $\beta_1\beta_2 \neq 0$ (i.e., $Y_1(0) = 0$ and $Y_m(h_m) = 0$). Then, similarly, the eigenvalues are

$$\lambda_n = \left(\frac{\pi n}{s_m}\right)^2, \qquad n \in \mathbb{Z},$$

with the corresponding eigenfunctions

$$Y_n(y) = C\cdot\begin{cases} y_{1n} = (-1)^{n+1}\sin\big(\dfrac{\pi n}{s_m}\,y\big), & 0<y<h_1,\\ y_{2n} = \sin\big(\dfrac{\pi n}{s_m}\,(h_2-y+h_3+\cdots+h_m)\big), & 0<y<h_2,\\ y_{3n} = \sin\big(\dfrac{\pi n}{s_m}\,(h_3-y+h_4+\cdots+h_m)\big), & 0<y<h_3,\\ \dots & \\ y_{m-1,n} = \sin\big(\dfrac{\pi n}{s_m}\,(h_{m-1}-y+h_m)\big), & 0<y<h_{m-1},\\ y_{mn} = \sin\big(\dfrac{\pi n}{s_m}\,(h_m-y)\big), & 0<y<h_m, \end{cases}$$

where C is an arbitrary constant.
3) Now let $\alpha_1\alpha_2=0,\ \ \alpha_1\beta_2k_m-\alpha_2\beta_1k_1\neq 0,\ \ \beta_1\beta_2=0$. Then from (15) we obtain

$$\big(\alpha_1\beta_2k_m-\alpha_2\beta_1k_1\big)\sqrt{\lambda}\,\cos\big(s_m\sqrt{\lambda}\big)=0.$$

At $\lambda=0$ equation (10) has only the trivial solution. From $\cos\big(s_m\sqrt{\lambda}\big)=0$ we find the eigenvalues

$$\lambda_n=\left(\frac{\pi(2n+1)}{2s_m}\right)^2,\qquad n\in\mathbb{Z}.$$

To determine the eigenfunctions, consider two possible cases.

Case 3.1: $\alpha_1=0$, $\alpha_2\neq 0$, $\beta_1\neq 0$, $\beta_2=0$ (i.e., $Y_1(0)=0$, $Y_m'(h_m)=0$). The corresponding eigenfunctions are

$$Y_n(y) = C\cdot\begin{cases} y_{1n} = (-1)^{n}\sin\big(\dfrac{\pi(2n+1)}{2s_m}\,y\big), & 0<y<h_1,\\ y_{2n} = \cos\big(\dfrac{\pi(2n+1)}{2s_m}\,(h_2-y+h_3+\cdots+h_m)\big), & 0<y<h_2,\\ y_{3n} = \cos\big(\dfrac{\pi(2n+1)}{2s_m}\,(h_3-y+h_4+\cdots+h_m)\big), & 0<y<h_3,\\ \dots & \\ y_{m-1,n} = \cos\big(\dfrac{\pi(2n+1)}{2s_m}\,(h_{m-1}-y+h_m)\big), & 0<y<h_{m-1},\\ y_{mn} = \cos\big(\dfrac{\pi(2n+1)}{2s_m}\,(h_m-y)\big), & 0<y<h_m, \end{cases}$$

where C is an arbitrary constant.
Case 3.2: $\alpha_1\neq 0$, $\alpha_2=0$, $\beta_1=0$, $\beta_2\neq 0$ (i.e., $Y_1'(0)=0$, $Y_m(h_m)=0$). Then the

eigenfunctions are

$$Y_n(y) = C \cdot \begin{cases} y_{1n} = (-1)^n \cos\Big(\dfrac{\pi(2n+1)}{2s_m}\, y\Big), & 0 < y < h_1, \\ y_{2n} = \sin\Big(\dfrac{\pi(2n+1)}{2s_m}\, (h_2 - y + h_3 + \dots + h_m)\Big), & 0 < y < h_2, \\ y_{3n} = \sin\Big(\dfrac{\pi(2n+1)}{2s_m}\, (h_3 - y + h_4 + \dots + h_m)\Big), & 0 < y < h_3, \\ \dots \\ y_{m-1,n} = \sin\Big(\dfrac{\pi(2n+1)}{2s_m}\, (h_{m-1} - y + h_m)\Big), & 0 < y < h_{m-1}, \\ y_{mn} = \sin\Big(\dfrac{\pi(2n+1)}{2s_m}\, (h_m - y)\Big), & 0 < y < h_m. \end{cases}$$

4) Consider the case $\alpha_1\alpha_2 \neq 0$, $\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1 \neq 0$, and $\beta_1\beta_2 = 0$. Then from (15) we have

$$\alpha_1\alpha_2\, \lambda \sin\big(s_m\sqrt{\lambda}\big) - \big(\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1\big)\sqrt{\lambda} \cos\big(s_m\sqrt{\lambda}\big) = 0.$$

It is easy to check that for $\lambda = 0$ equation (10) admits only the trivial solution. Hence the eigenvalues are given by the roots of

$$\tan\big(s_m\sqrt{\lambda}\big) = \begin{cases} \dfrac{k_m\,\beta_2}{\alpha_2\,\sqrt{\lambda}}, & \beta_1 = 0,\ \beta_2 \neq 0 \quad \Big(Y_1'(0) = 0,\ \dfrac{\alpha_2}{k_m} Y_m'(h_m) + \beta_2 Y_m(h_m) = 0\Big), \\ -\dfrac{k_1\,\beta_1}{\alpha_1\,\sqrt{\lambda}}, & \beta_1 \neq 0,\ \beta_2 = 0 \quad \Big(\dfrac{\alpha_1}{k_1} Y_1'(0) + \beta_1 Y_1(0) = 0,\ Y_m'(h_m) = 0\Big). \end{cases}$$

It is not possible to write the eigenvalues in explicit form. However, by Rouche's theorem one can obtain their asymptotics. Clearly, the zeros of the equation $\tan\big(s_m\sqrt{\lambda}\big) = 0$ are $\sqrt{\lambda} = \dfrac{\pi n}{s_m}$. Hence, by Rouche's theorem the zeros of

$$\tan\big(s_m\sqrt{\lambda}\big) = \frac{\alpha_1 k_m\beta_2 - \alpha_2 k_1\beta_1}{\alpha_1\alpha_2\sqrt{\lambda}}$$

have the form

$$\lambda_n = \left(\frac{\pi n}{s_m} + \delta_n\right)^2, \qquad n \in \mathbb{Z},$$

where $|\delta_n| \leq M$ and, moreover, $\delta_n = O\left(\dfrac{1}{n}\right)$.

If $\beta_1 \neq 0$ and $\beta_2 = 0$, the eigenfunctions are

$$Y_n(y) = C \cdot \begin{cases} \cos\big((h_1 - y + h_2 + \dots + h_m)\sqrt{\lambda_n}\big), & 0 < y < h_1, \\ \cos\big((h_2 - y + h_3 + \dots + h_m)\sqrt{\lambda_n}\big), & 0 < y < h_2, \\ \cos\big((h_3 - y + h_4 + \dots + h_m)\sqrt{\lambda_n}\big), & 0 < y < h_3, \\ \dots \\ \cos\big((h_{m-1} - y + h_m)\sqrt{\lambda_n}\big), & 0 < y < h_{m-1}, \\ \cos\big((h_m - y)\sqrt{\lambda_n}\big), & 0 < y < h_m, \end{cases}$$

corresponding to the boundary conditions $\frac{\alpha_1}{k_1}Y_1'(0)+\beta_1 Y_1(0)=0$ and $Y_m'(h_m)=0$.

If $\beta_1=0$ and $\beta_2\neq 0$, the eigenfunctions are

$$Y_n(y)=C\cdot\begin{cases}\cos\big(y\sqrt{\lambda_n}\big), & 0<y<h_1,\\ \cos\big((h_1+y)\sqrt{\lambda_n}\big), & 0<y<h_2,\\ \cos\big((h_1+h_2+y)\sqrt{\lambda_n}\big), & 0<y<h_3,\\ \dots & \\ \cos\big((h_1+h_2+\dots+h_{m-2}+y)\sqrt{\lambda_n}\big), & 0<y<h_{m-1},\\ \cos\big((h_1+h_2+\dots+h_{m-1}+y)\sqrt{\lambda_n}\big), & 0<y<h_m,\end{cases}$$

corresponding to the boundary conditions $Y_1'(0)=0$ and $\frac{\alpha_2}{k_m}Y_m'(h_m)+\beta_2 Y_m(h_m)=0$.

5) In the case $\alpha_1\alpha_2=0$, $\alpha_1\beta_2 k_m-\alpha_2\beta_1 k_1\neq 0$, $\beta_1\beta_2\neq 0$, an argument analogous to the previous one shows that the eigenvalues are the solutions of

$$\cot\big(s_m\sqrt{\lambda}\big)=\begin{cases}-\dfrac{k_m\beta_2}{\alpha_2\sqrt{\lambda}}, & \alpha_1=0,\ \alpha_2\neq 0,\\ \dfrac{k_1\beta_1}{\alpha_1\sqrt{\lambda}}, & \alpha_1\neq 0,\ \alpha_2=0.\end{cases}$$

explicit forms for the eigenvalues are not available. By Rouche's theorem we can, however, obtain their asymptotics. Since the zeros of $\cot\big(s_m\sqrt{\lambda}\big)=0$ are $\sqrt{\lambda}=\frac{\pi(2n+1)}{2s_m}$, it follows from Rouche's theorem that the zeros of

$$\cot\big(s_m\sqrt{\lambda}\big)=\frac{\alpha_2 k_1\beta_1-\alpha_1 k_m\beta_2}{\alpha_1\alpha_2\sqrt{\lambda}}$$

have the form

$$\lambda_n=\left(\frac{\pi(2n+1)}{2s_m}+\delta_n^*\right)^2,\qquad |\delta_n^*|\leq M,\qquad \delta_n^*=O\left(\frac{1}{n}\right).$$

If $\alpha_1\neq 0$ and $\alpha_2=0$, the eigenfunctions are

$$Y_n(y)=C\cdot\begin{cases}\sin\big((h_1-y+h_2+\dots+h_m)\sqrt{\lambda_n}\big), & 0<y<h_1,\\ \sin\big((h_2-y+h_3+\dots+h_m)\sqrt{\lambda_n}\big), & 0<y<h_2,\\ \sin\big((h_3-y+h_4+\dots+h_m)\sqrt{\lambda_n}\big), & 0<y<h_3,\\ \dots & \\ \sin\big((h_{m-1}-y+h_m)\sqrt{\lambda_n}\big), & 0<y<h_{m-1},\\ \sin\big((h_m-y)\sqrt{\lambda_n}\big), & 0<y<h_m,\end{cases}$$

corresponding to the boundary conditions

$$\frac{\alpha_1}{k_1}Y_1'(0)+\beta_1 Y_1(0)=0, \qquad Y_m(h_m)=0.$$

If $\alpha_1=0$ and $\alpha_2\neq 0$, the eigenfunctions are

$$Y_n(y)=C\cdot\begin{cases}\sin\big(y\sqrt{\lambda_n}\big), & 0<y<h_1,\\ \sin\big((h_1+y)\sqrt{\lambda_n}\big), & 0<y<h_2,\\ \sin\big((h_1+h_2+y)\sqrt{\lambda_n}\big), & 0<y<h_3,\\ \dots & \\ \sin\big((h_1+h_2+\dots+h_{m-2}+y)\sqrt{\lambda_n}\big), & 0<y<h_{m-1},\\ \sin\big((h_1+h_2+\dots+h_{m-1}+y)\sqrt{\lambda_n}\big), & 0<y<h_m,\end{cases}$$

corresponding to the boundary conditions $Y_1(0)=0$ and $\dfrac{\alpha_2}{k_m}Y_m'(h_m)+\beta_2 Y_m(h_m)=0$.

6) In the case $\alpha_1\alpha_2\neq 0$, $\alpha_1\beta_2 k_m-\alpha_2\beta_1 k_1=0$, and $\beta_1\beta_2\neq 0$, equation (15) reduces to

$$\Big(\alpha_1\alpha_2\lambda+\beta_1\beta_2 k_1 k_m\Big)\ \sin\big(s_m\sqrt{\lambda}\big)=0 \quad \Big(\text{equivalently } \big(\tfrac{\alpha_1\alpha_2}{k_1k_m}\lambda+\beta_1\beta_2\big)\sin\big(s_m\sqrt{\lambda}\big)=0\Big).$$

Thus, if $\sin\big(s_m\sqrt{\lambda}\big)=0$, the eigenvalues are

$$\lambda_n=\left(\frac{\pi n}{s_m}\right)^2.$$

The corresponding eigenfunctions have the form: The corresponding eigenfunctions (for $\sin(s_m\sqrt{\lambda})=0$) are

$$Y_n(y)=C\cdot\begin{cases} y_{1n}=\cos\Big(\dfrac{\pi n}{s_m}y\Big)-\dfrac{\beta_1 k_1 s_m}{\alpha_1\pi n}\ \sin\Big(\dfrac{\pi n}{s_m}y\Big), & 0<y<h_1,\\ y_{2n}=\cos\Big(\dfrac{\pi n}{s_m}(h_1+y)\Big)-\dfrac{\beta_1 k_1 s_m}{\alpha_1\pi n}\ \sin\Big(\dfrac{\pi n}{s_m}(h_1+y)\Big), & 0<y<h_2,\\ y_{3n}=\cos\Big(\dfrac{\pi n}{s_m}(h_1+h_2+y)\Big)-\dfrac{\beta_1 k_1 s_m}{\alpha_1\pi n}\ \sin\Big(\dfrac{\pi n}{s_m}(h_1+h_2+y)\Big), & 0<y<h_3,\\ \dots & \\ y_{m-1,n}=\cos\Big(\dfrac{\pi n}{s_m}(h_1+h_2+\dots+h_{m-2}+y)\Big)- & \\ \frac{\beta_1 k_1 s_m}{\alpha_1\pi n}\ \sin\Big(\frac{\pi n}{s_m}(h_1+h_2+\dots+h_{m-2}+y)\Big), & 0<y<h_{m-1},\\ y_{mn}=\cos\Big(\dfrac{\pi n}{s_m}(h_1+h_2+\dots+h_{m-1}+y)\Big)- & \\ \frac{\beta_1 k_1 s_m}{\alpha_1\pi n}\ \sin\Big(\frac{\pi n}{s_m}(h_1+h_2+\dots+h_{m-1}+y)\Big), & 0<y<h_m.\end{cases}$$

Here we have used the relation

$$\alpha_1\beta_2 k_m-\alpha_2\beta_1 k_1=0 \quad\Longrightarrow\quad \frac{\alpha_1}{\beta_1 k_1}=\frac{\alpha_2}{\beta_2 k_m}.$$

If

$$\frac{\alpha_1\alpha_2}{k_1k_m}\,\lambda+\beta_1\beta_2=0, \qquad \text{i.e.} \quad \lambda=-\frac{\beta_1\beta_2k_1k_m}{\alpha_1\alpha_2},$$

and taking into account that

$$\frac{\alpha_1\beta_2}{k_1}-\frac{\alpha_2\beta_1}{k_m}=0 \quad\Longrightarrow\quad \frac{\beta_1k_1}{\alpha_1}=\frac{\beta_2k_m}{\alpha_2},$$

we obtain the special eigenvalue

$$\lambda=\left(\frac{\beta_1k_1}{\alpha_1}\right)^2=\left(\frac{\beta_2k_m}{\alpha_2}\right)^2.$$

(The explicit form of the associated eigenfunction is given next.) For the special eigenvalue

$$\lambda=\left(\frac{\beta_1k_1}{\alpha_1}\right)^2=\left(\frac{\beta_2k_m}{\alpha_2}\right)^2,$$

an associated eigenfunction can be chosen as

$$Y(y)=C\cdot\begin{cases} e^{-\frac{\beta_1k_1}{\alpha_1}\,y}, & 0<y<h_1,\\ e^{-\frac{\beta_1k_1}{\alpha_1}\,(h_1+y)}, & 0<y<h_2,\\ e^{-\frac{\beta_1k_1}{\alpha_1}\,(h_1+h_2+y)}, & 0<y<h_3,\\ \dots & \\ e^{-\frac{\beta_1k_1}{\alpha_1}\,(h_1+h_2+\dots+h_{m-2}+y)}, & 0<y<h_{m-1},\\ e^{-\frac{\beta_1k_1}{\alpha_1}\,(h_1+h_2+\dots+h_{m-1}+y)} & 0<y<h_m. \end{cases}$$

7) In the last case, $\alpha_1\alpha_2\neq 0$, $\dfrac{\alpha_1\beta_2}{k_1}-\dfrac{\alpha_2\beta_1}{k_m}\neq 0$, and $\beta_1\beta_2\neq 0$, equation (15) applies. Introduce the functions

$$g(\lambda)=\alpha_1\alpha_2\lambda\,\sin\!\big(s_m\sqrt{\lambda}\big),$$

$$\psi(\lambda)=\big(\alpha_1\beta_2k_m-\alpha_2\beta_1k_1\big)\sqrt{\lambda}\,\cos\!\big(s_m\sqrt{\lambda}\big)-\beta_1\beta_2k_1k_m\,\sin\!\big(s_m\sqrt{\lambda}\big).$$

By Rouche's theorem, if $|g(\lambda)|\geq|\psi(\lambda)|$ for large λ, then $g(\lambda)$ and $g(\lambda)+\psi(\lambda)$ have the same number of zeros.

The eigenfunctions can be written as

$$Y_n(y)=C_n\cdot\begin{cases} \Phi\big(y\sqrt{\lambda_n}\big), & 0<y<h_1,\\ \Phi\big((s_1+y)\sqrt{\lambda_n}\big), & 0<y<h_2,\\ \Phi\big((s_2+y)\sqrt{\lambda_n}\big), & 0<y<h_3,\\ \dots & \\ \Phi\big((s_{m-2}+y)\sqrt{\lambda_n}\big), & 0<y<h_{m-1},\\ \Phi\big((s_{m-1}+y)\sqrt{\lambda_n}\big), & 0<y<h_m, \end{cases} \tag{16}$$

where $s_j = \sum_{i=1}^{j} h_i$ (with $s_0 = 0$) and

$$\Phi(z) = \alpha_1 \cos z - \beta_1 \frac{k_1}{\sqrt{\lambda_n}} \sin z. \tag{17}$$

A explicit-form expression for the eigenvalues is not available, but Rouche's theorem yields their asymptotics. Since the zeros of $\tan\big(s_m\sqrt{\lambda}\big) = 0$ are $\sqrt{\lambda} = \dfrac{\pi n}{s_m}$, it follows that the zeros of

$$\tan\big(s_m\sqrt{\lambda}\big) = \frac{\alpha_1 k_m \beta_2 - \alpha_2 k_1 \beta_1}{\alpha_1 \alpha_2 \sqrt{\lambda}} - \frac{\beta_1 \beta_2 k_1 k_m}{\lambda} \tan\big(s_m\sqrt{\lambda}\big)$$

have the form

$$\lambda_n = \left(\frac{\pi n}{s_m} + \delta_n\right)^2, \qquad |\delta_n| \le M, \qquad \delta_n = O\left(\frac{1}{n}\right).$$

Since $\{Y_n(y)\}$ are the eigenfunctions of the self-adjoint problem (10)-(12) (see Lemma 2), they form an orthonormal basis [18]. We choose C_n from the normalization condition; equivalently,

$$C_n = \left(\sum_{i=1}^{m} \frac{1}{k_i^2} \int_{l_{i-1}}^{l_i} \Phi^2\left(\left(s_{i-1} + \frac{x - l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right) dx\right)^{-\frac{1}{2}}.$$

Then the solution to problem (1)?(4) has the form

$$u_i(x,t) = \sum_{n=1}^{\infty} \varphi_n X_i(x)\, e^{-\lambda_n t} = \sum_{n=1}^{\infty} \varphi_n Y_n(y)\, e^{-\lambda_n t},$$

where

$$\varphi_n = \sum_{i=1}^{m} \int_0^{h_i} \varphi_i(k_i\eta + l_{i-1})\, Y_n(\eta)\, d\eta, \qquad y \text{ is defined by (5).}$$

Making the change of variables

$$\xi = k_i\eta + l_{i-1}, \qquad d\eta = \frac{d\xi}{k_i},$$

in the last integral we obtain

$$\varphi_n = \sum_{i=1}^{m} \frac{1}{k_i} \int_{l_{i-1}}^{l_i} \varphi_i(\xi)\, Y_n\left(\frac{\xi - l_{i-1}}{k_i}\right) d\xi. \tag{18}$$

Therefore, rewriting formula (16) we get

$$Y_n\left(\frac{x - l_{i-1}}{k_i}\right) = C_n \cdot \begin{cases} \Phi\big(\frac{x-l_{i-1}}{k_i}\sqrt{\lambda_n}\big), & l_0 < x < l_1, \\ \Phi\big((s_1 + \frac{x-l_{i-1}}{k_i})\sqrt{\lambda_n}\big), & l_1 < x < l_2, \\ \Phi\big((s_2 + \frac{x-l_{i-1}}{k_i})\sqrt{\lambda_n}\big), & l_2 < x < l_3, \\ \dots \\ \Phi\big((s_{m-2} + \frac{x-l_{i-1}}{k_i})\sqrt{\lambda_n}\big), & l_{m-2} < x < l_{m-1}, \\ \Phi\big((s_{m-1} + \frac{x-l_{i-1}}{k_i})\sqrt{\lambda_n}\big), & l_{m-1} < x < l_m, \end{cases} \tag{19}$$

where Φ is given by (17) and $s_j = \sum_{p=1}^{j} h_p$ (with $s_0 = 0$).

We now proceed to prove the main theorem.

Theorem 1. Let $\varphi(x)$ be a twice continuously differentiable function satisfying the boundary conditions (3) and the conjugation conditions (4), namely,

$$\alpha_1\varphi_1'(l_0) + \beta_1\varphi_1(l_0) = 0, \qquad \alpha_2\varphi_m'(l_m) + \beta_2\varphi_m(l_m) = 0, \tag{20}$$

$$\varphi_i(l_i - 0) = \varphi_{i+1}(l_i + 0), \qquad k_i\varphi_i'(l_i - 0) = k_{i+1}\varphi_{i+1}'(l_i + 0), \quad i = 1, 2, \ldots, m-1. \tag{21}$$

Then the function

$$u_i(x,t) = \sum_{n=1}^{\infty} \varphi_n\, Y_n\!\left(\frac{x - l_{i-1}}{k_i}\right) e^{-\lambda_n t}, \tag{22}$$

where the coefficients φ_n are defined by (18), is the unique classical solution of problem (1)-(4).

Proof. First we prove existence of the solution (22). Since $\left\{ Y_n\!\left(\frac{x-l_{i-1}}{k_i}\right) \right\}$ are the eigenfunctions and $\{\lambda_n\}$ are the eigenvalues of problem (1)-(4), it is straightforward to verify that the function $u(x,t)$ defined by (22) satisfies the equation, the initial condition, the boundary conditions, and the conjugation conditions of (1)-(4). The series (22) is a sum of the functions

$$u_n(x,t) = \varphi_n\, Y_n\!\left(\frac{x - l_{i-1}}{k_i}\right) e^{-\lambda_n t}. \tag{23}$$

We show that for any fixed $\varepsilon > 0$ the series

$$\sum_{n=1}^{\infty} u_n(x,t), \qquad \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}(x,t), \qquad \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2}(x,t)$$

converge uniformly on $\{(x,t) : l_0 < x < l_m,\ t \geq \varepsilon\}$. Clearly, $|\varphi| \leq K_1$, hence from (18) it follows that $|\varphi_n| \leq K_2$. Using (23) and the equalities

$$\frac{\partial u_n}{\partial t} = -\lambda_n\, \varphi_n\, Y_n\!\left(\frac{x - l_{i-1}}{k_i}\right) e^{-\lambda_n t}, \qquad \frac{\partial^2 u_n}{\partial x^2} = -\frac{\lambda_n}{k_i^2}\, \varphi_n\, Y_n\!\left(\frac{x - l_{i-1}}{k_i}\right) e^{-\lambda_n t},$$

we obtain, for $t \geq \varepsilon$,

$$|u_n(x,t)| \leq K_3\, e^{-\lambda_n \varepsilon}, \qquad \left\{ \left|\frac{\partial u_n}{\partial t}\right|, \left|\frac{\partial^2 u_n}{\partial x^2}\right| \right\} \leq K_4\, \lambda_n\, e^{-\lambda_n \varepsilon},$$

where the constants $K_i > 0$ ($i = 1, 2, 3, 4$) do not depend on n.

Therefore, using the asymptotics $\lambda_n \sim (\pi n / s_m)^2$, we have

$$\left\{ \sum_{n=1}^{\infty} |u_n(x,t)|,\ \sum_{n=1}^{\infty} \left|\frac{\partial u_n}{\partial t}(x,t)\right|,\ \sum_{n=1}^{\infty} \left|\frac{\partial^2 u_n}{\partial x^2}(x,t)\right| \right\} \;\leq\; \sum_{n=1}^{\infty} K\, n^2\, e^{-\left(\frac{\pi n}{s_m}\right)^2 \varepsilon},$$

for some constant $K > 0$ independent of n. Since the series on the right-hand side converges absolutely, the Weierstrass M test implies that the series for u, u_t, and u_{xx} converge uniformly for $t \geq \varepsilon$; hence $u(x,t)$, $\dfrac{\partial u(x,t)}{\partial t}$, and $\dfrac{\partial^2 u(x,t)}{\partial x^2}$ are continuous for $t \geq \varepsilon$. Now we must show that the series (22) converges uniformly on the whole domain Ω. Note that the n-th term of (22) is majorized by $|\varphi_n|$. Integrating by parts the integral in (18), we obtain

$$
\begin{aligned}
\varphi_n = {} & C_n\Bigg[-\frac{\beta_1 k_1\, \varphi_1(l_0)}{\lambda_n} + \frac{\varphi_1(l_0)}{\sqrt{\lambda_n}}\,\widetilde{\Phi}\big(s_1\sqrt{\lambda_n}\big) - \int_{l_0}^{l_1} \frac{\varphi_1'(\xi)}{\sqrt{\lambda_n}}\,\widetilde{\Phi}\left(\frac{\xi - l_0}{k_1}\sqrt{\lambda_n}\right) d\xi\Bigg] + \\
& C_n \frac{\varphi_2(l_2 - 0)}{\sqrt{\lambda_n}}\,\Phi\big(s_2\sqrt{\lambda_n}\big) - \frac{\varphi_2(l_1 + 0)}{\sqrt{\lambda_n}}\,\Phi\big(s_1\sqrt{\lambda_n}\big) - \\
& \int_{l_1}^{l_2} \frac{\varphi_2'(\xi)}{\sqrt{\lambda_n}}\,\Phi\left(\left(s_1 + \frac{\xi - l_1}{k_2}\right)\sqrt{\lambda_n}\right) d\xi \\
& \quad + \cdots \\
& C_n\Bigg[\frac{\varphi_{m-1}(l_{m-1} - 0)}{\sqrt{\lambda_n}}\,\Phi\big(s_{m-1}\sqrt{\lambda_n}\big) - \frac{\varphi_{m-1}(l_{m-2} + 0)}{\sqrt{\lambda_n}}\,\Phi\big(s_{m-2}\sqrt{\lambda_n}\big)\Bigg] - \\
& \int_{l_{m-2}}^{l_{m-1}} \frac{\varphi_{m-1}'(\xi)}{\sqrt{\lambda_n}}\,\Phi\left(\left(s_{m-2} + \frac{\xi - l_{m-2}}{k_{m-1}}\right)\sqrt{\lambda_n}\right) d\xi + \\
& C_n\Bigg[\frac{\varphi_m(l_m)}{\sqrt{\lambda_n}}\,\Phi\big(s_m\sqrt{\lambda_n}\big) - \frac{\varphi_m(l_{m-1} + 0)}{\sqrt{\lambda_n}}\,\Phi\big(s_{m-1}\sqrt{\lambda_n}\big)\Bigg] - \\
& \int_{l_{m-1}}^{l_m} \frac{\varphi_m'(\xi)}{\sqrt{\lambda_n}}\,\Phi\left(\left(s_{m-1} + \frac{\xi - l_{m-1}}{k_m}\right)\sqrt{\lambda_n}\right) d\xi,
\end{aligned}
$$

where Φ is given by (17) and

$$
\widetilde{\Phi}(z) = \alpha_1 \sin z + \beta_1 \frac{k_1}{\sqrt{\lambda_n}} \cos z. \tag{24}
$$

Taking into account the first relation in (21), $\varphi_i(l_i - 0) = \varphi_{i+1}(l_i + 0)$ for $i = 1, 2, \ldots, m-1$, and integrating once more, we obtain

$$
\begin{aligned}
\varphi_n = {} & C_n\Bigg[-\frac{\beta_1 k_1}{\lambda_n}\,\varphi_1(l_0) - \frac{k_1\alpha_1}{\lambda_n}\,\varphi_1'(l_0) + \frac{k_1}{\lambda_n}\,\varphi_1'(l_1 - 0)\,\Phi(s_1\sqrt{\lambda_n}) - \\
& \int_{l_0}^{l_1} \frac{k_1\varphi_1''(\xi)}{\lambda_n}\,\Phi\left(\frac{\xi - l_0}{k_1}\sqrt{\lambda_n}\right) d\xi\Bigg] + \\
& C_n\Bigg[\frac{k_2}{\lambda_n}\,\varphi_2'(l_2 - 0)\,\Phi(s_2\sqrt{\lambda_n}) - \frac{k_2}{\lambda_n}\,\varphi_2'(l_1 + 0)\,\Phi(s_1\sqrt{\lambda_n}) - \\
& \int_{l_1}^{l_2} \frac{k_2\varphi_2''(\xi)}{\lambda_n}\,\Phi\left(\left(s_1 + \frac{\xi - l_1}{k_2}\right)\sqrt{\lambda_n}\right) d\xi\Bigg] \\
& \quad + \cdots
\end{aligned} \tag{25}
$$

$$C_n\left[\frac{k_{m-1}}{\lambda_n}\,\varphi'_{m-1}(l_{m-1}-0)\,\Phi(s_{m-1}\sqrt{\lambda_n})-\frac{k_{m-1}}{\lambda_n}\,\varphi'_{m-1}(l_{m-2}+0)\,\Phi(s_{m-2}\sqrt{\lambda_n})-\right.$$
$$\left.\int_{l_{m-2}}^{l_{m-1}}\frac{k_{m-1}\varphi''_{m-1}(\xi)}{\lambda_n}\,\Phi\left(\left(s_{m-2}+\frac{\xi-l_{m-2}}{k_{m-1}}\right)\sqrt{\lambda_n}\right)d\xi\right]+$$
$$C_n\left[\frac{\varphi_m(l_m)}{\sqrt{\lambda_n}}\,\widetilde{\Phi}(s_m\sqrt{\lambda_n})+\frac{k_m}{\lambda_n}\,\varphi'_m(l_m)\,\Phi(s_m\sqrt{\lambda_n})-\right.$$
$$\left.\frac{k_m}{\lambda_n}\,\varphi'_m(l_{m-1}+0)\,\Phi(s_{m-1}\sqrt{\lambda_n})-\int_{l_{m-1}}^{l_m}\frac{k_m\varphi''_m(\xi)}{\lambda_n}\,\Phi\left(\left(s_{m-1}+\frac{\xi-l_{m-1}}{k_m}\right)\sqrt{\lambda_n}\right)d\xi\right],$$

where Φ is given by (17) and $\widetilde{\Phi}(z)=\alpha_1\sin z+\beta_1\dfrac{k_1}{\sqrt{\lambda_n}}\cos z$ (cf. (24)). Using the second relation in (21) together with (17) and (24), one checks that

$$\lambda_n\,\varphi_m(l_m)\,\widetilde{\Phi}\big(s_m\sqrt{\lambda_n}\big)+k_m\sqrt{\lambda_n}\,\varphi'_m(l_m)\,\Phi\big(s_m\sqrt{\lambda_n}\big)=\Delta(\lambda_n), \tag{26}$$

i.e., the left-hand side coincides with the characteristic equation evaluated at λ_n. Hence, using (20)-(21) and (26) in (25), we obtain

$$\varphi_n=-C_n\sum_{i=1}^{m}\frac{k_i}{\lambda_n}\int_{l_{i-1}}^{l_i}\varphi''_i(\xi)\,\Phi\left(\left(s_{i-1}+\frac{\xi-l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right)d\xi.$$

From this representation we derive the estimate

$$|\varphi_n|\le K\,\frac{|\alpha_n|}{n^2},\qquad K=\max_{1\le i\le m}k_i^2, \tag{27}$$

where α_n are the Fourier coefficients of the function $\varphi''(x)$ on the interval $[l_0,l_m]$ with respect to the orthonormal system of eigenfunctions $Y_n\big(\frac{x-l_{i-1}}{k_i}\big)$ defined by (19). From (27) it follows that

$$\sum_{n=1}^{\infty}|\varphi_n|\le K.$$

Thus the majorizing series converges absolutely; hence the series (22) converges uniformly on Ω and defines a continuous function $u(x,t)$ on Ω. This proves existence of a solution.

Uniqueness. Assume there are two solutions $\widetilde{u}(x,t)$ and $\widehat{u}(x,t)$. Let $v(x,t)=\widetilde{u}(x,t)-\widehat{u}(x,t)$. Then v solves

$$\frac{\partial v_i}{\partial t}=k_i^2\frac{\partial^2 v_i}{\partial x^2},\qquad (x,t)\in\Omega_i,\ i=1,2,\ldots,m, \tag{28}$$

with the initial condition

$$v(x,0)=0,\qquad l_0\le x\le l_m, \tag{29}$$

the boundary conditions

$$\begin{cases} \alpha_1 \dfrac{\partial v_1}{\partial x}(l_0, t) + \beta_1 v_1(l_0, t) = 0, \\ \alpha_2 \dfrac{\partial v_m}{\partial x}(l_m, t) + \beta_2 v_m(l_m, t) = 0, \end{cases} \qquad 0 \le t \le T, \tag{30}$$

and the conjugation conditions

$$\begin{cases} v_i(l_i - 0, t) = v_{i+1}(l_i + 0, t), \\ k_i \dfrac{\partial v_i}{\partial x}(l_i - 0, t) = k_{i+1} \dfrac{\partial v_{i+1}}{\partial x}(l_i + 0, t), \end{cases} \qquad 0 \le t \le T, \quad i = 1, 2, \ldots, m-1. \tag{31}$$

The solution of (28)-(31) can be expanded in the basis $Y_n\left(\frac{x - l_{i-1}}{k_i}\right)$; namely,

$$v_i(x, t) = \sum_{n=1}^{\infty} v_n(t)\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right), \tag{32}$$

where

$$v_n(t) = \sum_{i=1}^{m} \frac{1}{k_i} \int_{l_{i-1}}^{l_i} v_i(x, t)\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) dx. \tag{33}$$

Transforming (33) and differentiating with respect to t, we obtain

$$v_n'(t) = C_n \sum_{i=1}^{m} k_i \int_{l_{i-1}}^{l_i} \frac{\partial^2 v_i(x,t)}{\partial x^2}\, \Phi\left(\left(s_{i-1} + \frac{x - l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right) dx,$$

where Φ is given by (17) and C_n are the normalization constants.

Proceeding similarly, integrate twice by parts, using the boundary conditions (30), the conjugation conditions (31), and the identity

$$\Phi''\left(\left(s_{i-1} + \frac{x - l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right) = -\frac{\lambda_n}{k_i^2}\, \Phi\left(\left(s_{i-1} + \frac{x - l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right), \qquad i = 1, 2, \ldots, m.$$

We obtain

$$v_n'(t) = -\lambda_n v_n(t), \qquad \text{hence} \qquad v_n(t) = c_n e^{-\lambda_n t}, \quad n = 1, 2, \ldots.$$

Substituting this $v_n(t)$ into (33) gives

$$v_n(t) = \sum_{i=1}^{m} \frac{1}{k_i} \int_{l_{i-1}}^{l_i} v_i(x, t)\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) dx = c_n e^{-\lambda_n t}. \tag{34}$$

Passing to the limit in (34) as $t \to 0$ (which is permitted by the continuity of $v(x,t)$ on $\overline{\Omega}$), we obtain

$$\lim_{t \to 0} \sum_{i=1}^{m} \frac{1}{k_i} \int_{l_{i-1}}^{l_i} v_i(x, t)\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) dx = v_n(0) = c_n,$$

and therefore $c_n = 0$ for all $n = 1, 2, \ldots$. It follows from (32) that $v(x,t) \equiv 0$, whence $\widetilde{u}(x,t) = \widehat{u}(x,t)$. This completes the proof of the theorem.

4 Conclusion

In this paper, the solution of a multilayer problem for the heat equation with a discontinuous coefficient by the method of separation of variables is substantiated. The existence theorem of a unique classical solution of this problem is proved. The technique used here can also be applied to more general boundary problems and more general conjugation conditions.
Analytical solutions to such problems are very useful and necessary because they provide a higher level of understanding of the solution behavior and can be used for numerical solutions.

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