A note on Rogers semilattices of families of two embedded sets in the Ershov hierarchy

M. Manat[1], A. Sorbi[2]
[1]*Kazakh National University, Al-Farabi ave., 71, Almaty, 050038, Kazakhstan*
e-mail: manat.mustafa@gmail.com
[2]*Dipartimento di Scienze Matematiche ed Informatiche "Roberto Magari", Università di Siena, 53100 Siena, Italy*
e-mail: sorbi.andrea@gmail.com

Аннотация

Abstract

We show that for every ordinal notation a of a successor ordinal > 1, there is a Σ_a^{-1} family $\mathcal{A} = \{A, B\}$ with $A \subset B$ such that the Rogers semilattice of $\mathcal{A}$ has exactly one element. This extends a result of Badaev and Talasbaeva, proved for the case in which a is the ordinal notation of 2.

1 Introduction

In this paper we generalize a result of Badaev and Talasbaeva [2], stating that there is a family of d-c.e. sets $\mathcal{A} = \{A, B\}$, with $A \subset B$ (i.e., $A \subseteq B$, and $A \neq B$) such that the Rogers semilattice of $\mathcal{A}$ (in the sense of Goncharov and Sorbi [7]) consists of exactly one element. For unexplained notions and results on the theory of numberings, the reader is referred to [6].

In a nutshell, Goncharov and Sorbi's proposal for generalizing the theory of numberings to different notions of computability consists in the following. Let $\mathcal{C}$ be an abstract "notion" of computability, i.e. a countable class of sets of numbers, and let $\mathcal{A} \subseteq \mathcal{C}$: then a numbering $\pi : \omega \to \mathcal{A}$ is $\mathcal{C}$*–computable*, if $\{\langle k, x\rangle : x \in \pi(k)\} \in \mathcal{C}$. On numberings α, β of a same family, one defines $\alpha \leq \beta$ if there is a computable function f such that $\alpha = \beta \circ f$; and $\alpha \equiv \beta$ if $\alpha \leq \beta$ and $\beta \leq \alpha$; for $\mathcal{A} \subseteq \mathcal{C}$, we denote by $Com_{\mathcal{C}}(\mathcal{A})$ the set of $\mathcal{C}$–computable numberings of $\mathcal{A}$; we say that $\mathcal{A}$ is $\mathcal{C}$*–computable* if $Com_{\mathcal{C}}(\mathcal{A}) \neq \emptyset$; finally we denote by $\mathfrak{R}_{\mathcal{C}}(\mathcal{A})$ the set of *Rogers degrees* of the elements of $Com_{\mathcal{C}}(\mathcal{A})$, i.e. the set $Com_{\mathcal{C}}(\mathcal{A})/\equiv$; it can be shown that $\mathfrak{R}_{\mathcal{C}}(\mathcal{A})$, if nonempty, is an upper semilattice. When the class $\mathcal{C}$ is clearly understood from the context, it is customary to drop the prefix $\mathcal{C}$, and write simply "computable" instead of "$\mathcal{C}$–computable".

The motivation for Badaev and Talasbaeva's result lies in the fact that in the classical case, i.e. in the case $\mathcal{C} = \Sigma_1^0$–sets, it is well known ([6]) that the Rogers semilattice of any family $\{A, B\}$, with $A \subset B$, is infinite. This result holds of all abstract notions of computability (in the sense of Goncharov and Sorbi [7]) with reasonable "closure" properties. Indeed, let $\mathcal{A} = \{A, B\}$, with $A \subset B$: for every computably enumerable (c.e.) set U, one can define the numbering

$$\alpha_U(k) = \begin{cases} A, & \text{if } k \notin U; \\ B, & \text{if } k \in U. \end{cases}$$

which is computable, since

$$\{\langle k, x\rangle : x \in \alpha_U(k)\} = \{\langle k, x\rangle : x \in A \text{ or } [x \in B \text{ and } k \in U]\} \quad (1)$$

and the latter set is Σ^0_1. It is then easy to see that, for all pairs of c.e. sets U, V,

$$U \leq_m V \Leftrightarrow \alpha_U \leq \alpha_V$$

(where $\leq_m$ denotes many-one reducibility), and thus the upper semilattice of the c.e. m–degrees is embeddable (as an upper semilattice) into the Rogers semilattice of the family $\{A, B\}$, showing that this Rogers semilattice is infinite.

Clearly the argument remains valid for every notion of computability $\mathcal{C}$ for which, given any c.e. set U, the right-hand side of (1) is still in $\mathcal{C}$. In particular:

Theorem 1.1 *If* $\mathcal{C} \in \{\Sigma^0_n, \Pi^0_n : n \geq 1\} \cup \{\Sigma^1_1, \Pi^1_1, \Delta^1_1 : n \geq 0\}$ *then the Rogers semilattice of any* $\mathcal{C}$*-computable family* $\{A, B\}$*, with* $A \subset B$*, is infinite.*

Proof. The proof is immediate. Notice that in most cases one can embed upper semilattices of m–degrees, that are “bigger” than the upper semilattice of c.e. m–degrees: for instance, let $\mathcal{C} = \Sigma^0_n$, $n \geq 1$: if $A, B, U \in \Sigma^0_n$, then the right-hand side of (1) is still in Σ^0_n, and thus the upper semilattice of the Σ^0_n m–degrees is embeddable into the Rogers semilattice of $\{A, B\}$.

In this paper, we extend Badaev and Talasbaeva’s result showing that for every $n \geq 2$ there is a family $\mathcal{A} = \{A, B\}$ with $A \subset B$, such that $\mathfrak{R}_{\Sigma^{-1}_n}(\mathcal{A})$ has exactly one element. The result admits a further extension to the infinite levels of the Ershov hierarchy of Δ^0_2 sets, given by notations of successor ordinals. The proof is a straightforward generalization of [2].

2 Computable numberings for families of sets in the Ershov hierarchy

We refer to Kleene’s system O of ordinal notations for computable ordinals: for details, see [9]. In particular, for $a \in O$, the symbol $|a|_O$ represents the ordinal of which a is a notation. We now briefly recall the definition of the Ershov hierarchy, introduced in [3, 4, 5]. Our presentation is based on [8].

Definition 2.1 *If* a *is a notation for a computable ordinal, then a set of numbers* A *is said to be* Σ^{-1}_a *if there are a computable function* $f(z, t)$ *and a partial computable function* $\gamma(z, t)$ *such that, for all* z*,*

1. $A(z) = \lim_t f(z, t)$*, with* $f(z, 0) = 0$*; (here, given a set* X*, and a number* z*, the symbol* $X(z)$ *denotes the value of the characteristic function of* X *on* z*);*
2. *(a)* $\gamma(z, t) \downarrow \Rightarrow \gamma(z, t+1) \downarrow$*, and* $\gamma(z, t+1) \leq_O \gamma(z, t) <_O a$*;*
   (b) $f(z, t+1) \neq f(z, t) \Rightarrow \gamma(z, t+1) \downarrow \neq \gamma(z, t)$.

We call the partial function γ *the* mind–change function *for* A*, relatively to* f*.*

A Σ^{-1}_a–approximation *to a* Σ^{-1}_a*–set* A*, is a pair* $\langle f, \gamma\rangle$*, where* f *and* γ *are respectively a computable function and a partial computable function satisfying 1. and 2., above, for* A*.*

Following [7], we give the following:

Definition 2.2 *A* Σ^{-1}_a–computable numbering*, or simply a* computable numbering*, of a family* $\mathcal{A}$ *of* Σ^{-1}_a*–sets is an onto function* $\pi : \omega \longrightarrow \mathcal{A}$*, such that*

$$\{\langle k, x\rangle : x \in \pi(k)\} \in \Sigma^{-1}_a.$$

To be fully explicit, a computable numbering of a family $\mathcal{A}$ of Σ^{-1}_a–sets is an onto function $\pi : \omega \longrightarrow \mathcal{A}$ for which there exist a computable function $f(y, t)$ and a partial computable function $\gamma(y, t)$, such that

1. $\pi(k)(x) = \lim_t f(\langle k, x\rangle, t)$, with $f(y, 0) = 0$ for all y;

2. γ satisfies 2. of Definition 2.1 relatively to f.

In the rest of the paper we will write $Com_a^{-1}(\mathcal{A})$ for $Com_{\Sigma_a^{-1}}(\mathcal{A})$, and $\mathfrak{R}_a^{-1}(\mathcal{A})$ for $\mathfrak{R}_{\Sigma_a^{-1}}(\mathcal{A})$.

We recall (see e.g. [8]) that there is an effective indexing of the family of all Σ_a^{-1} sets. For instance, given a, let U_a be Σ_a^{-1}-complete (see e.g. [1]). Then $\{A_z\}_{z\in\omega}$, where

$$A_z = \phi_z^{-1}[U_a]$$

provides such an effective indexing, where "effective" means that $\{\langle x, z\rangle : x \in A_z\} \in \Sigma_a^{-1}$. From this, it is possible to define an effective indexing of all computable numberings of families of Σ_a^{-1} sets. Indeed, if π is a computable numbering of a family of Σ_a^{-1} sets, then by the above and the *s*-*m*-*n*-Theorem, there exists a total computable function f such that, for every k,

$$\pi(k) = \phi_{f(k)}^{-1}[U_a].$$

Now, let $u(e, k)$ be a partial computable universal function of two variables, and let

$$\pi_e(k) = \phi_{u(e,k)}^{-1}[U_a],$$

with the understanding that if $u(e, k)\uparrow$ then $\pi_e(k) = \emptyset$. It follows that $\{\pi_e\}_{e\in\omega}$ is the desired indexing: notice thatt

$$\{\langle e, k, x\rangle : x \in \pi_e(k)\} \in \Sigma_a^{-1}. \tag{2}$$

An indexing satisfying (2) is called a Σ_a^{-1}–*computable indexing* of all Σ_a^{-1}–computable numberings. From e one has (see e.g. [8]) an effective way of getting a computable function f_e and a partial computable function γ_e witnessing that $\{\langle k, x\rangle : x \in \pi_e(k)\}$ is Σ_a^{-1}, as in Definition 2.2.

Definition 2.3 *Let $a \in O$, and let $\{R_k\}_{k\in\omega}$ be a computable partition of a computable set into infinite sets. We say that a Σ_a^{-1}–computable numbering π has a* special approximation *with respect to $\{R_k\}_{k\in\omega}$ if there is a Σ_a^{-1}–approximation $\langle f, \gamma\rangle$ to π (in the sense of Definition 2.2) such that, for every k, s, x,*

1. $|\{y : f(\langle k, y\rangle, s + 1) \neq f(\langle k, y\rangle, s)\}| \leq 1$;

2. $f(\langle k, x\rangle, s + 1) \neq f(\langle k, x\rangle, s) \Rightarrow s \in R_k$,

where, for a given set X, the symbol $|X|$ denotes the cardinality of X.

The features of a special approximation (i.e. at most one change at each stage; and changes may be expected only at certain stages), for a numbering, will be used in the proof of Theorem 3.1. The following is a useful, although straightforward, lemma:

Lemma 2.4 *For every ordinal notation a, every Σ_a^{-1}–computable numbering π, and every computable partition $\{R_k\}_{k\in\omega}$ of a computable set into infinite sets, π has a special approximation with respect to $\{R_k\}_{k\in\omega}$.*

Proof. We show how to build a special approximation starting from a given one. Suppose we are given a Σ_a^{-1}–approximation to π, and let $\{R_k\}_{k\in\omega}$ be a computable partition of a computable set into infinite sets. The proof is by induction. At stage s we define for every z, $\hat{f}(z, s)$, $\hat{\gamma}(z, s)$, and two finite sets $L^i(z, s)$ and $L^f(z, s)$, with $L^f(z, s) \subseteq L^i(z, s)$. On $L^i(z, s)$

we define also a strict linear order $<_s$: we use the same symbol $<_s$ to denote its restriction to $L^f(z,s)$.

Stage 0). For every z,h, let $\hat{f}(z,0)=0$, $\hat{\gamma}(z,0)=\uparrow$, $L^i(h,0)=L^f(h,0)=\emptyset$, and $<_0=\emptyset$.

Stage $s+1$). For every h, let

$$L^i(h,s+1)=L^f(h,s)\cup\{\langle y,s\rangle : f(\langle h,y\rangle,s+1)\neq f(\langle h,y\rangle,s)\}.$$

Order $L^i(h,s+1)$ as follows: if $v,w\in L^i(h,s+1)$ then

$$v<_{s+1} w \Leftrightarrow [v,w\in L^f(h,s)\ \&\ v<_s w]\ \vee\ [v,w\notin L^f(h,s)\ \&\ v<w].$$

Next, we distinguish two cases:

1. If $s\notin\bigcup_k R_k$, then define for all z,h, $\hat{f}(z,s+1)=\hat{f}(z,s)$, $\hat{\gamma}(z,s+1)=\hat{\gamma}(z,s)$, and $L^f(h,s+1)=L^i(h,s+1)$.
2. Assume $s\in R_k$. If $L^i(k,s+1)=\emptyset$ then for every z,h define $\hat{f}(z,s+1)=\hat{f}(z,s)$, $\hat{\gamma}(z,s+1)=\hat{\gamma}(z,s)$, and $L^f(h,s+1)=L^i(h,s+1)$. If $L^i(k,s+1)\neq\emptyset$, and $\langle x,r\rangle$ is the $<_{s+1}$–least number of $L^i(k,s+1)$, then for every z define $\hat{f}(z,s+1)=\hat{f}(z,s)$ and $\hat{\gamma}(z,s+1)=\hat{\gamma}(z,s)$, unless $z=\langle k,x\rangle$, in which case define

$$\begin{aligned}\hat{f}(\langle k,x\rangle,s+1)&=f(\langle k,x\rangle,r+1),\\ \hat{\gamma}(\langle k,x\rangle,s+1)&=\gamma(\langle k,x\rangle,r+1).\end{aligned}$$

   For all h, let $L^f(h,s+1)=L^i(h,s+1)$, unless $h=k$, in which case define

$$L^f(k,s+1)=L^i(k,s+1)-\{\langle x,r\rangle\}.$$

In order to show that $\langle\hat{f},\hat{\gamma}\rangle$ is the desired Σ^{-1}_a–approximation, notice that for every u there are bijective finite sets $C,\hat{C}$, with

$$C=\{s_0<\cdots<s_{n_u}\},\quad \hat{C}=\{r_0<\cdots<r_{n_u}\},$$

such that

$$C=\{s:\hat{f}(u,s+1)\neq\hat{f}(u,s)\},\quad \hat{C}=\{r:f(u,r+1)\neq f(u,r)\},$$

and

$$\hat{f}(u,s_i+1)=f(u,r_i+1),\quad \hat{\gamma}(u,s_i+1)=\gamma(u,r_i+1).$$

Finally, notice that $\hat{f}(z,s+1)\neq\hat{f}(z,s)$ only if $z=\langle k,x\rangle$, and for some r, $\langle x,r\rangle$ is $<_{s+1}$–least of $L^i(k,s+1)$, and $s\in R_k$. Hence $\langle\hat{f},\hat{\gamma}\rangle$ is special.

A similar argument shows:

Corollary 2.5 *Let $\{\pi_e\}_{e\in\omega}$ be a Σ^{-1}_a–computable indexing of all computable numberings of all Σ^{-1}_a–computable families, and let $\{R_{e,k}\}_{e,k\in\omega}$ be a computable partition of a computable set into infinite sets. For every e, one can uniformly find a special approximation to π_e with respect to the partition $\{R_{e,k}\}_{k\in\omega}$; in other words, there is a pair $\langle f,\gamma\rangle$ of functions having three variables, with f computable and γ partial computable, such that, for every e, the pair $\langle f_e,\gamma_e\rangle$ is a special approximation to π_e with respect to $\{R_{e,k}\}_{k\in\omega}$, where $f_e(\langle k,x\rangle,s)=f(e,\langle k,x\rangle,s)$ and $\gamma_e(\langle k,x\rangle,s)=\gamma(e,\langle k,x\rangle,s)$.*

Corollary 2.6 *For every $A\in\Sigma^{-1}_a$ and any computable infinite set R there is a Σ^{-1}_a–approximation to A which is special with respect to R, meaning a Σ^{-1}_a–approximation $\langle f,\gamma\rangle$ to A, such that for every s,z, $|\{y:f(y,s+1)\neq f(y,s)\}|\leq 1$, and $f(z,s+1)\neq f(z,s)$ or $\gamma(z,s+1)\neq\gamma(z,s)$ only if $s\in R$.*

3 The theorem

We are now in a position to prove the theorem:

Theorem 3.1*For every ordinal notation a, with $|a|_O > 1$ and $|a|_O$ successor, there exists a Σ^{-1}_a–computable family $\mathcal{A} = \{A, B\}$, with $A \subset B$ such that $|\mathcal{R}^{-1}_a(\mathcal{A}))| = 1$.*

Proof. Given a, with $|a|_O > 1$ and $|a|_O$ successor, we build $A \subseteq B$, $A \neq B$, and a Σ^{-1}_a–computable numbering α of $\mathcal{A} = \{A, B\}$, such that for every Σ^{-1}_a–computable numbering π of $\mathcal{A}$, we have $\pi \equiv \alpha$.

We will define $\alpha(0) = A$, $\alpha(1) = B$, and $\alpha(k) = B$ for all $k \geq 1$.

Let $\{\pi_e\}_{e\in\omega}$ be a Σ^{-1}_a–computable indexing of all Σ^{-1}_a–computable numberings, and by Corollary 2.5 let us refer to uniform special approximations to these numberings with respect to the partition $\{R_{e,k}\}_{e,k\in\omega}$, where

$$R_{e,k} = \{\langle e, k, x\rangle + 1 : x \in \omega\} :$$

if $\langle f, \gamma\rangle$ is such a uniform special approximation, write $f_e(\langle k, x\rangle, s) = f(e, \langle k, x\rangle, s)$ and $\gamma_e(\langle k, x\rangle, s) = \gamma(e, \langle k, x\rangle, s)$. We will define α so that, for every e, k, the following requirement is satisfied:

$$Q_{e,k} : \qquad \pi_e(k) \in \{A, B\} \Rightarrow\ g_e(k) \text{ defined and } \pi_e(k) = \alpha(g_e(k)),$$

where g_e is a partial computable function defined by us.

The construction is by stages. At stage $s+1 > 1$ with $s \in R_{e,k}$ (notice that for every $t > 0$ there is a unique e, k such that $t \in R_{e,k}$)) our action aims at making $\pi_e(k) \notin \mathcal{A}$ (so that π_e is not a numbering of $\mathcal{A}$), or we define $g_e(k) \in \{0, 1\}$ so as to ensure that $\pi_e(k) = \alpha(g_e(k))$, if eventually $\pi_e(k) \in \{A, B\}$. This is enough to show the claim since, if $\pi_e \in Com^{-1}_a(\mathcal{A})$ then trivially $\alpha \leq \pi_e$.

Our attempts at diagonalizing $\pi_e(k)$ against A, B at stage $s + 1$ with $s \in R_{e,k}$ make use of (computably given) witnesses $a_0(e, k), a_1(e, k)$: we assume that $a_0(e, k) \neq a_1(e, k)$, and $\{a_0(e, k), a_1(e, k)\} \cap \{a_0(e', k'), a_1(e', k')\} = \emptyset$ if $\langle e, k\rangle \neq \langle e', k'\rangle$. So at this stage we define $f_A(a_0(e, k), s + 1)$, $f_A(a_1(e, k), s + 1)$, and $\gamma_A(a_0(e, k), s + 1)$, $\gamma_A(a_1(e, k), s + 1)$; and likewise $f_B(a_0(e, k), s + 1)$, $f_B(a_1(e, k), s + 1)$ and $\gamma_B(a_0(e, k), s + 1)$, $\gamma_B(a_1(e, k), s + 1)$. The pair $\langle f_A, \gamma_A\rangle$ will be a Σ^{-1}_a–approximation to A; the pair $\langle f_B, \gamma_B\rangle$ will be a Σ^{-1}_a–approximation to B. From these two pairs we will also get a Σ^{-1}_a–approximation $\langle \hat{f}, \hat{\gamma}\rangle$ to α, by letting $\hat{f}(\langle 0, x\rangle, s + 1) = f_A(x, s + 1)$, $\hat{f}(\langle k, x\rangle, s + 1) = f_B(x, s + 1)$, $\hat{\gamma}(\langle 0, x\rangle, s + 1) = \gamma_A(x, s + 1)$, and $\hat{\gamma}(\langle k, x\rangle, s + 1) = \gamma_B(x, s + 1)$, for $k \geq 1$. It is understood that all values of $f_A, f_B, \gamma_A, \gamma_B$ that are not explicitly defined maintain the same values as at the preceding stage, the values at $s = 0$ being 0 for f_A, f_B, and undefined for γ_A, γ_B. Since A and B are disjoint, it is straightforward to see that $\langle \hat{f}, \hat{\gamma}\rangle$ is a Σ^{-1}_a–approximation to α. Finally, at stage $s+1$, $s \in R_{e,k}$, if $g_{e,s}(k) =\uparrow$ (i.e. the value of $g_e(k)$ is still undefined), we might define also $g_{e,s+1}(k) = 0$ or $g_{e,s+1}(k) = 1$. After defining $g_e(k)$ our only worry will be to make sure that $\pi_e(k) \neq A$ if $g_e(k) = 1$, and $\pi_e(k) \neq B$ if $g_e(k) = 0$. If π_e is a numbering of $\mathcal{A}$, then eventually g_e is total, and $\pi_e \leq \alpha$ via g_e.

Without loss of generality we may also assume that our uniform special approximation $\langle f, \gamma\rangle$ also satisfies, for every e, k, x, $f(e, \langle k, x\rangle, 1) = 0$.

Let $s \in R_{e,k}$: at stage $s+1$ we monitor the *initial* $\pi_e(k)$*-setup* at $s+1$, meaning the table

$$\left| \begin{array}{ccc} \pi_e(k) & (u,i) & (v,j) \\ A & (u',i') & (v',j') \\ B & (u'',i'') & (v'',j'') \\ g_e(k) & w & \end{array} \right| \tag{3}$$

where $u,u',u'',v,v',v'' \in \{0,1\}$, $w \in \{0,1,\uparrow\}$, and $i,i',i'',j,j',j'' \in \omega$. The table has the following meaning: for simplicity, let $a_0 = a_0(e,k)$, and $a_1 = a_1(e,k)$):

1. (first line) $f_e(\langle k,a_0\rangle, s+1) = u$, $f_e(\langle k,a_1\rangle, s+1) = v$; moreover, for $r \leq s+1$, $f_e(\langle k,a_0\rangle, r)$ has already made i changes, and $f_e(\langle k,a_1\rangle, r)$ has already made j changes;
2. (second line): $f_A(a_0,s) = u'$, $f_A(a_1,s) = v'$; moreover, for $r \leq s$, $f_A(a_0,r)$ has already made i' changes, and $f_A(a_1,r)$ has already made j' changes;
3. (third line): $f_B(a_0,s) = u''$, $f_B(a_1,s) = v''$; moreover, for $r \leq s$, $f_B(a_0,r)$ has already made i'' changes, and $f_B(a_1,r)$ has already made j'' changes;
4. (fourth line) w denotes the value of $g_e(k)$ at the end of stage s.

At the end of the stage, as a result of our action performed during the stage we have the *final* $\pi_e(k)$*-setup* at $s+1$, i.e. the table

$$\left| \begin{array}{ccc} \pi_e(k) & (u,i) & (v,j) \\ A & (\overline{u'},\overline{i'}) & (\overline{v'},\overline{j'}) \\ B & (\overline{u''},\overline{i''}) & (\overline{v''},\overline{j''}) \\ g_e(k) & \overline{w} & \end{array} \right|.$$

Notice that the first line of the final setup is just the same as in the initial setup; the overlined symbols denote the (possibly new) values of A, B, and of $g_e(k)$, at the end of the stage.

The advantage of working with uniform special approximations is that we can take complete care of the requirement $Q_{e,k}$ only by looking at the behavior of $\pi_e(k)$ at stages $s+1$ with $s \in R_{e,k}$: moreover at each such stage at most one of a_0, a_1 may change its membership status in $\pi_e(k)$.

The construction Stage 0): Let $f_A(z,0) = f_B(z,0) = 0$ and let $\gamma_A(z,0) = \gamma_B(z,0) =\uparrow$.

Stage 1): for every e,k, let $f_A(a_0(e,k),1) = 1$, $\gamma_A(a_0(e,k),1) = 2$ (remember that $|2|_O = 1$); let $f_B(a_0(e,k),1) = f_B(a_1(e,k),1) = 1$, $\gamma_B(a_0(e,k),1) = b$ (where $2^b = a$, i.e. b is the unique ordinal notation, with $b <_O a$, of the predecessor of $|a|_O$), and $\gamma_B(a_1(e,k),1) = 1$ (remember that $|1|_O = 0$). So for every e,k, the final $\pi_e(k)$–setup at stage 1 is

$$\left| \begin{array}{ccc} \pi_e(k) & (0,0) & (0,0) \\ A & (1,1) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & \uparrow & \end{array} \right|$$

Stage $s+1), s > 0$. Suppose $s \in R_{e,k}$, and assume that the initial $\pi_e(k)$–setup at this stage is $\sigma^i(s+1)$. For simplicity let $a_i = a_i(e,k)$. We distinguish the following cases:

1. If

$$\sigma^i(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (u,i) & (1,1) \\ A & (u',i') & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & \uparrow & \end{array} \right|$$

(with $(u,i) = (0,0)$ and $(u',i') = (1,1)$, or $(u,i) = (1,1)$ and $(u',i') = (0,2)$) then define $g_e(k) = 1$, $f_A(a_0, s+1) = 0$, $\gamma_A(a_0, s+1) = 1$; so the final setup is

$$\sigma^f(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (u,i) & (1,1) \\ A & (0,2) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & 1 & \end{array} \right|$$

2. If

$$\sigma^i(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (1,1) & (0,0) \\ A & (1,1) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & \uparrow & \end{array} \right|$$

then extract a_0 from A, i.e. define $f_A(a_0, s+1) = 0$, and $\gamma_A(a_0, s+1) = 1$, so the final setup is

$$\sigma^f(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (1,1) & (0,0) \\ A & (0,2) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & \uparrow & \end{array} \right|$$

3. If

$$\sigma^i(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (0,2) & (0,0) \\ A & (0,2) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & \uparrow & \end{array} \right|$$

then define $g_e(k) = 0$: so the final setup is

$$\sigma^f(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (0,2) & (0,0) \\ A & (0,2) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & 0 & \end{array} \right|$$

4. If none of the above cases applies then, as we argue in the verification, $g_e(k)$ is defined. If σ^i_{s+1} is *not winning* (meaning that either $g_e(k) = 1$ and $v = v'$, or $g_e(k) = 0$ and $u = u''$: here u, u'', v, v' are as in table (3); also we assume by induction that at the end of the previous relevant stage the final $\pi_e(k)$-setup is winning) then:

   (a) if $g_e(k) = 1$ then (by properties of a special approximation) we have $f_e(\langle k, a_1\rangle, s+1) \neq f_e(\langle k, a_1\rangle, s)$, and the initial setup at $s+1$ is of the form:

$$\sigma^i(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (u,i) & (v,j) \\ A & (0,2) & (v,j-2) \\ B & (1,1) & (1,1) \\ g_e(k) & 1 & \end{array} \right|$$

let $A(a_1, s+1) = 1 - A(a_1, s)$ and $\gamma_A(a_1, s+1) = \gamma_e(\langle k, a_1\rangle, s)$: the final setup is

$$\sigma^f(s+1) = \begin{vmatrix} \pi_e(k) & (u,i) & (v,j) \\ A & (0,2) & (1-v, j-1) \\ B & (1,1) & (1,1) \\ g_e(k) & 1 & \end{vmatrix}$$

(b) if $g_e(k) = 0$ then, similarly, we have $f_e(\langle k, a_0\rangle, s+1) \neq f_e(\langle k, a_0\rangle, s)$, and

$$\sigma^i(s+1) = \begin{vmatrix} \pi_e(k) & (u,i) & (v,j) \\ A & (0,2) & (0,0) \\ B & (u, i-2) & (1,1) \\ g_e(k) & 1 & \end{vmatrix}$$

let $B(a_0, s+1) = 1 - B(a_0, s)$ and $\gamma_B(a_0, s+1) = \gamma_e(\langle k, a_0\rangle, s+1)$: the final setup is

$$\sigma^f(s+1) = \begin{vmatrix} \pi_e(k) & (u,i) & (v,j) \\ A & (0,2) & (0,0) \\ B & (1-u, i-1) & (1,1) \\ g_e(k) & 0 & \end{vmatrix}$$

If none of the above cases apply, or after acting through one of the above cases, move to stage $s+2$.

This concludes the construction, with A, B eventually given by $A(z) = \lim_s f_A(z, s)$ and $B(z) = \lim_s f_B(z, s)$, and with $\alpha \in Comp{-}1a(\{A, B\})$ defined as explained earlier.

Verification. Let $\mathcal{A} = \{A, B\}$, let e, k be given, and $a_i = a_i(e, k)$. We want to show that either $\pi_e(k) \notin \{A, B\}$, or $g_e(k)$ is defined and $\pi_e(k) = \alpha(g_e(k))$.

If $f_e(\langle k, a_0\rangle, s) = f_e(\langle k, a_1\rangle, s) = 0$ for all s, then the claim is true, since in this case $\pi_e(k) \notin \{A, B\}$, as $a_0 \in A$ and $a_0, a_1 \in B$. Since $a_1 \notin A$, we trivially have in this case that $A \cap \{a_0, a_1\} \subset B \cap \{a_0, a_1\}$.

Otherwise, by definition of a special approximation, there is a least $s_0 > 0$ such that $f_e(\langle k, a_0\rangle, s_0+1) \neq f_e(\langle k, a_0\rangle, s_0)$ or $f_e(\langle k, a_1\rangle, s_0+1) \neq f_e(\langle k, a_1\rangle, s_0)$, but noth both, and $s_0 \in R_{e,k}$.

1. If $f_e(\langle k, a_1\rangle, s_0+1) \neq f_e(\langle k, a_1\rangle, s_0)$, then at stage s_0+1 we act as in case 1 of the construction, and get the final setup

$$\sigma^f(s_0+1) = \begin{vmatrix} \pi_e(k) & (0,0) & (1,1) \\ A & (0,2) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & 1 & \end{vmatrix}$$

   From now on we may act on behalf of $Q_{e,k}$ only through part 4a of the construction. Every time we act in this way (due to a change of $f_e(\langle k, a_1\rangle, s+1) \neq f_e(\langle k, a_1\rangle, s)$, at stages $s+1$, with s in $R_{e,k}$, we have a not winning initial setup of the form

$$\sigma^i(s+1) = \begin{vmatrix} \pi_e(k) & (u,i) & (v,j) \\ A & (0,2) & (v, j-2) \\ B & (1,1) & (1,1) \\ g_e(k) & 1 & \end{vmatrix}$$

ending with a final setup of the form

$$\sigma^f(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (u,i) & (v,j) \\ A & (0,2) & (1-v,j-1) \\ B & (1,1) & (1,1) \\ g_e(k) & 1 & \end{array} \right|$$

after which we define $\gamma_A(a_1, s+1) = \gamma_e(\langle k, a_1\rangle, s)$. But as $\lim_t f_e(\langle k, a_1\rangle, t)$ exists, we eventually achieve that if $\pi_e(k) \in \mathcal{A}$ then $\pi_e(k) = \alpha(g_e(k))$ as $\pi_e(k) = \alpha(1)$. Notice that for every t, $\gamma_A(a_1, t)$ is correctly defined making this function a correct mind–change function, since for every $t+1$ starting from the first change of the memberhip status of a_1 in $\pi_e(k)$, we define $\gamma_A(a_1, t+1) = \gamma_e(\langle k, a_1\rangle, t)$. The values $\gamma_A(a_0, t)$, $\gamma_B(a_0, t)$ and $\gamma_B(a_1, t)$ are trivially correctly defined, since there is no further changes in the membership status of a_0 in A, and of a_0, a_1 in B.

2. If $f_e(\langle k, a_0\rangle, s_0 + 1) \neq f_e(\langle k, a_0\rangle, s_0)$ then we first act through 2, and then, if we act again, we may assume that we act through 3, since otherwise action through 1 would yield a final setup of the form

$$\sigma^f(s+1) = \left| \begin{array}{ccc} \pi_e(k) & (1,1) & (1,1) \\ A & (0,2) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & 1 & \end{array} \right|$$

but then by an argument similar to the one used in the previous case, we may conclude that suitably changing $f_A(a_1, s+1)$ in response to, and diagonalizing against, corresponding changes of $f_e(\langle k, a_1\rangle, s+1)$ we eventually achieve success of our strategy. So assume that when we act again at, say $s_1 + 1$, we do it through 3 because an initial setup of the form

$$\sigma^i(s_1+1) = \left| \begin{array}{ccc} \pi_e(k) & (0,2) & (0,0) \\ A & (0,2) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & \uparrow & \end{array} \right|$$

At the same stage, we get the final setup

$$\sigma^f(s_1+1) = \left| \begin{array}{ccc} \pi_e(k) & (0,2) & (0,0) \\ A & (0,2) & (0,0) \\ B & (1,1) & (1,1) \\ g_e(k) & 0 & \end{array} \right|$$

From now on we may act on behalf of $Q_{e,k}$ only through part 4b of the construction, when finding initial $\pi_e(k)$–setups that are not winning: we start by suitably changing $f_B(a_0, s+1)$ in response to, and diagonalizing against, corresponding changes of $f_e(\langle k, a_0\rangle, s+1)$. By Δ^0_2-ness, the process eventually stops, at the end of which we have correctly defined the values of the mind–change functions γ_A, γ_B on a_0 and a_1. To see for instance that the values $\gamma_B(a_0, t)$ are correctly defined remember that we define $\gamma_B(a_0, 1) = b$ (where $b <_O a$ is a notation for the predecessor of $|a|_O$), and then we may redefine the values only at a stage $s_2 + 1$ when $f_e(\langle k, a_0\rangle, s+1)$ has already made two changes, and so, by construction, $\gamma_B(a_1, s_2 + 1) = \gamma_e(\langle k, a_0\rangle, s_2 + 1) <_O b$. From this stage on, the next values of γ_B on a_1 will be defined through the corresponding values of γ_e on $\langle k, a_1\rangle$.

In either case 1 or 2, we have $a_0 \notin A$, $a_1 \in B$; if $g_e(k)$ is not defined, or $g_e(k) = 0$ then $a_1 \notin A$; if $g_e(k) = 1$ then $a_0 \in B$. In any case we have that $A \cap \{a_0, a_1\} \subset B \cap \{a_0, a_1\}$. Since this holds of every e, k, we conclude that $A \subset B$.

Remark *A closer look at the construction shows that if* $|a|_O = n \in \omega$, $n \geq 2$, *then for every* e, k, $f_A(a_0(e,k), t)$ *changes at least once and at most* 2 *times;* $f_A(a_1(e,k), t)$ *and* $f_B(a_0(e,k), t)$ *change at most* $n-1$ *times (and* $f_B(a_0(e,k), t)$ *at least once);* $f_B(a_1(e,k), t)$ *changes exactly once. Thus in general* B *is* $n-1$*-c.e., and* A *is* $n-1$*-c.e. or* A *is* 2*-c.e. if* $n = 2$, *in accordance with a similar remark made for* $n = 2$ *by Badaev and Talasbaeva [2].*

Problem. We do not know if Theorem 3.1 is true also of ordinal notations of limit ordinals, although we conjecture that it is so.

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