1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

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ON PROPER EXPANSIONS AND PROPER CONTRACTIONS OF NONLINEAR OPERATORS REPRESENTED IN THE FORM OF A PRODUCT

Today there are many works devoted to the questions of expansion and contraction of operators [1–12]. In all these works the questions of expansion of the additive “minimal” operator and the questions of contraction of the additive “maximal” operator are considered. In this paper it is shown that these restrictions on the additivity of the corresponding operators are not essential. In [10] the questions of proper contraction of a maximal operator represented as a product are considered, i.e., the relationship between the set of proper contractions of the operator $A = LM$ and the sets of proper contractions of the operators L and M is established. Here, an abstract theorem is proved which allows us to establish the relationship between the set of proper extensions of the operator $A_0 = L_0M_0$ and the sets of proper extensions of the operators L_0 and M_0. In this connection, we prove an abstract theorem that allows us to describe the correct contractions of one class of nonlinear operators represented as a product.
Key words: operator, correct expansion, correct contraction, regular expansion, Bitsadze-Samarskii type problem.

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Көбейтінді түрінде берілген сызықтық емес операторлардың дұрыс кеңейтілімдері мен дұрыс тарылымдары туралы.

Осы таңда операторлардың дұрыс тарылымы мен дұрыс кеңейтілімі бойынша көптеген жұмыстар жарық көрген [1–12]. Бұл жұмыстарда аддитивті "минимальды" операторлардың кеңейтілімдері мен аддитивті "максимальды" операторлардың тарылымдары қарастырылған. Бұл жұмыста қарастырылатын операторлардың аддитивтілігі маңызды болмайтыны көрсетілген. Автордың [10] еңбегінде көбейтінді түрінде берілген сызықтық максимальды операторлардың дұрыс тарылымдары қарастырылған, яғни аталмыш еңбекте $A = LM$ операторының барлық дұрыс тарылымдары жиыны мен L және M операторларының барлық дұрыс тарылымдары жиындары арасында өзара бірмәнді сәйкестік орнатылған. Бұл жұмыста $A_0 = L_0M_0$ операторының барлық дұрыс кеңейтілімдері мен L_0 және M_0 операторларының барлық дұрыс кеңейтілімдері арасында тығыз байланыс барын көрсететін абстракциялы теорема дәлелденген. Осы орайда дәлелденген теорема көбейтінді түрінде берілген қайсыбір сызықтық емес операторлар санатынаның дұрыс кеңейтілімдерін сипаттауға болатыны мысал арқылы көрсетілген.
Түйін сөздер: оператор, дұрыс кеңейтілім, дұрыс тарылым, регулярлы кеңейтілім, Бицадзе-Самарский типтес есептер.

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О корректных расширениях и коррекных сужениях нелинейных операторов, представленных в виде произведения

На сегодня имеются множество работ, посвященных вопросам расширения и сужения операторов [1–12]. Во всех этих работах рассматриваются вопросы расширения аддитивного "минимального" оператора и вопросы сужения аддитивного "максимального" оператора. В данной работе показано, что эти ограничения аддитивности соответствующих операторов не существенны. В работе [10] рассмотрены вопросы корректного сужения максимального оператора, представимого в виде произведения, т.е. установлено взаимосвязь между множеством правильных сужении оператора $A = LM$ и множествами правильных сужении операторов L и M. Здесь доказана абстрактная теорема, позволяющая установить взаимосвязь между множеством правильных расширении оператора $A_0 = L_0 M_0$ и множествами правильных расширении операторов L_0 и M_0. В этой связи, доказывается абстрактная теорема, позволяющая описать правильные сужения одного класса нелинейных операторов, представимых в виде произведения.
Ключевые слова: оператор, корректное расширения, корректное сужение, регулярное расширение, задача типа Бицадзе-Самарского.

1 Introduction

In this work we consider following PDE

$$u^{2n}\frac{\partial^2 u}{\partial x \partial y} + 2n \cdot u^{(2n-1)}\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = f(x, y), f(x, y) \in C(G)$$

A condition of univariate solvability of one problem of Bitsadze-Samarsky type is shown. Here $C([0, 1] \times [0, 1]) \equiv C(G)$.

Keywords: operator, proper extension, proper restriction, regular extension, Bitsadze-Samarsky type problem.

Let us briefly recall some provisions of [10].

Let the operator $A = LM$ act in the Banach space B. Here, L is a certain additive closed operator for which $D(L) \subset B$ and $R(L) = B$. In the domain of definition $D(L)$, we introduce the norm

$$\|u\|_{\mathfrak{M}} = \|u\|_{\mathbb{B}} + \|Lu\|_{\mathbb{B}}, \quad u \in D(L). \tag{1}$$

The closure of the manifold $D(L)$ in the norm (1) will be denoted by $\mathfrak{M}$. It is evident that $\mathfrak{M}$ is a Banach space. Now, let the operator $\mathfrak{M}$ map the manifold $D(M)$ onto the space $\mathfrak{M}$, i.e., $R(M) = \mathfrak{M}$. Then, we define the operator A by the equality $A = LM$. Clearly, $D(A) = D(M)$ and $R(A) = R(L) = B$. If $\tilde{L}$ and $\tilde{M}$ are certain proper restrictions of the operators L and M, respectively, then the following holds:

Theorem 1 *The operator* $\tilde{A}^{-1} = \tilde{M}^{-1}\tilde{L}^{-1}$ *is invertible, and its inverse* $\tilde{A}$ *is a proper restriction of the operator* A.

Additionally, the following lemma is proven:

Лемма 1 $\ker A = B$, *where*

$$\mathfrak{B} = \{g = \tilde{M}^{-1}g_1 + g_2, g_1 \in \ker L,\ g_2 \in \ker M\}.$$

Using this lemma, an abstract theorem is proved, which provides a complete description of the set of all proper restrictions of the operator A in terms of the sets of all proper restrictions of the operators L and M.

Now, we will show that this method can also be applied to a certain class of nonlinear operators that can be represented as a product.

Within the previously used notation, let us additionally consider a (generally nonlinear) bijective mapping $N : \mathbb{B} \to \mathbb{B}$ such that $N(0) = 0$. Then, we can define the product

$$A = LMN \tag{2}$$

Clearly, $D(A) = D(MN) = D(M)$ and $R(A) = R(L) = \mathbb{B}$. Let $\tilde{L}$ and $\tilde{M}$ still be certain proper restrictions of the operators L and M, respectively. Then, the following holds:

Theorem 2 *The operator $\tilde{A}^{-1} = N^{-1}\tilde{M}^{-1}\tilde{L}^{-1}$ is invertible, and its inverse $\tilde{A}$ is a proper restriction of the operator A.*

Proof 1 *The product $N^{-1}\tilde{M}^{-1}\tilde{L}^{-1}$ defines a certain operator $\tilde{A}^{-1}$. Indeed, by definition, we have $R(\tilde{L}) = D(\tilde{L}^{-1}) = \mathbb{B}$ and $R(\tilde{L}^{-1}) = D(\tilde{L}) \subset \mathfrak{M}, D(N^{-1}\tilde{M}^{-1}) = D(\tilde{M}^{-1}) = \mathfrak{M}$. Therefore, the operator $\tilde{A}^{-1} = N^{-1}\tilde{M}^{-1}\tilde{L}^{-1}$ is well-defined with domain $D(\tilde{A}^{-1}) = \mathbb{B}$ and range $R(\tilde{A}^{-1}) = N^{-1}\tilde{M}^{-1}D(\tilde{L}) \subset \mathbb{B}$.*

Now, if for some $f \in \mathbb{B}$ the equality $\tilde{A}^{-1}f = 0$ holds, then

$$f = LMN(N^{-1}\tilde{M}^{-1}\tilde{L}^{-1}f) = A(\tilde{A}^{-1}f) = 0,$$

which means that the operator $\tilde{A}^{-1}$ has an inverse operator $\tilde{A}$. Since the operator $\tilde{A}^{-1}$ is continuous, it follows that $\tilde{A}$ is a proper restriction of the operator A. The theorem is proved.

Previously, we considered proper restrictions of operators that can be represented as a product. Now, we will show that it is also possible to consider proper extensions of such operators.

Let L be a certain closed additive operator with domain $D(L) \subset \mathbb{B}$ and range $R(L) = \mathbb{B}$, where $\mathbb{B}$ is a Banach space. Let L_0 be a restriction of the operator L, which has a continuous inverse L_0^{-1} on $R(L_0)$ and satisfies $\overline{R(L_0)} \neq \mathbb{B}$, i.e., the operator L_0 has a continuous left inverse.

By taking the closure of the manifold $D(L)$ in the norm (4), we obtain the Banach space $\mathfrak{M}$. Let $\mathfrak{M}_0$ denote the closure of the manifold $D(L_0)$ in the norm (1).

Let the operator M_0 satisfy the following conditions:

a) $D(M_0) \subset B, R(M_0) \subset M_0$;

b) On the set $R(M_0)$, the operator M_0 has a continuous inverse M_0^{-1}.

Then, the product $A_0 = L_0M_0$ is well-defined, and we have

$D(A_0) = D(M_0) \subset B, \quad R(A_0) = R(L_0) \subset B.$

Clearly, the inverse operator $A_0^{-1} = M_0^{-1}L_0^{-1}$ is well-defined on the set $R(A_0)$.

Let $\tilde{L}$ be a regular extension of the operator L_0, i.e., $L_0 \subset \tilde{L} \subset L$. Let $\tilde{M}$ be a proper extension of the operator M_0. Then, the following holds:

Theorem 3 *The operator $\tilde{A}^{-1} = \tilde{M}^{-1}\tilde{L}^{-1}$ is invertible, and its inverse $\tilde{A}$ is the correct extension of the operator A_0.*

Proof 2 *It is evident that the operator $\tilde{A}^{-1}$ is defined on the entire space B. Now, let us show that the operator $\tilde{A}^{-1}$ is invertible. Indeed, if for some $f \in B$ we have $\tilde{A}^{-1}f = 0$, then $\tilde{M}^{-1}\tilde{L}^{-1}f = 0$. Since the operators $\tilde{M}^{-1}$ and $\tilde{L}^{-1}$ have inverses, we obtain $\tilde{L}^{-1}f = 0$, which implies $f = 0$. Therefore, the operator $\tilde{A} = (\tilde{A}^{-1})^{-1}$ exists. Now, it is sufficient to show that $A_0 \subset \tilde{A}$.*

Indeed, if $u_0 \in D(A_0)$, then $L_0 u_0 \in D(M_0)$, i.e., there exists an element $f_0 \in R(A_0)$ such that $f_0 = A_0 u_0$ and $u_0 = A_0^{-1} f_0$. Then,

$$\tilde{A}^{-1} f_0 = \tilde{M}^{-1}\tilde{L}^{-1} f_0 = \tilde{M}^{-1} L_0^{-1} f_0 = M_0^{-1} L_0^{-1} f_0 = u_0.$$

Therefore, $u_0 \in D(\tilde{A})$, i.e., $\tilde{A}u_0 = f_0$. The theorem is proven.

Next, using this theorem, as an example, let's consider the proper extensions of a certain nonlinear differential operator.

In the space $C([0,1] \times [0,1]) \equiv C(G)$, we consider the following differential equation:

$$u^{2n}\frac{\partial^2 u}{\partial x \partial y} + 2n \cdot u^{2n-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in C(G), \tag{3}$$

Let L denote the operator acting as the differential expression u'_y with the domain of definition:

$$D(L) = \{u \in C(G) : \frac{\partial u}{\partial y} \in C(G)\}$$

Let M denote a Banach space obtained by closing the manifold $D(L)$ with respect to the norm:

$$\|u\|_M = \|u\|_{C(G)} + \|Lu\|_{C(G)}. \tag{4}$$

Let L_0 be the restriction of the operator L with the domain of definition:

$$D(L_0) = \{u \in D(L) : u(x,0) = 0, u(x,1) = 0\}.$$

Then,

$$R(L_0) = \{f(x,y) \in C(G) : \int_0^1 f(x,\tau)d\tau = 0\} \subset C(G).$$

In the set $R(L_0)$, there exists a continuous inverse L_0^{-1}:

$$L_0^{-1} f = \int_0^y f(x,\tau)d\tau.$$

Let M_0 denote the Banach space obtained by closing $D(L_0)$ with respect to the norm (4).

Let M_0 denote the operator $M_0 : D(M_0) \to R(M_0)$, where $D(M_0) \subset C(G), R(M_0) \subset B_0$, and:

$$D(M_0) = \{u \in C(G) : u^{2n}\frac{\partial u}{\partial x} \in B_0, u(x,0) = u(x,1) = 0\},$$

$$R(M_0) = \{f \in B_0 : \int_0^1 f(t,y)dt = 0\}.$$

In the set $R(M_0)$, there exists a continuous inverse M_0^{-1}

$$M_0^{-1}f = \sqrt{2n+1}\left[\int_0^x f(t,y)dt\right]^{1/(2n+1)}.$$

Let the operator $\tilde{L}$ be generated by the following boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in C(G), \\ u(x,0) = 0, \end{cases} \tag{5}$$

The operator $\tilde{L}$ is a regular extension of the operator L_0: $L_0 \subset \tilde{L} \subset L$, and:

$$\tilde{L}^{-1}f = \int_0^x f(x,\tau)d\tau.$$

Also, the operator $\tilde{M}$, generated by the following boundary value problem:

$$\begin{cases} u^{2n}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in B, \\ u(0,y) = 0, \end{cases} \tag{6}$$

is a proper extension of the operator M_0, and its inverse is:

$$\tilde{M}^{-1}f = \sqrt{2n+1}\left[\int_0^x f(t,y)dt\right]^{1/(2n+1)}.$$

Лемма 2 *The unique solution to the boundary value problem*

$$\begin{cases} u^{2n}\frac{\partial^2 u}{\partial x \partial y} + 2n \cdot u^{2n-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in C(G), \\ u(x,0) = 0, \quad u(0,y) = 0, \end{cases} \tag{7}$$

has the form:

$$u(x,y) = \sqrt{2n+1}\left[\int_0^x \int_0^y f(t,\tau)dtd\tau\right]^{1/(2n+1)}. \tag{8}$$

Proof 3 *According to Theorem 3, the operator*

$$\tilde{A}^{-1} = \tilde{M}^{-1}\tilde{L}^{-1}$$

is invertible, and its inverse operator $\tilde{A}$ is a proper extension of the operator $A_0 = L_0M_0$. Therefore, from (5) and (7), we conclude that the boundary value problem

$$u^{2n}\frac{\partial^2 u}{\partial x \partial y} + 2n \cdot u^{2n-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in C(G), \tag{9}$$

$$u(0,y) = 0, \tag{10}$$

$$\frac{\partial}{\partial x}u^{2n+1}(x,0)=0 \tag{11}$$

has a unique solution of the form (8). The boundary value problems (7) and (9)–(11) are equivalent. Indeed, integrating the condition (11) with respect to x, we get $u^{2n+1}(x,0)-u^{2n+1}(0,0)=0$, and from (8), we have $u(0,0)=0$, so we obtain that $u(x,0)=0$.

Now, let L_{10} be the operator generated by the following boundary value problem:

$$\frac{\partial u}{\partial x}=f(x,y), \quad f(x,y)\in C(G),$$

$$a_1(y)u(0,y)+a_2(y)u(\phi(y),y)+a_3(y)u(1,y)=0,$$

$$\frac{\partial u(0,y)}{\partial x}=0,$$

where $x=\phi(y)$ is a smooth curve located in the region G, $a_i(y)\in C[0,1]$, and

$$a^*(y)=a_1(y)+a_2(y)+a_3(y)\neq 0, y\in[0,1] \tag{12}$$

It is clear that:

$$L_{10}^{-1}f=\int_0^x f(t,y)dt-\frac{a_2(y)}{a^*(y)}\int_0^{\phi(y)} f(t,y)dt-\frac{a_3(y)}{a^*(y)}\int_0^1 f(t,y)dt, \tag{13}$$

and

$$R(L_{10})=\{f(x,y)\in C(G): f(0,y)=0\}.$$

As a regular extension of the operator L_{10}, we take the operator L_1, generated by the following boundary value problem:

$$\frac{\partial u}{\partial x}=f(x,y), \quad f(x,y)\in C(G),$$

$$a_1(y)u(0,y)+a_2(y)u(\phi(y),y)+a_3(y)u(1,y)=0.$$

For this problem to have a unique solution, it is necessary and sufficient to fulfill the condition (12) (i.e., $a^*(y)\neq 0$), and the unique solution is given by (13).

As the operator M_0, we take the previously considered operator, i.e., the operator $M_0: D(M_0)\to R(M_0)$, where $D(M_0)\subset C(G)$, $R(M_0)\subset B_0$, and:

$$D(M_0)=\{u\in C(G): u^{2n}\frac{\partial u}{\partial x}\in B_0, u(x,0)=u(x,1)=0\}.$$

We also consider the operator $\tilde{M}$, generated by the boundary value problem (6). This operator is a proper extension of the operator M_0, and:

$$\tilde{M}^{-1}f=\sqrt{2n+1}\left[\int_0^x f(t,y)dt\right]^{1/(2n+1)}.$$

Then, by Theorem 3, we have that the operator $A_1=L_1\tilde{M}$ is a proper extension of the operator $A_{10}=L_{10}M_0$. Thus, we have proven the following theorem:

Theorem 4 *In order for the problem*

$$u^{2n}\frac{\partial^2 u}{\partial x\partial y} + 2n\cdot u^{2n-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y)\in C(G),$$

$$a_1(y)u(0,y) + a_2(y)u(\phi(y),y) + a_3(y)u(1,y) = 0,$$

$$u(x,0) = 0,$$

to be uniquely solvable, it is necessary and sufficient to fulfill the inequality $a^*(y)\neq 0$, $y\in[0,1]$, *and its unique solution is given by:*

$$u(x,y) = \sqrt{2n+1}\left[\int_0^y\int_0^x f(t,\tau)dtd\tau - \int_0^y \frac{a_2(\tau)}{a^*(\tau)}\int_0^{\phi(\tau)} f(t,\tau)dtd\tau\right]^{1/(2n+1)} -$$

$$-\sqrt{2n+1}\left[\int_0^y \frac{a_3(\tau)}{a^*(\tau)}\int_0^1 f(t,\tau)dtd\tau\right]^{1/(2n+1)},$$

where

$$a^*(y) = a_1(y) + a_2(y) + a_3(y) \neq 0, \quad y\in[0,1].$$

Remark 1 *: The results of Theorem 4 can also be obtained by applying Theorem 2. In this case, as the bijective map* $N : C(G)\to C(G)$, *we take the operator* N *acting as* $N(u) = u^{2n+1}$, $u\in C(G)$.

2 Conclusion

In this work an abstract theorem is proved which allows us to establish the relationship between the set of proper extensions of the operator $A_0 = L_0M_0$ and the sets of proper extensions of the operators L_0 and M_0. In this connection, author proves an abstract theorem that allows us to describe the correct contractions of one class of nonlinear operators represented as a product.

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