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IMPROVEMENT IN VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS BY ADOMIAN DECOMPOSITION METHOD

The Adomian Decomposition Method (ADM) is widely recognized as a powerful and versatile semi-analytical tool designed to solve a broad range of problems, including linear and nonlinear differential equations, as well as integral equations. This method has been extensively applied across various scientific and engineering disciplines due to its simplicity and efficiency in generating accurate approximate solutions. In this note, we introduce an enhanced and refined scheme based on the ADM framework to obtain approximate solutions for Volterra-Fredholm integro-differential equations (IDEs) with specified initial conditions. Our proposed scheme not only simplifies the computational process but also ensures improved accuracy and convergence. Additionally, we rigorously prove the uniqueness of the solutions to the Volterra-Fredholm IDEs by leveraging the mathematical foundation of Banach's Fixed Point Theorem, providing theoretical validity to our approach. To validate the effectiveness of the enhanced scheme, we apply it to a diverse set of linear and nonlinear Volterra-Fredholm IDEs with initial conditions. The numerical results obtained are systematically compared with those from existing methods reported in the literature. Our findings reveal that the proposed approach demonstrates remarkable accuracy, efficiency, and reliability in solving complex IDEs. Consequently, this method represents a significant advancement in the field of integro-differential equations.

Key words: Adomian decomposition method, Volterra-Fredholm Integrodifferential equation, Approximate solution, Uniqueness solution, Adomian poliynomials, Banach's fixed point theorem.

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Адомианның жіктеу әдісі арқылы Вольтерра–Фредгольм интегро-дифференциалдық теңдеулерін жетілдіру

Адомиандық декомпозиция әдісі (ADM) желілік және сызықтық емес дифференциалдық теңдеулерді, сондай-ақ интегралдық теңдеулерді қоса алғанда, есептердің кең ауқымын шешуге арналған қуатты және әмбебап жартылай аналитикалық құрал ретінде кеңінен танылды. Бұл әдіс нақты жуық шешімдерді шығарудағы қарапайымдылығы мен тиімділігіне байланысты әртүрлі ғылыми және инженерлік пәндерде кеңінен қолданылады. Бұл жазбада біз белгіленген бастапқы шарттармен Вольтерра-Фредгольм интегро-дифференциалдық теңдеулеріне (IDE) жуық шешімдерді алуға арналған ADM негізіне негізделген жетілдірілген және нақтыланған схеманы ұсынамыз. Біздің ұсынылған схема есептеу процесін жеңілдетіп қана қоймайды, сонымен қатар жоғарылатылған дәлдік пен конвергенцияны қамтамасыз етеді.

Сонымен қатар, біз Вольтерра-Фредхольм IDE шешімдерінің бірегейлігін Банах бекітілген нүкте теоремасының математикалық негізін пайдалана отырып, біздің көзқарасымызды теориялық негіздей отырып, қатаң дәлелдейміз. Жақсартылған схеманың тиімділігін тексеру үшін біз оны әр түрлі сызықтық және сызықтық емес Вольтерра-Фредхольм бастапқы мән теңдеулерінің жиынтығына қолданамыз. Алынған сандық нәтижелер әдебиетте сипатталған бар әдістердің нәтижелерімен жүйелі түрде салыстырылады. Нәтижелеріміз ұсынылған тәсіл күрделі ӨЖБ шешуде керемет дәлдік, тиімділік және беріктік көрсететінін көрсетеді. Демек, бұл әдіс интегро-дифференциалдық теңдеулер саласындағы айтарлықтай ілгерілеушілікті білдіреді.

Түйін сөздер: Адомиандық кеңейту әдісі, Вольтерра-Фредгольм интегро-дифференциалдық теңдеуі, жуық шешім, бірегей шешім, Адомиян полиномдары, Банахтың қозғалмайтын нүкте теоремасы

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Улучшение интегро-дифференциальных уравнений Вольтерра–Фредгольма методом разложения Адомиана

Метод разложения Адомиана (ADM) широко признан как мощный и универсальный полуаналитический инструмент, предназначенный для решения широкого круга задач, включая линейные и нелинейные дифференциальные уравнения, а также интегральные уравнения. Этот метод широко применяется в различных научных и инженерных дисциплинах благодаря своей простоте и эффективности в создании точных приближенных решений. В этой заметке мы представляем усовершенствованную и улучшенную схему, основанную на структуре ADM, для получения приближенных решений для интегро-дифференциальных уравнений (ИДУ) Вольтерры-Фредгольма с указанными начальными условиями. Наша предлагаемая схема не только упрощает вычислительный процесс, но и обеспечивает повышенную точность и сходимость. Кроме того, мы строго доказываем уникальность решений ИДУ Вольтерры-Фредгольма, используя математическую основу теоремы Банаха о неподвижной точке, обеспечивая теоретическую обоснованность нашего подхода. Чтобы подтвердить эффективность усовершенствованной схемы, мы применяем ее к разнообразному набору линейных и нелинейных ИДУ Вольтерры-Фредгольма с начальными условиями. Полученные численные результаты систематически сравниваются с результатами существующих методов, описанных в литературе. Наши результаты показывают, что предложенный подход демонстрирует замечательную точность, эффективность и надежность при решении сложных ИДУ. Следовательно, этот метод представляет собой значительный прогресс в области интегро-дифференциальных уравнений.

Ключевые слова: Метод разложения Адомиана, Интегродифференциальное уравнение Вольтерра-Фредгольма, Приближенное решение, единственное решение, многочлены Адомиана, теорема Банаха о неподвижной точке

1 Introduction

In this paper, we consider a class of Volterra-Fredholm integro-differential equations of the type:

$$\sum_{j=0}^{m} \delta_j(x) u^{(j)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t) F_1(u(t)) dt + \lambda_2 \int_a^b K_2(x,t) F_2(u(t)) dt, \quad (1)$$

with the initial conditions

$$u^{(j)}(a) = b_j, \quad j = 0, 1, 2, \ldots (m-1) \quad (2)$$

Where $u^j(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined $K_r(x,t)$, $r = 1,2$ are the kernels of the equation (1), $f(x)$ and $\delta_j(x)$ are analytic functions, $F_1(u(t))$ and $F_2(u(t))$ are nonlinear continuous functions and $a, b, \lambda_1, \lambda_2, b_j$ are constants.

Linear and nonlinear IDEs are the class of mathematical problems appearing in many engineering and science areas. These types of equations are challenging to solve analytically. Therefore, numerical methods are required to obtain approximate solutions [1–12]. The Adomian Decomposition Method (ADM) is one of the powerful tools to obtain semi-analytical solutions of operator equations and has been widely used in recent years due to its simplicity and accuracy in solving linear and nonlinear differential equations, integral equations and IDEs with initial and boundary conditions [13–28].

Besides that we have achieved very good results in this direction in our previous articles [29–31].

Our main aim is to obtain numerical solution of Eq. (1)-(2) by standard ADM and modified ADM and analysis the behaviour of error terms for large number of iterations. The proposed scheme is tested on various linear and nonlinear IDEs and compared with other existing methods, to demonstrate its high accuracy and efficiency. This paper provides a valuable tool for researchers in engineering and science, enabling them to obtain high accurate approximation solutions of Volterra-Fredholm IDEs with initial conditions using the standard and modified ADM.

2 Methodology

2.1 Application of adomian decomposition method to nonlinear volterra-fredholm integro-differential equations

Now, we can rewrite Eq. (1) in the form

$$u^{(m)}(x) = \frac{f(x)}{\delta_m(x)} + \lambda_1 \int_0^x \frac{K_1(x,t)}{\delta_m(x)} F_1(u(t)) dt + \lambda_2 \int_a^b \frac{K_2(x,t)}{\delta_m(x)} F_2(u(t)) dt - \sum_{j=0}^{m-1} \frac{\delta_j(x)}{\delta_m(x)} u^{(j)}(x), \quad (3)$$

Since $Lu = \frac{d^m u}{dx^m}$ is the differential operator of order m., to reduce IDEs (1) into integral equations (IEs), we integrate both sides of equation (3) m-times in the interval $[a, x]$ with respect to x to obtain

$$
\begin{gathered}
u(x)=\sum_{r=0}^{m-1}\tfrac{1}{r!}(x-a)^r b_r-\sum_{j=0}^{m-1}L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)}u^{(j)}(x)\right)+L^{-1}\left(\tfrac{f(x)}{\delta_m(x)}\right) \\
+\lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)}F_1(u(t))dt\right)+\lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)}F_2(u(t))dt\right)
\end{gathered}
\tag{4}
$$

where L^{-1} is the inverse operator of L and can be computed by Leibniz rule

$$
L^{-1}(F)=\int_a^x\int_a^{x_1}\ldots\int_a^{x_{m-1}}(F)dx_m dx_{m-1}\ldots dx_1=\frac{1}{(m-1)!}\int_0^x (x-t)^{m-1}(F)dt. \tag{5}
$$

For nonlinear terms in (4), we apply Adomian polynomial series

$$
M_1(u(x))=\sum_{n=0}^{\infty}A_n, \quad M_2(u(x))=\sum_{n=0}^{\infty}B_n. \tag{6}
$$

where A_n, B_n; $n\geq 0$ are the Adomian polynomials determined formally as follows:

$$
A_n=\frac{1}{n!}\left[\frac{d^n}{d\gamma^n}F_1\left(\sum_{i=0}^{\infty}\gamma^i u_i\right)\right]\Bigg|_{\gamma=0}, \qquad B_n=\frac{1}{n!}\left[\frac{d^n}{d\gamma^n}F_2\left(\sum_{i=0}^{\infty}\gamma^i u_i\right)\right]\Bigg|_{\gamma=0} \tag{7}
$$

The Adomian polynomials were introduced in [1-3] as:

$$
\begin{gathered}
A_0=F_1(u_0), \\
A_1=u_1 F_1{}'(u_0), \\
A_2=u_2 F_1{}'(u_0)+\tfrac{1}{2!}u_1{}^2 F_1{}''(u_0), \\
A_3=u_3 F_1{}'(u_0)+u_1 u_2 F_1{}''(u_0)+\tfrac{1}{3!}u_1{}^2 F_1{}'''(u_0), \\
A_4=u_4 F_1{}'(u_0)+u_1 u_3 F_1{}''(u_0)+\tfrac{1}{2!}u_2{}^2 F_1{}''(u_0)+\tfrac{1}{2!}u_1{}^2 u_2 F_1{}'''(u_0)+\tfrac{1}{4!}u_1{}^{(IV)}F_1{}^{(IV)}(u_0), \\
\ldots\ldots\ldots\ldots
\end{gathered}
\tag{8}
$$

and

$$
\begin{gathered}
B_0=F_2(u_0), \\
B_1=u_1 F_2{}'(u_0), \\
B_2=u_2 F_2{}'(u_0)+\tfrac{1}{2!}u_1{}^2 F_2{}''(u_0), \\
B_3=u_3 F_2{}'(u_0)+u_1 u_2 F_2{}''(u_0)+\tfrac{1}{3!}u_1{}^2 F_2{}'''(u_0), \\
B_4=u_4 F_2{}'(u_0)+u_1 u_3 F_2{}''(u_0)+\tfrac{1}{2!}u_2{}^2 F_2{}''(u_0)+\tfrac{1}{2!}u_1{}^2 u_2 F_2{}'''(u_0)+\tfrac{1}{4!}u_1{}^{(IV)}F_2{}^{(IV)}(u_0), \\
\ldots\ldots\ldots\ldots
\end{gathered}
\tag{9}
$$

The standard decomposition technique represents the solution of u as the following series:

$$
u=\sum_{i=0}^{\infty}u_i \tag{10}
$$

By substituting (6) and (10) into (4), we obtain

$$\begin{aligned}\sum_{i=0}^{\infty} u_i(x) = \sum_{k=0}^{m-1} \tfrac{1}{r!}(x-a)^r b_r - \sum_{i=0}^{\infty}\sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_i}^{(j)}(x)\right) + L^{-1}\left(\tfrac{f(x)}{\delta_m(x)}\right) \\ +\lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_i(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_i(t)dt\right)\end{aligned} \tag{11}$$

The components u_0, u_1, $u_2, \ldots$ are usually determined recursively by

$$\begin{aligned}
u_0(x) &= L^{-1}\left(\tfrac{f(x)}{\delta_m(x)}\right) + \sum_{k=0}^{m-1} \tfrac{1}{r!}(x-a)^r b_r, \\
u_1(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_0(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_0(t)dt\right) \\
&\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_0}^{(j)}(x)\right), \\
u_2(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_1(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_1(t)dt\right) \\
&\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_1}^{(j)}(x)\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ , \\
u_n(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_{n-1}(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_{n-1}(t)dt\right) \\
&\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_{n-1}}^{(j)}(x)\right).
\end{aligned} \tag{12}$$

For the modified ADM, we decompose $f(x) = f_1(x) + f_2(x)$ in Eq. (11) and do search solution as (10) and performing operations similarly as (12) we arrive at

$$\begin{aligned}
u_0(x) &= L^{-1}\left(\tfrac{f_1(x)}{\delta_m(x)}\right) + \sum_{k=0}^{m-1} \tfrac{1}{r!}(x-a)^r b_r, \\
u_1(x) &= L^{-1}\left(\tfrac{f_2(x)}{\delta_m(x)}\right) + \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_0(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_0(t)dt\right) \\
&\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_0}^{(j)}(x)\right), \\
u_2(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_1(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_1(t)dt\right) \\
&\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_1}^{(j)}(x)\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \ , \\
u_n(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_{n-1}(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_{n-1}(t)dt\right) \\
&\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_{n-1}}^{(j)}(x)\right), \ \ n \geq 1.
\end{aligned} \tag{13}$$

Then, $u(x) = \sum_{i=0}^{n} u_i(x)$ will be taken as the approximate solution. Eqs (12) and (13) are called standard and modified ADM respectively.

2.2 Uniqueness solution of ides

In this section, we show that under which conditions the solution of the Eq. (1)-(2) is unique. In Eshkuvatov [30] proved the following lemma and theorem regarding to the problem (1)-(2). Before starting and proving the main results, we introduce the following hypotheses:

H_1: There exist two constants $L_1 > 0,\ L_2 > 0$ such that for any $u_1,\, u_2 \in C(J, R),\quad J = [a, b]$

$$\begin{gathered} |F_1(u_1(t)) - F_1(u_2(t))| \le L_1 |u_1 - u_2|, \\ |F_2(u_1(t)) - F_2(u_2(t))| \le L_2 |u_1 - u_2|, \\ |D^j(u_1(t)) - D^j(u_2(t))| \le \gamma_j |u_1 - u_2|,\ \ j = \{0, 1, \dots, m-1\}, \end{gathered}$$

where D^j is a differential operator and γ_j are Lipschitz constants.

H_2: There exist two functions K_1^*, K_2^* for the kernel functions $K_1, K_2 \in C(D, R)$ the set of all positive functions continuous on $D = \{(t, x) \in R : a \le x \le t \le b\}$ such that

$$K_1^* = \sup_{t\in[a,b]} \int_a^t |K_1(t,x)dx| < \infty,\ \ K_2^* = \sup_{t\in[a,b]} \int_a^b |K_2(t,x)dx| < \infty.$$

H_3: The functions $\delta_j(t),\ j = \{1, 2, \dots, m-1\}$ and $f(t)$ mapping $J \to R$ are continuous functions.

Theorem 1 ***(Banach's Fixed Point Theorem)***. *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x \in X$ such that $T(x) = x$. The following lemma is proven in Eshkuvatov [30].*

Lemma 1 *Let $\varphi(t) \in C(J, R^+)$ then $u(t) \in C(J, R^+)$ is a solution of the problem (1)-(2) if u is satisfying.*

$$\begin{cases} u(t) = \varphi(t) - \sum\limits_{j=1}^{m-1} \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} (\delta_j(x) D^j(u(x)))dx \\ \qquad + \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} \lambda_1 \left[\int\limits_0^x K_1(x,r) F_1(u(r))dr \right] dx \\ \qquad + \frac{1}{(m-1)!} \int\limits_a^b (t-x)^{m-1} \lambda_2 \left[\int\limits_0^x K_2(x,r) F_2(u(r))dr \right] dx, \end{cases} \tag{14}$$

for $t \in J = [a, b]$ and

$$\varphi(t) = \sum_{j=1}^{m-1} \frac{\delta_k}{k!}(t-a)^k + \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} f(t)dx \tag{15}$$

Let us prove the following theorems based on the Lemma 1.

Theorem 2 *Assume that the hypothesizes H_1, H_2 and H_3 hold. If*

$$\varepsilon^* = \left[\frac{\gamma^*\delta^*}{m!} + \frac{\lambda_1 K_1^*(L_1)}{m!} + \frac{\lambda_2 K_2^*(L_2)}{m!}\right](b-a)^m < 1, \tag{16}$$

where $\gamma^* = \max_{1\le j\le m-1}\gamma_j$ and $\delta^* = \max_{1\le j\le m-1}|\delta_j(t)|$ then there exists a unique solution $u(x) \in C(J)$ for Eq. (1)-(2).

Proof of Theorem 2. Let the operator $T : C(J,R) \to C(J,R)$ be defined by

$$\begin{cases} (Tu)(t) = \varphi(t) - \sum\limits_{j=1}^{m-1} \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} (\delta_j(x) D^j(u(x)))dx \\ \qquad + \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} \left[\lambda_1 \int\limits_0^x K_1(x,r) F_1(u(r))\, dr\right] dx \\ \qquad + \frac{1}{(m-1)!} \int\limits_a^b (t-x)^{m-1} \left[\lambda_2 \int\limits_0^x K_2(x,r) F_2(u(r))\, dr\right] dx, \end{cases}$$

where $\varphi(t)$ is defined by (15).

It is known by Lemma 1, that a function u is a solution to (1)-(2) if u satisfies Eq.(14). Now we prove that T has a fixed point u in $C(J,R)$ under condition (16). To do this end, let $u_1, u_2 \in C(J,R)$ then for any $t \in [a,b]$.

$$\begin{aligned} \|(Tu_1)(t) - (Tu_2)(t)\| &\le \sum_{j=1}^{m-1} \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} \|\delta_j(x)\| \|D^j u_1(x) - D^j u_2(x)\| dx \\ &+ \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} \left[\lambda_1 \int\limits_0^x \|K_1(x,r)\| \|F_1(u_1(r)) - F_1(u_2(r))\|\, dr\right] dx \\ &+ \frac{1}{(m-1)!!} \int\limits_a^b (t-x)^{m-1} \left[\lambda_2 \int\limits_0^x \|K_2(x,r)\| \|F_2(u_1(r)) - F_2(u_2(r))\|\, dr\right] dx, \\ &\le \frac{\delta^*}{(m-1)!} \sum_{j=1}^{m-1} \gamma_j \|u_1 - u_2\| \int\limits_a^t (t-x)^{m-1} dx + \frac{\lambda_1}{(m-1)!} [K_1^* L_1 \|u_1 - u_2\|] \int\limits_a^t (t-x)^{m-1} dx \\ &+ \frac{\lambda_2}{(m-1)!} [K_2^* L_2 \|u_1 - u_2\|] \int\limits_a^b (t-x)^{m-1} dx \\ &\le \left[\frac{\gamma^*\delta^*(b-a)^m}{m!} + \frac{\lambda_1 K_1^* L_1 (b-a)^m}{m!} + \frac{\lambda_2 K_2^* L_2 (b-a)^m}{m!}\right] \|u_1 - u_2\| \\ &= \varepsilon \|u_1 - u_2\|, \ \varepsilon < 1. \end{aligned}$$

Thus, operator T is the contraction map. By the Banach contraction principle we can conclude that T has a unique fixed point u in $C(J,R)$.

3 Results

3.1 Illustrative examples

Example 1 (Hamoud et al. [23]). Consider the following Fredholm IDEs of order one with the initial condition.

$$u^{'}(x)=e^x(1+x)-x+\int\limits_0^1 xu(t)dt, \qquad u(0)=0. \tag{17}$$

The exact solution of (17) is $u(x)=xe^x$

Solution: To apply ADM, we need to convert Eq. (17) into integral equations which yields $u(x)=xe^x-\frac{x^2}{2}+\frac{x^2}{2}\int\limits_0^1 u(\tau)d\tau$

Using the standard ADM we get $U_2=xe^x-\frac{x}{72}$; $U_5=xe^x-\frac{x}{15552}$; $U_{10}=xe^x-\frac{x}{120932352}$; $U_n=xe^x-\frac{x}{2\cdot 6^n}$.

The numerical results of Example 1 are given in Table 1.

Table 1: Numerical results of Example 1

X	Exact solution	Error for ADM [9] (n=3)	Error for ADM (n=10)	Error for ADM (n=30)
0.1	0.11051709	1.3×10^{-4}	8.27×10^{-11}	2.26×10^{-26}
0.2	0.24428055	5.56×10^{-4}	3.31×10^{-10}	9.05×10^{-26}
0.3	0.40495764	1.25×10^{-3}	7.44×10^{-10}	2.04×10^{-25}
0.4	0.59672987	2.22×10^{-3}	1.32×10^{-9}	3.62×10^{-25}
0.5	0.82436063	3.47×10^{-3}	2.07×10^{-9}	5.65×10^{-25}
0.6	1.09327128	5.00×10^{-3}	2.98×10^{-9}	8.14×10^{-25}
0.7	1.40962689	6.80×10^{-3}	4.05×10^{-9}	1.11×10^{-24}
0.8	1.78043274	8.89×10^{-3}	5.29×10^{-9}	1.45×10^{-24}
0.9	2.21364280	1.12×10^{-2}	6.70×10^{-9}	1.83×10^{-24}
1.0	2.71828182	–	8.27×10^{-9}	2.26×10^{-24}

Remark 1. From the Table 1, we can conclude that ADM is very high accurate semi-analytical method to solve linear IDEs of order one. Hamoud et al. [23] found error of ADM for two iteration only. We are able to run the iteration upto 30 and improve accuracy upto 10^{-24}.

Example 2 (Alao et al. [20]). Solve the following Fredholm integrodifferential equation

$$u^{'}(x)=1-\frac{x}{3}+\int\limits_0^1 xtu(t)dt, \qquad u(0)=0. \tag{18}$$

The exact solution is $u(x)=x$.

Solution: Convert IDEs (18) into integral equations $u(x)=x-\frac{x^2}{6}+\frac{x^2}{2}\int\limits_0^1 \tau u(\tau)d\tau$

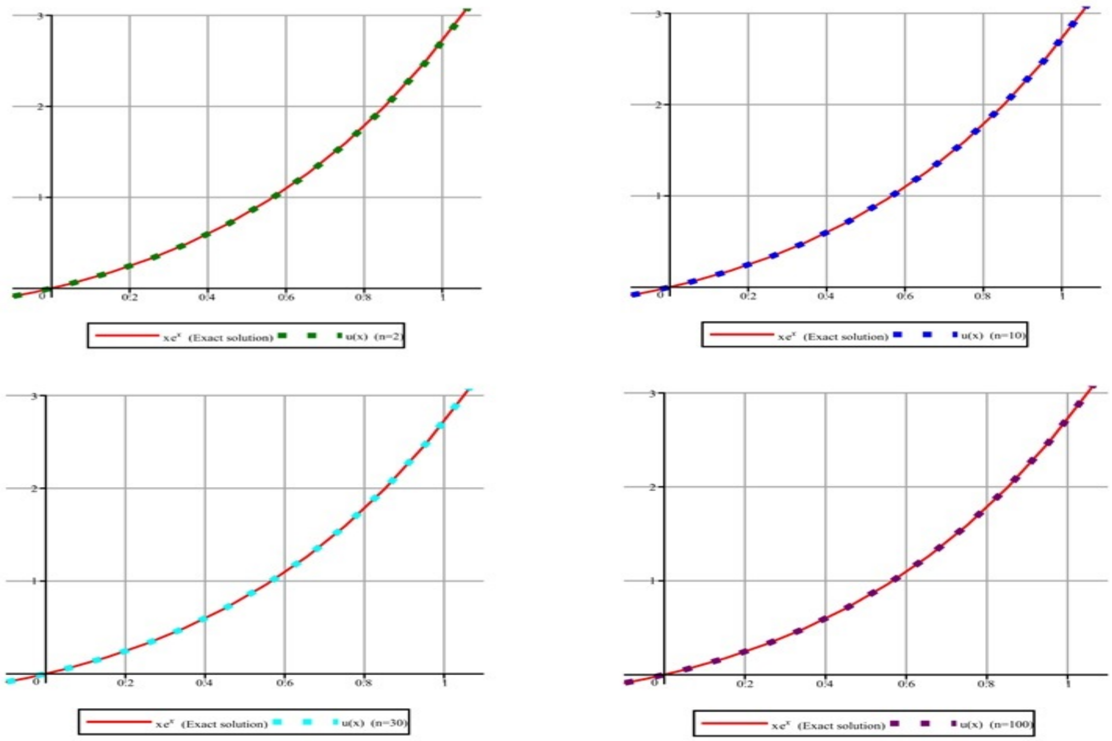

Figure 1: Graph for Example 1.

Using the standard ADM we get

$$
\begin{aligned}
U_2 &= x - \tfrac{x^2}{384},\\
U_5 &= x - \tfrac{x^2}{196608},\\
U_{10} &= x - \tfrac{x^2}{6442450944};\\
U_n &= x - \tfrac{x^2}{6\cdot 8^n}.
\end{aligned}
$$

The numerical results of Example 2 are given in Table 2.

Remark 2. Table 2 shows that ADM is again very high accurate semi-analytical method to solve linear IDEs of order one. Alao et al. [8] found error of ADM for seven iteration only. We are able to run the iteration upto 30 and improve accuracy is 10^{-28}.

Example 3. (Hamoud [24]). Consider the following non-linear Volterra-Fredholm integro-differential equation.

$$
\begin{gathered}
u'(x) + xu(x) = 2x + x^3 - \tfrac{x^5}{5} - \tfrac{0.9^7}{7}x + \int_0^x u^2(t)dt + \int_0^{0.9} xu^3(t)dt,\\
u(0) = 0.
\end{gathered} \tag{19}
$$

with the exact solution is $u(x) = x^2$.

Solution: To apply ADM, we convert it to integral equations

$$
u(x) = x^2 + \frac{x^4}{4} - \frac{x^6}{30} - \frac{0.9^7}{14}x^2 + \int_0^t u^2(\tau)d\tau + \frac{x^2}{2}\int_0^{0.9} u^3(\tau)d\tau - \int_0^x tu(t)dt
$$

Table 2: Numerical results of Example 2

X	Exact solution	Error for ADM [17] (n=7)	Error for ADM (n=10)	Error for ADM (n=30)
0.1	0.10	7.95×10^{-10}	1.55×10^{-12}	1.35×10^{-30}
0.2	0.20	3.18×10^{-9}	1.62×10^{-12}	5.39×10^{-30}
0.3	0.30	7.15×10^{-9}	1.40×10^{-11}	1.21×10^{-29}
0.4	0.40	1.27×10^{-8}	2.48×10^{-11}	2.15×10^{-29}
0.5	0.50	1.99×10^{-8}	3.88×10^{-11}	3.37×10^{-29}
0.6	0.60	2.86×10^{-8}	5.59×10^{-11}	4.85×10^{-29}
0.7	0.70	3.89×10^{-8}	7.61×10^{-11}	6.60×10^{-29}
0.8	0.80	5.09×10^{-8}	9.93×10^{-11}	8.62×10^{-29}
0.9	0.90	6.44×10^{-8}	1.26×10^{-10}	1.09×10^{-28}
1.0	1.00	7.95×10^{-8}	1.55×10^{-10}	1.35×10^{-28}

The summary of numerical results of Example 3 are given in Table 3.

Table 3: Numerical results of Example 3

X	Exact solution	Error for ADM [17] (n=2)	Error for ADM (n=3)	Error for ADM (n=10)
0.1	0.0100	2.40×10^{-5}	2.55×10^{-6}	4.7×10^{-11}
0.2	0.0400	3.94×10^{-4}	9.87×10^{-6}	1.9×10^{-10}
0.3	0.0900	1.96×10^{-3}	2.07×10^{-5}	4.4×10^{-10}
0.4	0.1600	8.72×10^{-3}	3.27×10^{-5}	9.0×10^{-10}
0.5	0.2500	6.24×10^{-3}	4.16×10^{-5}	1.2×10^{-9}
0.6	0.3600	9.12×10^{-3}	4.17×10^{-5}	1.7×10^{-9}
0.7	0.4900	6.31×10^{-3}	2.78×10^{-5}	2.4×10^{-9}
0.8	0.6400	9.74×10^{-3}	9.61×10^{-5}	3.1×10^{-9}
0.9	0.8100	8.53×10^{-3}	3.63×10^{-5}	4.0×10^{-9}

Remark 3. In Hamoud [24] consider nonlinear IDEs with initial conditions and demonstrated the error terms for two iteration only. In Table 3, we are able to run Maple coding until 10 iterations and got favourable decreasing the error terms. ADM is again demonstrated the suitable and high accurate semi-analytical method to solve nonlinear IDEs of order one.

Example 4. (Olayiwola et al. [28]). Consider the following non-linear Volterra IDEs.

$$\begin{gathered} u'''(x) = 1 + x + \frac{x^3}{6} + \int_0^x (x-t)\, u(t)dt, \\ u(0) = 1,\ u'(0) = 0,\ u''(0) = 1. \end{gathered} \tag{20}$$

The exact solution of Eq. (17) is $u(x) = e^x - x$.

Solution. Convert (20) into integral equations

$$u(x) = 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{1}{2}\int\limits_0^x (x-t)^2 \left(\int\limits_0^t (t-\tau)\, u(\tau) d\tau \right) dt.$$

The numerical results of Example 4 are given in Table 4.

Table 4: Numerical results of Example 4

X	Exact solution	Error for ADM [17] (n=5)	Error for ADM (n=5)	Error for ADM (n=30)
0.0	1.0051709	1.0×10^{-10}	0.0	0.0
0.1	1.0051709	3.2×10^{-5}	3.7×10^{-63}	1.2×10^{-149}
0.2	1.0214027	2.3×10^{-4}	4.0×10^{-54}	2.0×10^{-149}
0.3	1.0498588	6.8×10^{-4}	7.8×10^{-49}	1.0×10^{-149}
0.4	1.0918246	1.4×10^{-3}	4.3×10^{-45}	3.0×10^{-149}
0.5	1.1487212	2.4×10^{-3}	3.5×10^{-42}	5.7×10^{-199}
0.6	1.2221188	3.7×10^{-3}	8.3×10^{-40}	1.6×10^{-199}
0.7	1.3137527	5.3×10^{-3}	8.5×10^{-38}	1.9×10^{-199}
0.8	1.4255409	7.3×10^{-3}	4.6×10^{-36}	3.2×10^{-199}
0.9	1.5596031	1.2×10^{-2}	1.5×10^{-34}	4.7×10^{-199}
1.0	1.7182818	1.2×10^{-2}	3.7×10^{-33}	2.2×10^{-274}

Remark 4. In Example 4, M.O.Olayiwola et al. [28] consider linear Volterra IDEs with initial conditions and solved in the Collocation method. In Table 4, we are able to get results until 5 iterations and got the gradually decreasing error terms. Numerical results revealed that ADM is very accurate semi-analytical method to solve linear IDEs of order one.

Example 5. Consider the following non-linear Volterra integro-differential equation.

$$\begin{aligned} u''(x) - u'(x) &= \tfrac{3}{2} - x + \tfrac{e^{-2x}}{2} + \int\limits_0^x u^2(t)dt \\ u(0) = 0,& \quad u'(0) = -1. \end{aligned} \tag{21}$$

with the exact solution is $u(x) = e^{-x} - 1$.

Solution: To apply ADM, we convert it to integral equations

$$u(x) = -\frac{1}{8} - \frac{3}{4}x + \frac{3}{4}x^2 - \frac{x^3}{6} + \frac{e^{-2x}}{8} + \int\limits_0^x (x-t) \left(\int\limits_0^t u^2(\tau) d\tau \right) dt + \int\limits_0^x u(t)dt$$

This example was solved by SADM and MADM.

The solution with SADM is as follows

$$u_0(x) = -\frac{1}{8} - \frac{3}{4}x + \frac{3}{4}x^2 - \frac{x^3}{6} + \frac{e^{-2x}}{8};$$
$$u_1(x) = \int_0^x (x-t)\left(\int_0^t A_0 d\tau\right) dt + \int_0^x u_0(t)dt$$
$$\dots$$
$$u_n(x) = \int_0^x (x-t)\left(\int_0^t A_{n-1} d\tau\right) dt + \int_0^x u_{n-1}(t)dt$$

So that n-term approximate solution
$U_n(x) = u_0(x) + u_1(x) + u_2(x) + \ldots + u_n(x)$
Also, the solution with MADM is as follows

$$u_0(x) = -\frac{1}{8} - \frac{3}{4}x + \frac{e^{-2x}}{8};$$
$$u_1(x) = \frac{3}{4}x^2 - \frac{x^3}{6} + \int_0^x (x-t)\left(\int_0^t A_0 d\tau\right) dt + \int_0^x u_0(t)dt;$$
$$u_2(x) = \int_0^x (x-t)\left(\int_0^t A_1 d\tau\right) dt + \int_0^x u_1(t)dt$$
$$\dots$$
$$u_n(x) = \int_0^x (x-t)\left(\int_0^t A_{n-1} d\tau\right) dt + \int_0^x u_{n-1}(t)dt$$

The n-term approximate solution for this also looks the same

$$U_n(x) = u_0(x) + u_1(x) + u_2(x) + \ldots + u_n(x)$$

The summary of numerical results of Example 5 are given in Table 5.

Table 5: Numerical results of Example 5

X	Exact solution	Error for ADM (n=5)	Error for MADM (n=5)	Error for ADM (n=10)	Error for MADM (n=10)
0.1	-0.09516258	-1.95×10^{-11}	-9.91×10^{-12}	-2.06×10^{-21}	1.04×10^{-21}
0.2	-0.18126924	-2.47×10^{-9}	-9.92×10^{-9}	-8.23×10^{-18}	-9.91×10^{-17}
0.3	-0.25918177	-4.14×10^{-8}	-9.41×10^{-8}	-1.0×10^{-15}	-4.28×10^{-17}
0.4	-0.32967995	-3.03×10^{-7}	-4.69×10^{-7}	-2.82×10^{-14}	-9.10×10^{-14}
0.5	-0.39346934	-1.4×10^{-6}	-1.57×10^{-6}	-3.36×10^{-13}	-9.54×10^{-13}
0.6	-0.45118836	-4.87×10^{-6}	-4.04×10^{-6}	-2.08×10^{-12}	-6.46×10^{-12}
0.7	-0.50341469	-1.30×10^{-5}	-8.60×10^{-6}	-5.95×10^{-12}	-3.27×10^{-11}
0.8	-0.55067103	-3.31×10^{-5}	-1.56×10^{-5}	-1.39×10^{-11}	-1.35×10^{-10}
0.9	-0.59343034	-7.10×10^{-5}	-2.47×10^{-5}	-2.60×10^{-10}	-4.78×10^{-10}

Remark 5. This example was established by the authors and solved by ADM and MADM. It can be seen that the error in both methods worked well and not much difference each other. When $n = 10$ iterations, error of both methods reached to 10^{-10}. By increasing number of iterations we get more decreasing errors.

4 Conclusion

In this study, we have developed an enhanced scheme based on the Adomian Decomposition Method (ADM) for solving both linear and nonlinear integro-differential equations (IDEs) of arbitrary order nnn. Through a series of numerical experiments, we observed that the approximation error associated with ADM decreases progressively as the number of iterations increases, indicating improved convergence behavior (see Tables 1–4). Notably, our results demonstrate that ADM performs particularly well for linear IDEs when a larger number of iterations is employed (see Tables 1–2), with a significant improvement in the accuracy of the solution. To evaluate the effectiveness of the proposed method, we compared our results with those obtained using existing approaches, including the collocation method and the variational iteration method. The comparison reveals that our improved ADM-based scheme achieves superior accuracy across various test problems. Looking ahead, we intend to extend this work by applying the improved ADM-based scheme in combination with the Series Solution Method (SSM) to tackle systems of complex Volterra-Fredholm integro-differential equations. This future direction aims to further enhance the applicability and efficiency of analytical and semi-analytical methods for solving advanced IDE systems arising in applied mathematics and engineering contexts.

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