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ON THE SOLVABILITY OF BOUNDARY VALUE PROBLEMS WITH GENERAL CONDITIONS FOR THE TRIHARMONIC EQUATION IN A BALL

The need to study boundary value problems for elliptic and parabolic equations is dictated by numerous practical applications in the theoretical study of processes in hydrodynamics, electrostatics, mechanics, heat conduction, elasticity theory, and quantum physics. This paper investigates the solvability of a boundary value problem with general conditions for the triharmonic equation in a unit ball.The validity of the analogue of the Almansi representation is proved. For completeness of presentation, a representation of the Green's functions of the Dirichlet-2 problem is given. This article indicates the difference between the Green's function of the real Dirichlet problem and the Green's function of the Dirichlet-2 problem. It is known that the results of differential equations with partial derivatives in the entire space or differential equations without boundary conditions are in a sense final. The theory of boundary value problems for general differential operators is currently a relevant and rapidly developing part of the theory of differential equations. However, there is a shortage of explicitly solvable problems on the path of further development of the theory of boundary value problems of differential equations. Over the past decades, sufficient material has been accumulated on the constructive construction of solutions to boundary value problems for model equations with partial derivatives. This article relates to this topical issue.

Key words: Green's function, triharmonic equation, Dirichlet-2 problem, boundary value problem with general conditions, integral representation of the solution.

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Көп өлшемді шарда үш гармоникалық теңдеу үшін жалпы шарттары бар шеттік есептердің шешімділігі туралы

Эллиптикалық және параболалық теңдеулер үшін шекаралық есептерді зерттеу қажеттілігі гидродинамика, электростатика, механика, жылу өткізгіштік, серпімділік теориясы және кванттық физика процестерін теориялық зерттеуде көптеген практикалық қолданулардан туындайды.Бұл жұмыста көп өлшемді бірлік шарда үш гармоникалық теңдеу үшін жалпы шарттары бар шеттік есептердің шешімділігі зерттеледі. Альманси өрнегінің аналогы дұрыстығы дәлелденеді. Материалдың толықтығы үшін Дирихле-2 есебінің Грин функциясының өрнегі келтірілген. Бұл мақалада нағыз Дирихле есебі мен Дирихле-2 есебінің Грин функцияларының айырмашылығы көрсетілген. Дербес туындылы дифференциалдық теңдеулерге бүкіл кеңістіктегі немесе шекаралық шарттары жоқ дифференциалдық теңдеулерге қатысты нәтижелер белгілі бір мағынада түпкілікті дәрежеде зерттелген болып табылады. Жалпы дифференциалдық операторлар үшін шекаралық есептер теориясы қазіргі уақытта дифференциалдық теңдеулер теориясының өзекті және қарқынды дамып келе жатқан саласы болып табылады. Алайда дифференциалдық теңдеулердің шекаралық есептерінің теориясын одан әрі дамыту жолында анық шешілетін есептердің тапшылығы байқалады.

Соңғы онжылдықтарда жеке туындылары бар модельдік теңдеулер үшін шекаралық есептердің шешімдерін конструктивті құру бойынша жеткілікті материал жинақталды. Бұл мақала осы өзекті тақырыпқа арналады.
Түйін сөздер: Грин функциясы, үш гармоникалық теңдеу, Дирихле-2 есебі, жалпы шарттары бар шеттік есептер, шешімнің интегралдық өрнегі.

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О разрешимости краевых задач с общими условиями для тригармонического уравнения в шаре

Необходимость исследования краевых задач для эллиптических и параболических уравнений диктуется многочисленными практическими приложениями при теоретическом исследовании процессов гидродинамики, электростатики, механики, теплопроводности, теории упругости, квантовой физики. В данной работе исследуется разрешимость краевой задачи с общими условиями для тригармонического уравнения в единичном шаре. Доказана справедливости аналога представление Альманси. Для полноты изложения приведен представление функций Грина задачи Дирихле-2. В данной статье указана разница между Функцией Грина настоящей задачи Дирихле с функцией Грина задачи Дирихле-2. Известно, что результаты дифференциальных уравнений с частными производными во всем пространстве или дифференциальных уравнений без краевых условий являются в некотором смысле окончательными. Теория краевых задач для общих дифференциальных операторов в настоящее время является актуальной и бурно развивающейся частью теории дифференциальных уравнений. Однако ощущается дефицит явнорешаемых задач на пути дальнейшего развития теории краевых задач дифференциальных уравнений. За последние десятилетия накоплен достаточный материал по конструктивному построению решений краевых задач для модельных уравнений с частными производными. К этой актуальной теме относится данная статья.
Ключевые слова: Функция Грина, тригармоническое уравнение, задача Дирихле-2, краевая задача с общими условиями, интегральное представление решения.

1 Introduction

One of the effective methods of representing solutions to boundary value problems for elliptic equations is a method based on constructing the Green's function of the problem. Many works are devoted to constructing the Green's function in explicit form for various classical boundary value problems. The explicit form of the Green's function of the Dirichlet problem for the polyharmonic equation in the unit ball is constructed in various ways in the works [1–6]. In [7, 8] the solvability of some local and nonlocal boundary value problems with involution for the biharmonic equation is investigated and Green's functions are constructed. Solvability conditions for some versions of boundary value problems for the biharmonic equation in a ball are also obtained in [9]. In [10], solutions to the Dirichlet and Neumann problems for a homogeneous polyharmonic equation were found without using the Green's function. In [11], the Green's functions of the Navier [12] and Riquier-Neumann problems for a biharmonic equation in a ball are given, and in [13], the Green's functions of such problems for a polyharmonic equation are constructed. In [14, 15] the conditions for solvability of some boundary value problems for the polyharmonic equation are found and examples are given for the biharmonic and triharmonic equations. In [16, 17] the Fredholm solvability was investigated and the index formulas for the generalized Neumann problem for high-order elliptic equations containing powers of normal derivatives in the boundary conditions were calculated.

2 Statement of the problem and the main result

In this paper, we study the following boundary value problem with general conditions for the triharmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^3 u(x) = 0, \quad x \in S, \tag{1}$$

$$\begin{cases} a_{00}u + a_{01}\frac{\partial}{\partial\nu}u + a_{02}\Delta u + a_{03}\frac{\partial}{\partial\nu}\Delta u + a_{04}\Delta^2 u = \varphi_1(x),\ x \in \partial S, \\ a_{11}\frac{\partial}{\partial\nu}u + a_{12}\Delta u + a_{13}\frac{\partial}{\partial\nu}\Delta u + a_{14}\Delta^2 u + a_{15}\frac{\partial}{\partial\nu}\Delta^2 u = \varphi_2(x),\ x \in \partial S, \\ a_{21}\frac{\partial}{\partial\nu}u + a_{22}\Delta u + a_{23}\frac{\partial}{\partial\nu}\Delta u + a_{24}\Delta^2 u + a_{25}\frac{\partial}{\partial\nu}\Delta^2 u = \varphi_3(x),\ x \in \partial S, \end{cases} \tag{2}$$

where $\frac{\partial}{\partial\nu}$ is the outer normal derivative to ∂S, a_{ij}, $(i = 0,\ j = \overline{0,4},\ i = 1,2,\ j = \overline{1,5})$and are some constants.

This problem generalizes the Dirichlet problem ($a_{00} \neq 0$, $a_{11} \neq 0$, $a_{22} \neq 0$, $a_{ij} = 0$ for the remaining i, j), the Riquier problem ($a_{00} \neq 0$, $a_{12} \neq 0$, $a_{23} \neq 0$, $a_{ij} = 0$ for the remaining i, j), but does not generalize the Neumann problem.

Theorem 1. *a) The solution of problem (1)-(2) from the class $C^5(\overline{S}) \cap C^6(S)$ for arbitrary functions $\varphi_1(x) \in C^4(\partial S)$, $\varphi_2(x) \in C^3(\partial S)$, $\varphi_3(x) \in C^3(\partial S)$ exists and is unique if and only if the polynomial*

$$\det P(\lambda) =$$

$$\begin{vmatrix} a_{00} + \lambda a_{01} & 2[a_{01} + 2(2\lambda + n)(a_{02} + \lambda a_{03})] & a_{02}^* \\ \lambda a_{11} & 2[a_{11} + 2(2\lambda + n)(a_{12} + \lambda a_{13})] & a_{12}^* \\ \lambda a_{21} & 2[a_{21} + 2(2\lambda + n)(a_{22} + \lambda a_{23})] & a_{22}^* \end{vmatrix} \tag{3}$$

does not have integer roots in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where

$$a_{02}^* = 8[a_{02} + a_{03} + (2 + 2\lambda + n)(2\lambda + n)a_{04}],$$

$$a_{i2}^* = 8\left[a_{i2} + a_{i3} + (2 + 2\lambda + n)(2\lambda + n)(a_{i4} + \lambda a_{i5})\right],\ i = 1, 2.$$

b) If $\det P(m) = 0$, then the homogeneous problem (1)-(2) has a solution

$$u(x) = \left[C_1 - C_2 + (C_2 - C_3)|x|^2 + (C_3 - C_2)|x|^4\right] H_m(x),$$

where $H_m(x)$ is a homogeneous harmonic polynomial of degree m [18], and the constants C_1, C_2, C_3 are found from the system of equations

$$P(m)\overrightarrow{C} = 0, \quad \overrightarrow{C} = (C_1,\ C_2,\ C_3)^\top. \tag{4}$$

Proof. Let us prove that the homogeneous problem (1)-(2) has only a zero solution. Any triharmonic function in S $u(x) \in C^5(\overline{S})$ can be expanded in a power series [18] and therefore the solution of problem (1)-(2) can be represented in the form

$$u(x) = u_0(x) + |x|^2 u_1(x) + |x|^4 u_2(x) = \sum_{m=0}^{\infty} \sum_{i=1}^{h_m} \left[u_{0m}^{(i)} + |x|^2 u_{1m}^{(i)} + |x|^4 u_{2m}^{(i)}\right] H_m^{(i)}(x), \tag{5}$$

where $h_m = \frac{2m+n-2}{n-2}(m+n-3...n-3)$, a $H_m^{(i)}(x)$, $m \in \mathbb{N}_0$, $i = \overline{1, h_m}$– is a complete orthogonal system of harmonic polynomials on ∂S [18].
Series (5) converges uniformly at $|x| \le \varepsilon < 1$.

Let's look at the operators

$$L_0 = a_{00} + a_{01}\Lambda + a_{02}\Delta + a_{03}\Lambda\Delta + a_{04}\Delta^2,$$

$$L_j = a_{j1}\Lambda + a_{j2}\Delta + a_{j3}\Lambda\Delta + a_{j4}\Delta^2 + a_{j5}\Lambda\Delta^2,\ j = 1, 2,$$

where $\Lambda = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$.
Since $u(x) \in C^5(\overline{S})$, it follows from the properties of the operator Λ that

$$L_0 u(x) \to a_{00}u + a_{01}\frac{\partial}{\partial\nu}u + a_{02}\Delta u + a_{03}\frac{\partial}{\partial\nu}\Delta u + a_{04}\Delta^2 u, \quad x \to s \in \partial S,$$

$$L_i u(x) \to a_{i1}\frac{\partial}{\partial\nu}u + a_{i2}\Delta u + a_{i3}\frac{\partial}{\partial\nu}\Delta u + a_{i4}\Delta^2 u + a_{i5}\frac{\partial}{\partial\nu}\Delta^2 u,\ i = 1, 2, \quad x \to s \in \partial S, \quad (6)$$

and the limit is uniform in $s \in \partial S$.
It is easy to see that the polynomials $L_j(u_{0m}^{(i)} + |x|^2 u_{1m}^{(i)} + |x|^4 u_{2m}^{(i)})\, H_m^{(i)}(x)|_{x=s}$ are orthogonal on ∂S for fixed $j = 0, 1, 2$ and for all $m \in \mathbb{N}_0$, $i = \overline{1, h_m}$.

Let for some $m \in \mathbb{N}_0$, $i = \overline{1, h_m}$ either $u_{0m}^{(i)} \ne 0$, or $u_{1m}^{(i)} \ne 0$, or $u_{2m}^{(i)} \ne 0$ in expansion (5). Then, due to the uniform convergence of the series from (5) for $|x| \le \varepsilon < 1$, we have

$$\int_{|x|=\varepsilon} H_m^{(i)}(x)\, L_j u(x)\, ds_x = \int_{|x|=\varepsilon} H_m^{(i)}(x)\, L_j \sum_{p=0}^{\infty}\sum_{k=1}^{h_p} \left[u_{0p}^{(k)} + |x|^2 u_{1p}^{(k)} + |x|^4 u_{2p}^{(k)}\right] H_p^{(k)}(x)\, ds_x =$$

$$\int_{|x|=\varepsilon} H_m^{(i)}(x)\, L_j \left[u_{0m}^{(i)} + |x|^2 u_{1m}^{(i)} + |x|^4 u_{2m}^{(i)}\right] H_m^{(i)}(x)\, ds_x.$$

Directing $\varepsilon \to 1$ in the resulting equality and using (6) we obtain

$$\int_{|x|=1} H_m^{(i)}(x)\, L_j \left[u_{0m}^{(i)} + |x|^2 u_{1m}^{(i)} + |x|^4 u_{2m}^{(i)}\right] H_m^{(i)}(x)\, ds_x = 0,\ j = 0, 1, 2. \quad (7)$$

Let's calculate the integrands. Let's use the following properties of operators Λ, Δ:

$$\Lambda(u\,v) = u\Lambda v + v\Lambda u, \quad \Delta\big(|x|^{2k+2}\, Q_s(x)\big) = (2k+2)(2k+2s+n)|x|^{2k}\, Q_s(x).$$

Then on ∂S we have

$$L_0\big(u_{0m}^{(i)} + |x|^2 u_{1m}^{(i)} + |x|^4 u_{2m}^{(i)}\big)\, H_m^{(i)}(x) =$$

$$\big(a_{00} + a_{01}\Lambda + a_{02}\Delta + a_{03}\Lambda\Delta + a_{04}\Delta^2\big)[u_{0m}^{(i)} + |x|^2 u_{1m}^{(i)} + |x|^4 u_{2m}^{(i)}]\, H_m^{(i)}(x) =$$

$$[u_{0m}^{(i)}(a_{00} + m a_{01}) + u_{1m}^{(i)}\big(a_{00}|x|^2 + (2+m)a_{01}|x|^2 + 2(2m+n)a_{02} + 2(2m+n)m a_{03}\big) +$$

$$u_{2m}^{(i)}\big(a_{00}|x|^4 + (4+m)|x|^4 a_{01} + 4(2+2m+n)|x|^2 a_{02} + 4(2+2m+n)m|x|^2 a_{03} +$$

$$4(2+2m+n)2(2m+n)a_{04}\big)]\, H_m^{(i)}(x) = |x \in \partial S| =$$

$$[u_{0m}^{(i)}(a_{00} + m a_{01}) + u_{1m}^{(i)}\big(a_{00} + (2+m)a_{01} + 2(2m+n)a_{02} + 2(2m+n)m a_{03}\big) +$$

$u_{2m}^{(i)}\big(a_{00}+(4+m)a_{01}+4(2+2m+n)a_{02}+4(2+2m+n)ma_{03}+4(2+2m+n)2(2m+n)a_{04}\big)\big]\, H_m^{(i)}(x);$

$$L_j\big(u_{0m}^{(i)}+|x|^2\,u_{1m}^{(i)}+|x|^4\,u_{2m}^{(i)}\big)\,H_m^{(i)}(x)=$$
$$\big(a_{j1}\Lambda+a_{j2}\Delta+a_{j3}\Lambda\Delta+a_{j4}\Delta^2\big)[u_{0m}^{(i)}+|x|^2\,u_{1m}^{(i)}+|x|^4\,u_{2m}^{(i)}]\,H_m^{(i)}(x)=$$
$$\big[u_{0m}^{(i)}\,ma_{j1}+u_{1m}^{(i)}\big((2+m)a_{j1}+2(2m+n)a_{j2}+2(2m+n)ma_{j3}\big)+$$
$$u_{2m}^{(i)}\big((4+m)a_{j1}+4(2+2m+n)a_{j2}+4(2+2m+n)ma_{j3}+4(2+2m+n)2(2m+n)a_{j4}+$$
$$+4(2+2m+n)2(2m+n)ma_{j5}\big)\big]\,H_m^{(i)}(x),\; j=1,2;$$

Therefore (7) can be rewritten as follows

$$\begin{cases}\big(u_{0m}^{(i)}\,(a_{00}+ma_{01})+u_{1m}^{(i)}a_{01}^*+u_{2m}^{(i)}a_{02}^*\big)\,\|H_m^{(i)}\|^2_{L_2(\partial S)}=0,\\ \big(u_{0m}^{(i)}ma_{11}+a_{11}^*u_{1m}^{(i)}+a_{12}^*u_{2m}^{(i)}\big)\,\|H_m^{(i)}\|^2_{L_2(\partial S)}=0,\\ \big(u_{0m}^{(i)}ma_{21}+u_{1m}^{(i)}a_{21}^*+u_{2m}^{(i)}a_{22}^*\big)\,\|H_m^{(i)}\|^2_{L_2(\partial S)}=0,\end{cases}$$

where

$$a_{01}^*=a_{00}+(2+m)a_{01}+2(2m+n)a_{02}+2(2m+n)ma_{03},$$
$$a_{j1}^*=(2+m)a_{j1}+2(2m+n)a_{j2}+2(2m+n)ma_{j3},$$
$$a_{02}^*=a_{00}+(4+m)a_{01}+4(2+2m+n)a_{02}+4(2+2m+n)ma_{03}+4(2+2m+n)2(2m+n)a_{04},$$
$$a_{j2}^*=(4+m)a_{j1}+4(2+2m+n)a_{j2}+4(2+2m+n)ma_{j3}+$$
$$4(2+2m+n)2(2m+n)a_{j4}+4(2+2m+n)2(2m+n)ma_{j5},\; j=1,2.$$

Since $\|H_m^{(i)}\|^2_{L_2(\partial S)}\neq 0$, we get

$$\begin{pmatrix} a_{00}+ma_{01} & a_{01}^* & a_{02}^*\\ ma_{11} & a_{11}^* & a_{12}^*\\ ma_{21} & a_{21}^* & a_{22}^*\end{pmatrix}\overrightarrow{U}=0. \tag{8}$$

Let us calculate the determinant of this system. It is equal to $\det P(m)$.
If $\det P(m)\neq 0$, then system (8) has only zero $\overrightarrow{U}=(u_{0m}^{(i)},u_{1m}^{(i)},u_{2m}^{(i)})^\top=0$. This contradicts the assumption that either $u_{0m}^{(i)}\neq 0$, or $u_{1m}^{(i)}\neq 0$, or $u_{2m}^{(i)}\neq 0$ in the expansion (5). Thus, problem (1)-(2) has only the zero solution.

If $\det P(m)=0$, then system (4) has a non-zero solution $\overrightarrow{C}=(C_1,C_2,C_3)^\top$, which means

$$P(m)\begin{pmatrix} C_1-C_2\\ C_2-C_3\\ C_3-C_2\end{pmatrix}=0. \tag{9}$$

Therefore, on ∂S the equalities

$$L_j\big[C_1-C_2+(C_2-C_3)\,|x|^2+(C_3-C_2)\,|x|^4\big]\,H_m(x)=0,\; j=0,1,2,$$

are true and therefore

$$u(x) = \left[C_1 + C_2(-1 + |x|^2 - |x|^4) + C_3(-|x|^2 + |x|^4)\right] H_m(x)$$

is a solution to the homogeneous problem (1)-(2). Theorem 1 is proven.

Theorem 2. *If $u(x)$ is a triharmonic function in S, then for harmonic functions in S*

$$u_0(x) = \Delta^2 u(x) - \frac{|x|^2}{2} \int_0^1 t^{n-1} \Delta^2 u(t^2 x)\, dt,$$

$$u_1(x) = \frac{1}{2} \int_0^1 t^{n-1} \Delta^2 u(t^2 x)\, dt - \frac{|x|^2}{2} \int_0^1 t^{n-1} \Delta^2 u(t^2 x)\, dt,$$

$$u_2(x) = \frac{1}{2} \int_0^1 t^{n-1} \Delta^2 u(t^2 x)\, dt \tag{10}$$

Almansi's representation is fair

$$u(x) = u_0(x) + |x|^2\, u_1(x) + |x|^4\, u_2(x). \tag{11}$$

Proof. If the functions $u_0(x)$, $u_1(x)$, $u_2(x)$ are defined by equalities (10), then the representation (11) is true. Let us prove that the functions $u_0(x)$, $u_1(x)$, $u_2(x)$ from (10) are harmonic in S. Obviously, the functions $u_1(x), u_2(x)$ are harmonic in S if the function $u(x)$ is triharmonic in S.

Further, since the equality $\Delta(|x|^2\, v(x)) = (2 + 4\Lambda)\, v(x)$ is true for the harmonic function $v(x)$, and the chain of equalities

$$\Delta(|x|^2\, u_2) = (2m + 4\Lambda) \frac{1}{2} \int_0^1 t^{n-1} w(t^2 x)\, dt =$$

$$n \int_0^1 t^{n-1} w(t^2 x)\, dt + 2 \int_0^1 t^{n+1} \sum_{i=1}^{n} x_i w_{x_i}(t^2 x)\, dt = n \int_0^1 t^{n-1} w(t^2 x)\, dt + \int_0^1 t^n w_t(t^2 x)\, dt =$$

$$n \int_0^1 t^{n-1} w(t^2 x)\, dt + t^n w(t^2 x)|_0^1 - n \int_0^1 t^{n-1} w(t^2 x)\, dt = w(x),$$

then we have

$$\Delta u_0 = \Delta^3 u - \frac{1}{2} \Delta \left(|x|^2 \int_0^1 t^{n-1} \Delta^2 u(t^2 x)\, dt \right) = \Delta^3 u - (2m + 4\Lambda) \frac{1}{2} \int_0^1 t^{n-1} \Delta^2 u(t^2 x)\, dt =$$

$$\Delta^3 u(x) - \Delta^3 u(x) = 0.$$

This means that the function $u_0(x)$ is harmonic in S. Theorem 2 is proven.

For the sake of completeness, we present the results of V.V. Karachik [20] on the representation of the solution of the following boundary value problem for the triharmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^3 u(x) = 0, \quad x \in S, \tag{12}$$

$$u|_{\partial S} = \varphi_0, \quad , \frac{\partial u}{\partial \nu}|_{\partial S} = \varphi_1, \quad , \Delta u|_{\partial S} = \varphi_2, \tag{13}$$

which can be called the Dirichlet-2 problem. This problem turned out to be close to the Dirichlet problem, they have the same Green's function. The solution to the problem is sought in the class $u \in C^5(\overline{S}) \cap C^6(S)$.

The Green's function of the Dirichlet problem for the Poisson equation in the ball S for $n \geq 2$ has the form

$$G_2(x,\xi) = E_2(x,\xi) - E_2(\frac{x}{|x|}, |x|\xi), \tag{14}$$

where $E_2(x,\xi)$ is an elementary solution of the Laplace equation, as A.V. Bitsadze called it [19]. In the work [20], an elementary solution of the biharmonic equation was determined

$$E_4(x,\xi) = \begin{cases} \frac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n>4,\, n=3 \\ -\frac{1}{4}\ln|x-\xi|, & n=4 \\ \frac{|x-\xi|^2}{4}(\ln|x-\xi|-1) & n=2, \end{cases} \tag{15}$$

was determined, and in the paper [21] for the 3-harmonic equation

$$E_6(x,\xi) = \begin{cases} \frac{|x-\xi|^{6-n}}{2\cdot 4(n-2)(n-4)(n-6)}, & n\geq 3,\, n\neq 4,6 \\ -\frac{1}{64}\ln|x-\xi|, & n=6 \\ \frac{|x-\xi|^2}{32}(\ln|x-\xi|-\frac{3}{4}), & n=4 \\ -\frac{|x-\xi|^4}{64}(\ln|x-\xi|-\frac{3}{2})\ln|x-\xi|, & n=2. \end{cases} \tag{16}$$

In addition, the Green functions $G_4(x,\xi)$ and $G_6(x,\xi)$ corresponding to the Dirichlet problems in S were found.

If we denote $E_k^*(x,\xi) = E_k(\frac{x}{|x|}, |x|\xi)$, then $G_6(x,\xi)$ has the form

$$G_6(x,\xi) = E_6(x,\xi) - E_6^*(x,\xi) - \frac{1}{2}\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E_4^*(x,\xi) - \frac{1}{4}\frac{(|x|^2-1)^2}{4}\frac{(|\xi|^2-1)^2}{4}E_2^*(x,\xi). \tag{17}$$

Based on the functions $E_4(x,\xi)$ and $E_6(x,\xi)$, an elementary solution of the m-harmonic equation $\Delta^m u = 0$ was introduced in [23]. If $m \in \mathbb{N}$, then $\mathbb{N}\backslash\{1\}$ can be partitioned into two disjoint sets $\mathbb{N}_m = \{n \in \mathbb{N} : n > 2m > 1\} \cup (2\mathbb{N}+1)$ and its complement $\mathbb{N}_m^c = \{2,4,...,2m\}$. Since the set $\mathbb{N}_m^c$ is finite, $\mathbb{N}_m$ is infinite. It is clear that $\mathbb{N}_{m-1}^c \subset \mathbb{N}_m^c$, and therefore $\mathbb{N}_m \subset \mathbb{N}_{m-1}$. We define the elementary solution $E_{2m}(x,\xi)$ as

$$E_{2m}(x,\xi) = \begin{cases} \frac{(-1)^m\,|x-\xi|^{2m-n}}{(2-n,2)_m(2,2)_{m-1}}, & n\in\mathbb{N}_m, \\ \frac{(-1)^m\,|x-\xi|^{2m-n}}{(2-n,2)_m^*(2,2)_{m-1}}(\ln|x-\xi| - \sum_{k=1}^{m-n/2}\frac{1}{2k} - \sum_{k=n/2}^{m-1}\frac{1}{2k}), & n\in\mathbb{N}_m^c, \end{cases} \tag{18}$$

where $(a,b)_k = a(a+b)...(a+kb-b)$ is a generalized Pochhammer symbol with the convention $(a,b)_0 = 1$, and the symbol $(a,b)_k^*$ means that if among the factors $a, (a+b), ..., (a+kb-b)$, included in $(a,b)_k$, there is 0, then it should be replaced by 1, for example, $(-2,2)_3^* = (-2)\cdot 1\cdot 2 = -4$. In addition, if in the sums included in (18) the upper index becomes less than the lower index, then the sum is considered to be equal to zero. Note that $(2-n,2)_m = (2-n)(4-n)...(2m-n) \neq 0$ for $n \in \mathbb{N}_m$ and therefore the right-hand side of formula (18) is defined correctly.

In [23] for $n \in \mathbb{N}_{m-1}^c$ the Green's function was constructed

$$G_{2m}(x,\xi) = E_{2m}(x,\xi) - \sum_{k=0}^{m-1}\frac{(|x|^2-1)^k(|\xi|^2-1)^k}{(2m-2,-2)_k(2,2)_k}E_{2m-2k}^*(x,\xi). \tag{19}$$

In [22] the elementary solution $E_{2m}(x,\xi)$ was slightly corrected and another function $\mathcal{E}_{2m}(x,\xi)$ was introduced, which is related to the function $E_{2m}(x,\xi)$ by the formula

$$\mathcal{E}_{2m}(x,\xi)=\begin{cases} E_{2m}(x,\xi), & n\in\mathbb{N}_{m-1}, \\ E_{2m}(x,\xi)+\frac{(-1)^m\,|x-\xi|^{2m-n}}{(2-n,2)^*_m(2,2)_{m-1}}\sum_{k=n/2}^{m-1}\frac{1}{2k}, & n\in\mathbb{N}^c_{m-1}. \end{cases} \tag{20}$$

It is clear that $\mathcal{E}_{2m}(x,\xi)=E_{2m}(x,\xi)$ for all $n\geq 2$. For $m=2$ we have $\mathbb{N}^c_1=\{2\}$ and, therefore, in formula (15) only the last line will change

$$\mathcal{E}_4(x,\xi)=\begin{cases} \frac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n>4,\, n=3 \\ -\frac{1}{4}\ln|x-\xi|, & n=4 \\ \frac{|x-\xi|^2}{4}(\ln|x-\xi|-1/2) & n=2, \end{cases} \tag{21}$$

If $m=3$, then $\mathbb{N}^c_2=\{2,4\}$ and therefore the last two lines will change

$$\mathcal{E}_6(x,\xi)=\begin{cases} \frac{|x-\xi|^{6-n}}{2\cdot4(n-2)(n-4)(n-6)}, & n\geq 3,\, n\neq 4,6 \\ -\frac{1}{64}\ln|x-\xi|, & n=6 \\ \frac{|x-\xi|^2}{32}(\ln|x-\xi|-\frac{1}{2}), & n=4 \\ -\frac{|x-\xi|^4}{64}(\ln|x-\xi|-\frac{3}{4})\ln|x-\xi|, & n=2. \end{cases} \tag{22}$$

Replacing in (17) $E_{2m}(x,\xi)$ with $\mathcal{E}_{2m}(x,\xi)$ we obtain a new function

$$\mathcal{G}_6(x,\xi)=\mathcal{E}_6(x,\xi)-\mathcal{E}^*_6(x,\xi)-\frac{1}{2}\,\frac{|x|^2-1}{2}\,\frac{|\xi|^2-1}{2}\,\mathcal{E}^*_4(x,\xi)-\frac{1}{4}\,\frac{(|x|^2-1)^2}{4}\,\frac{(|\xi|^2-1)^2}{4}\,\mathcal{E}^*_2(x,\xi). \tag{23}$$

If we put $m=3$ and $n=4$ in it, then in relation to (20)

$$\mathcal{E}_6(x,\xi)=E_6(x,\xi)+\frac{1}{4}\,\frac{|x-\xi|^2}{32},\quad \mathcal{E}_4(x,\xi)=E_4(x,\xi),$$

and therefore

$$\begin{aligned}\mathcal{G}_6(x,\xi)&=\mathcal{E}_6(x,\xi)-\mathcal{E}^*_6(x,\xi)-\frac{1}{2}\,\frac{|x|^2-1}{2}\,\frac{|\xi|^2-1}{2}\,\mathcal{E}^*_4(x,\xi)-\frac{1}{4}\,\frac{(|x|^2-1)^2}{4}\,\frac{(|\xi|^2-1)^2}{4}\,\mathcal{E}^*_2(x,\xi)\\ &=G_6(x,\xi)+\frac{1}{4}\,(\frac{|x-\xi|^2}{32}-\frac{|x/|\xi|-\xi|x||^2}{32})\\ &=G_6(x,\xi)-\frac{1}{4}\,\frac{(|\xi|^2-1)(|x|^2-1)}{32}.\end{aligned}$$

It turns out that the function $\mathcal{G}_6(x,\xi)$ obtained for $m=3$ and $n=4$ coincides with the Green's function for the 3-harmonic Dirichlet problem (12)-(13) from the works [21].

Theorem 3. [20] *If a solution to problem (12)-(13) exists, then it can be written in the form*

$$u(x)=\frac{1}{\omega_n}\int_{\partial S}\big(-\frac{\partial\Delta^2\mathcal{G}_6(x,\xi)}{\partial\nu}\varphi_0(\xi)+\Delta^2\mathcal{G}_6(x,\xi)\varphi_1(\xi)-\frac{\partial\Delta\mathcal{G}_6(x,\xi)}{\partial\nu}\varphi_0(\xi)\big)ds_\xi$$
$$-\frac{1}{\omega_n}\int_S\mathcal{G}_6(x,\xi)\,f(\xi)\,d\xi, \tag{24}$$

where $\omega_n=|\partial S|$ is the area of an unit sphere in $\mathbb{R}$ and ν is the outward unit normal to ∂S, the Green's function $\mathcal{G}_6(x,\xi)$ is defined in (23).

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