1-бөлім Раздел 1 Section 1

Математика Математика Mathematics

IRSTI 27.31.21; 27.31.55 DOI: https://doi.org/10.26577/JMMCS2025126201

G.F. Azhmoldaev[1*] **K.A. Bekmaganbetov**[2,3] **G.A. Chechkin**[4,5] **V.V. Chepyzhov**[6]

[1] L.N. Gumilyov Eurasian national university, Astana, Kazakhstan
[2] Kazakhstan branch of M.V. Lomonosov Moscow state university, Astana, Kazakhstan
[3] Institute of mathematics and mathematical modelling, Almaty, Kazakhstan
[4] M.V. Lomonosov Moscow state university, Russia, Moscow
[5] Institute of mathematics with computing center subdivision of the Ufa Federal research center, Ufa, Rassia
[6] Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia
*e-mail: azhmoldaevgf@gmail.com

HOMOGENIZATION OF ATTRACTORS TO THE REACTION-DIFFUSION SYSTEM IN A DOMAIN WITH ROUGH BOUNDARY

In this paper, we consider the homogenization problem in a micro inhomogeneous domain with a rapidly oscillating boundary. It is assumed that a system of nonlinear reaction–diffusion equations with rapidly oscillating terms and dissipation is considered in the domain. On the locally periodic oscillating part of the boundary, the third boundary condition with rapidly oscillating coefficients and a small parameter characterizing the oscillation of the boundary to some degree is imposed. Depending on the degree of the small parameter in the boundary condition, various homogenized (limit) problems are obtained and the convergence of the trajectory attractors of the given system to the attractors of the homogenized system is proved. Critical, subcritical and supercritical cases of attractor behavior as the small parameter tends to zero are carefully studied. The paper also considers problems in a domain with a random rapidly oscillating boundary. In this case, a homogenized system of reaction–diffusion equations with deterministic coefficients is obtained in the case of a statistically homogeneous random structure of the boundary. A theorem on the convergence of random trajectory attractors of the initial given system of reaction-diffusion equations to a deterministic attractor of the homogenized (limit) system of reaction–diffusion equations is also proved. The paper also proves the convergence of global attractors in the case of uniqueness of solutions, which in turn is proved for nonlinearity in a system of equations of a special type.
Key words: attractors, homogenization, reaction-diffusion equations, non-linear equations, weak convergence, rapidly oscillating boundary.

Г.Ф. Ажмолдаев[1*], К.А. Бекмаганбетов[2,3], Г.А. Чечкин [4,5], В.В. Чепыжов[6]
[1]Л.Н. Гумилёв атындағы Еуразия ұлттық университеті, Астана, Қазақстан
[2]М.В. Ломоносов атындағы Мәскеу мемлекеттік университетінің Қазақстандағы филиалы, Астана, Қазақстан
[3]Математика және математикалық модельдеу институты, Алматы, Қазақстан
[4]М.В. Ломоносов атындағы Мәскеу мемлекеттік университеті, Мәскеу, Ресей
[5]Компьютерлік орталығы бар математика институты — Ресей Ғылым академиясының Уфа федеральды зерттеу орталығының бөлімшесі, Уфа, Ресей
[6]Ресей Ғылым академиясының А.А. Харкевич атындағы Ақпарат беру мәселелері институты, Мәскеу, Ресей
*e-mail: azhmoldaevgf@gmail.com

Тез тербелмелі шекарасы бар аймақтардағы реакция-диффузия теңдеулерінің аттракторларының орташалауы

Бұл жұмыста шекарасы тез тербелетін микро біртекті емес аймақта орташалау мәселесі қарастырылады. Аймақта жылдам тербелетін мүшелері мен диссипациясы бар сызықты емес реакция-диффузия теңдеулер жүйесі зерттелінген. Шекараның локальды тербелмелі бөлігінде жылдам тербелмелі коэффициенттері бар үшінші шекаралық шарт және шекараның белгілі бір дәрежеде тербелісін сипаттайтын шағын параметр белгіленеді. Шектік жағдайдағы кіші параметр дәрежесіне байланысты әртүрлі орташаланған (шектік) есептер алынды және бастапқы жүйенің траекториялық аттракторларының орташаланған жүйенің аттракторларына жинақталуы дәлелденді. Кіші параметр нөлге ұмтылған кезде аттракторлардың ерекшеліктері критикалық, субкритикалық және суперкритикалық жағдайлары мұқият зерттелінді. Мақалада сонымен қатар кездейсоқ, жылдам тербелетін шекарасы бар аймақтағы мәселелер қарастырылады. Бұл жағдайда шекараның статистикалық біртекті кездейсоқ құрылымы жағдайында детерминирленген коэффициенттері бар реакция-диффузия теңдеулерінің орташаланған жүйесі алынды. Реакция-диффузия теңдеулерінің бастапқы жүйесінің кездейсоқ траекториялық аттракторларының орташаланған (шектік) реакция-диффузия теңдеулер жүйесінің кездейсоқ емес есебінің аттракторына жинақталуы туралы теоремасы дәлелденген. Жұмыс сондай-ақ бірегей шешімдер жағдайында глобалды аттракторлардың жинақталуын дәлелденді, бұл жағдай сызықтық емес мүшелерге қосымша шарт қойылған кезде пайда болады.
Түйін сөздер: аттракторлар, орташалау, реакция-диффузия теңдеулері, сызықтық емес теңдеулер, әлсіз жинақтылық, тез тербелмелі шекара.

Г.Ф. Ажмолдаев[1*], К.А. Бекмаганбетов[2,3], Г.А. Чечкин[4,5], В.В. Чепыжов[6]
[1]Евразийский национальный университет им. Л.Н. Гумилева, Астана, Казахстан
[2]Казахстанский филиал Московского государственного университета им. М.В. Ломоносова, Астана, Казахстан
[3]Институт математики и математического моделирования, Алматы, Казахстан
[4]Московский государственный университет им. М.В. Ломоносова, Москва, Россия
[5]Институт математики с вычислительным центром. Подразделение Уфимского федерального исследовательского центра Российской Академия наук, Уфа, Россия
[6]Институт проблем передачи информации РАН, Москва, Россия
[*]e-mail: azhmoldaevgf@gmail.com

Об усреднении аттракторов уравнений реакции-диффузии в области с шероховатой границей

В данной работе рассматривается задача усреднения в микро неоднородной области с быстро осциллирующей границей. Предполагается, что в области задана система нелинейных уравнений реакции–диффузии с быстро осциллирующими членами и диссипацией. На локально периодической осциллирующей части границы выставлено третье краевое условие с быстро осциллирующими коэффициентами и малым параметром, характеризующим осцилляцию границы, в некоторой степени. В зависимости от степени малого параметра в краевом условии получены различные усреднённые (предельные) задачи и доказана сходимость траекторных аттракторов исходной системы к аттракторам усреднённой системы. Аккуратно исследованы критический, субкритический и суперкритический случаи поведения аттракторов при стремлении малого параметра к нулю. В статье рассмотрены также задачи в области со случайной быстро осциллирующей границей. При этом получена усреднённая система уравнений реакции–диффузии с детерминированными коэффициентами в случае статистически однородной случайной структурой границы. Также доказана теорема о сходимости случайных траекторных аттракторов исходной системы уравнений реакции–диффузии к детерминированному аттрактору усреднённой (предельной) системы уравнений реакции–диффузии.В работе также доказана сходимость и глобальных аттракторов в случае единственности решений, которая в свою очередь доказана для нелинейности в системе уравнений специального вида.

Ключевые слова: аттракторы, усреднение, уравнения реакции-диффузии, нелинейные уравнения, слабая сходимость, быстро осциллирующая граница.

1 Introduction

In this paper, we present a review of results on homogenization of initial–boundary value problems for a system of reaction–diffusion equations in domains with a rapidly oscillating boundary (for detailed geometric formulations, see [3] and [40]). We consider nonlinear systems of reaction–diffusion equations in such a domain with a locally periodic and random rapidly oscillating boundary, and investigate the case of dissipative coefficients in the equations. We prove the existence of trajectory attractors, construct a limit (homogenized) system of reaction–diffusion equations both in the case of a locally periodic and in the case of a statistically homogeneous random boundary, and prove the convergence of the attractors of the original system as the small parameter characterizing the boundary oscillation, tends to zero, i.e. prove the Hausdorff convergence of the attractors of the original system to the attractors of the homogenized (limit) system as the small parameter tends to zero. In many purely mathematical works one can find asymptotic analysis of problems in domains with oscillating (rough) boundaries (see, for example, [1–10]). We also mention here the fundamental works on this topic [11–14], where one can find a detailed bibliography. A special feature of the second part of the work is the random geometry of the domain (see some examples in [37–39]). It is assumed that the random structure is statistically homogeneous. This fact allows us to obtain a deterministic limit problem (see [40]), which does not depend on the choice of an element of the probability space. Theoretical results on attractor averaging can be found, for example, in [15–17], and see references therein. Attractor averaging was also studied in [17–20] (see also [21–24]).

In this paper, we establish weak convergence (in the sense of "almost surely" in the probabilistic case, i.e. with probability one) of the trajectory attractor $\mathfrak{A}_\varepsilon$ of reaction–diffusion systems in domains with an oscillating boundary, for $\varepsilon \to 0$, to the trajectory attractors $\overline{\mathfrak{A}}$ of homogenized systems in some natural functional space. Here the small parameter ε characterizes the period and amplitude of the oscillations. The parameter ε also appears to some power in the third boundary condition on a part of the locally periodic boundary, and in the limit in the locally periodic case we obtain 3 different homogenized problems (critical, subcritical and supercritical cases) depending on the ratio between the powers of the small parameter. In the random formulation of the problem ε also characterizes the microinhomogeneity on the boundary.

In the second section of the paper one can find the main preliminary results on attractors and random sets, the third section is devoted to homogenization in the locally periodic case. In the fourth section we present the results of homogenization when the boundary has a random structure.

2 Preliminary information.

2.1 Trajectory attractors of evolution equations

This section is devoted to the construction of trajectory attractors to autonomous evolution equations (see details in [17]).

Consider an autonomous evolution equation of the form for $t > 0$

$$\frac{\partial u}{\partial t} = A(u). \tag{1}$$

Here $A(\cdot) : E_1 \to E_0$ is a nonlinear operator, E_1, E_0 are Banach spaces and $E_1 \subseteq E_0$. As an example one can consider $A(u) = \lambda \Delta u - a(\cdot) f(u) + h(\cdot)$.

We study weak solutions $u(t)$ to (1) as functions $t \in \mathbb{R}_+$ as a whole. The set of solutions of (1) is said to be *a trajectory space* $\mathcal{K}^+$ of equation (1). Now, we describe the trajectory space $\mathcal{K}^+$ in detail.

Consider solutions $u(t)$ of (1) defined on $[t_1, t_2] \subset \mathbb{R}$. We consider solutions to problem (1) in a Banach space $\mathcal{F}_{t_1,t_2}$. The space $\mathcal{F}_{t_1,t_2}$ is a set $f(s), s \in [t_1, t_2]$ satisfying $f(t) \in E$ for almost all $t \in [t_1, t_2]$, where E is a Banach space, satisfying $E_1 \subseteq E \subseteq E_0$.

For instance, $\mathcal{F}_{t_1,t_2}$ can be considered as the intersection spaces $C([t_1, t_2]; E)$, or $L_p(t_1, t_2; E)$, for $p \in [1, \infty]$. Suppose that $\Pi_{t_1,t_2} \mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and $\|\Pi_{t_1,t_2} f\|_{\mathcal{F}_{t_1,t_2}} \leq C(t_1, t_2, \tau_1, \tau_2) \|f\|_{\mathcal{F}_{\tau_1,\tau_2}}$ $\forall f \in \mathcal{F}_{\tau_1,\tau_2}$. Here $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Π_{t_1,t_2} denotes the restriction operator onto $[t_1, t_2]$, constant $C(t_1, t_2, \tau_1, \tau_2)$ does not depend on f.

Denote by $S(\tau)$ for $\tau \in \mathbb{R}$ the translation operator $S(\tau) f(t) = f(\tau + t)$. It is easy to see, that if the argument t of $f(\cdot)$ belongs to the segment $[t_1, t_2]$, then the argument t of $S(\tau) f(\cdot)$ belongs to $[t_1 - \tau, t_2 - \tau]$ for $\tau \in \mathbb{R}$. Suppose that the mapping $S(\tau)$ is an isomorphism from F_{t_1,t_2} to $F_{t_1-\tau,t_2-\tau}$ and $\|S(\tau) f\|_{\mathcal{F}_{t_1-\tau,t_2-\tau}} = \|f\|_{\mathcal{F}t_1,t_2}$ $\forall f \in \mathcal{F}_{t_1,t_2}$. It is easy to see that this assumption is natural.

Suppose that if $f(t) \in \mathcal{F}_{t_1,t_2}$, then $A(f(t)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{D}_{t_1,t_2}$ is a Banach space, which is larger, $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\frac{\partial f(t)}{\partial t}$ is a distribution with values in E_0, $\frac{\partial f}{\partial t} \in D'((t_1, t_2); E_0)$ and we suppose that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(t) \in \mathcal{F}_{t_1,t_2}$ is a *solution* of (1), if $\frac{\partial u}{\partial t}(t) = A(u(t))$ in the sense of $D'((t_1, t_2); E_0)$.

Let us define the space $\mathcal{F}_+^{loc} = \{f(t),\, t \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(t) \in \mathcal{F}_{t_1,t_2}, \quad \forall\, [t_1, t_2] \subset \mathbb{R}_+\}$. For instance, if $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$, then $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$ and if $\mathcal{F}_{t_1,t_2} = L_p(t_1, t_2; E)$, then $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(t) \in \mathcal{F}_+^{loc}$ is a solution of (1), if $\Pi_{t_1,t_2} u(t) \in \mathcal{F}_{t_1,t_2}$ and $u(t)$ is a solution of (1) for every $[t_1, t_2] \subset \mathbb{R}_+$.

Let $\mathcal{K}^+$ be a set of solutions to (1) from $\mathcal{F}_+^{loc}$. Note, that $\mathcal{K}^+$ in general is not the set of *all* solutions from $\mathcal{F}_+^{loc}$. The set $\mathcal{K}^+$ consists on elements, which are *trajectories* and the set $\mathcal{K}^+$ is the *trajectory space* of the equation (1).

Suppose that the trajectory space $\mathcal{K}^+$ is *translation invariant*, i.e., if $u(t) \in \mathcal{K}^+$, then $u(\tau + t) \in \mathcal{K}^+$ for every $\tau \geq 0$.

Consider the translation operators $S(\tau)$ in $\mathcal{F}_+^{loc} : S(\tau) f(t) = f(\tau + t), \tau \geq 0$. It is easy to see that the map $\{S(\tau), \tau \geq 0\}$ forms a semigroup in $\mathcal{F}_+^{loc}$: $S(\tau_1) S(\tau_2) = S(\tau_1 + \tau_2)$ for $\tau_1, \tau_2 \geq 0$ and in addition $S(0)$ is the identity operator. The *translation semigroup* $\{S(\tau), \tau \geq 0\}$ maps the trajectory space $\mathcal{K}^+$ to itself: $S(\tau) \mathcal{K}^+ \subseteq \mathcal{K}^+$ for all $\tau \geq 0$.

We investigate attracting properties of the translation semigroup $\{S(\tau)\}$ acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. Next step is to define a topology in the space $\mathcal{F}_+^{loc}$.

Let some metrics $\rho_{t_1,t_2}(\cdot, \cdot)$ be defined on $\mathcal{F}_{t_1,t_2}$ for every $[t_1, t_2] \subset \mathbb{R}$. Suppose that

$$\rho_{t_1,t_2}(\Pi_{t_1,t_2} f, \Pi_{t_1,t_2} g) \leq D(t_1, t_2, \tau_1, \tau_2) \rho_{\tau_1,\tau_2}(f, g) \quad \forall f, g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1, t_2] \subseteq [\tau_1, \tau_2],$$

$$\rho_{t_1-\tau,t_2-\tau}(S(\tau) f, S(\tau) g) = \rho_{t_1,t_2}(f, g) \quad \forall f, g \in \mathcal{F}_{t_1,t_2},\ [t_1, t_2] \subset \mathbb{R},\ \tau \in \mathbb{R}.$$

Now, we denote by Θ_{t_1,t_2} metric spaces on $\mathcal{F}_{t_1,t_2}$. For instance, ρ_{t_1,t_2} is metric associated with the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ of $\mathcal{F}_{t_1,t_2}$. On the other hand, in application ρ_{t_1,t_2} generates the topology Θ_{t_1,t_2} that is weaker than the strong one of the $\mathcal{F}_{t_1,t_2}$.

The *projective limit* of the spaces Θ_{t_1,t_2} defines the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$, that is, by definition, a sequence $\{f_k(t)\} \subset \mathcal{F}_+^{loc}$ tends to $f(t) \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\rho_{t_1,t_2}(\Pi_{t_1,t_2} f_k, \Pi_{t_1,t_2} f) \to 0$ as $k \to \infty$ for all $[t_1,t_2] \subset \mathbb{R}_+$. It is possible to show that the topology Θ_+^{loc} is metrizable. For this aim we use, for example, the Frechet metric

$$\rho_+(f_1,f_2) := \sum_{m\in\mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{2}$$

The translation semigroup $\{S(\tau)\}$ is continuous in Θ_+^{loc}. This statement follows from the definition of Θ_+^{loc}.

We also define the following Banach space $\mathcal{F}_+^b := \{f(t) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\}$, where the norm $\|f\|_{\mathcal{F}_+^b} := \sup_{\tau\geq 0} \|\Pi_{0,1} f(\tau+t)\|_{\mathcal{F}_{0,1}}$.

We remember that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. We need from our Banach space $\mathcal{F}_+^b$ only one fact that it should define bounded subsets in the trajectory space $\mathcal{K}^+$. For constructing a trajectory attractor in $\mathcal{K}^+$, instead of considering the corresponding uniform convergence topology of the Banach space $\mathcal{F}_+^b$, we use much weaker topology, i.e. the local convergence topology Θ_+^{loc}.

Assume that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, that is, every trajectory $u(t) \in \mathcal{K}^+$ of equation (1) has a finite norm. We define an attracting set and a trajectory attractor of the translation semigroup $\{S(\tau)\}$ acting on $\mathcal{K}^+$.

Definition 1 *A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an* attracting set *of the semigroup $\{S(\tau)\}$ acting on $\mathcal{K}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(\tau)\mathcal{B}$ as $\tau \to +\infty$ in the topology Θ_+^{loc}, i.e., for any ε-neighbourhood $O_\varepsilon(\mathcal{P})$ in Θ_+^{loc} there exists $\tau_1 \geq 0$ such that $S(\tau)\mathcal{B} \subseteq O_\varepsilon(\mathcal{P})$ for all $\tau \geq \tau_1$.*

It is easy to see that the attracting property of $\mathcal{P}$ can be formulated equivalently: we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \longrightarrow 0 \quad (\tau \to +\infty),$$

where $\mathrm{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x,Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x,y)$ is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$. We remember that the Hausdorff semidistance is not symmetric, for any $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$.

Definition 2 ([17]) *A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the* trajectory attractor *of the translation semigroup $\{S(\tau)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if*

(i) $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc},

(ii) the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(\tau)\mathfrak{A} = \mathfrak{A}$ for all $\tau \geq 0$,

(iii) $\mathfrak{A}$ is an attracting set for $\{S(\tau)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$ we have $\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(\tau)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (\tau \to +\infty)$.

Let us formulate the main assertion on the trajectory attractor for equation (1).

Theorem 1 ([16, 17]) *Assume that the trajectory space $\mathcal{K}^+$ corresponding to equation* (1) *is contained in $\mathcal{F}_+^b$. Suppose that the translation semigroup $\{S(t)\}$ has an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$*

which is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(\tau), \tau \geq 0\}$ acting on $\mathcal{K}^+$ has the trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}.

Let us describe in detail, i.e., in terms of complete trajectories of the equation, the structure of the trajectory attractor $\mathfrak{A}$ to equation (1). We study the equation (1) on the time axis,i.e. $t \in \mathbb{R}$.

Note that the trajectory space $\mathcal{K}^+$ of equation (1) on $\mathbb{R}_+$ have been defined. We need this notion on the entire $\mathbb{R}$. If a function $f(t),\ s \in \mathbb{R}$, is defined on the entire time axis, then the translations $S(\tau)f(t) = f(\tau + t)$ are also defined for negative τ. A function $u(t), t \in \mathbb{R}$ is a *complete trajectory* of equation (1) if $\Pi_+ u(\tau+t) \in \mathcal{K}^+$ for all $\tau \in \mathbb{R}$. Here $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator to $\mathbb{R}_+$.

We have $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc}. Let us define spaces $\mathcal{F}^{loc}, \mathcal{F}^b$, and Θ^{loc} in the same way:

$$\mathcal{F}^{loc} := \{f(t), t \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}\ \forall\ [t_1,t_2] \subseteq \mathbb{R}\}; \quad \mathcal{F}^b := \{f(t) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h \in \mathbb{R}} \|\Pi_{0,1} f(\tau + t)\|_{\mathcal{F}_{0,1}}. \tag{3}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by definition, $f_k(t) \to f(t)$ $(k \to \infty)$ in Θ^{loc} if $\Pi_{t_1,t_2} f_k(t) \to \Pi_{t_1,t_2} f(t)$ $(k \to \infty)$ in Θ_{t_1,t_2} for each $[t_1,t_2] \subseteq \mathbb{R}$. It is easy to see that Θ^{loc} is a metric space as well as Θ_+^{loc}.

Definition 3 *The* kernel $\mathcal{K}$ *in the space $\mathcal{F}^b$ of equation* (1) *is the union of all complete trajectories $u(t), t \in \mathbb{R}$, of equation* (1) *that are bounded in the space $\mathcal{F}^b$ with respect to the norm* (3)*, i.e. $\|\Pi_{0,1} u(\tau + t)\|_{\mathcal{F}_{0,1}} \leq C_u \quad \forall \tau \in \mathbb{R}$.*

Theorem 2 *Assume that the hypotheses of Theorem* 1 *holds. Then $\mathfrak{A} = \Pi_+ \mathcal{K}$, the set $\mathcal{K}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.*

To prove this assertion one can use the approach from [17].

In this paper we investigate evolution equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$.

Definition 4 *We say that the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon \to 0$ in the topological space Θ_+^{loc} if for any neighbourhood $\mathcal{O}(\overline{\mathfrak{A}})$ in Θ_+^{loc} there is an $\varepsilon_1 \geq 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, that is, for each $M > 0$ we have* $\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0$ $(\varepsilon \to 0)$.

2.2 The probabilistic framework and main assumptions

Throughout the paper, we assume that all the random fields and random variables are defined on a probability space $(\Omega, \mathcal{A}, \mu)$. The random fields considered in the paper are statistically homogeneous.

Definition 5 A family of measurable maps $T_x : \Omega \to \Omega,\ \ x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, is called a *d-dynamical system* if the following properties hold true:

- Group property:

$$T_{x+y} = T_x T_y \;\; \forall x, y \in \mathbb{R}^d, \;\; T_0 = Id \;\; (Id \text{ is the identical mapping});$$

- Isometry property:

$$T_x \mathcal{U} \in \mathcal{A}, \quad \mu(T_x \mathcal{U}) = \mu(\mathcal{U}), \;\; \forall x \in \mathbb{R}^d, \;\; \forall \mathcal{U} \in \mathcal{A};$$

- Measurability: for any measurable functions $\phi(\omega)$ on Ω, the function $\phi(T_x\omega)$ is measurable on $\Omega \times \mathbb{R}^d$, where the space $\mathbb{R}^d$ is equipped with the Borel σ-algebra $\mathcal{B}$.

Definition 6 Let $\phi(\omega)$ be a measurable function (i.e. a random variable) on Ω. The function $\phi(T_x\omega)$ of $x \in \mathbb{R}^d$ and $\omega \in \Omega$ is called *statistically homogeneous random field*, and, for fixed $\omega \in \Omega$, $\phi(T_x\omega)$ is called the *realization* of the random field ϕ.

Let $L_q(\Omega)$ $(q \geq 1)$ be the space of measurable functions and integrable in the power q with respect to the measure μ. The following assertion holds, see [14] and [13] for the proof.

Proposition 1 *Assume that $\phi \in L_q(\Omega)$. Then almost all realizations $\phi(T_x\omega)$ belong to $L_q^{loc}(\mathbb{R}^d)$. If the sequence $\{\phi_k\} \subset L_q(\Omega)$ converges in $L_q(\Omega)$ to the function ϕ, then there exists a subsequence $\{\phi_{k'}\}$ such that almost all realizations $\phi_{k'}(T_x\omega)$ converge in $L_q^{loc}(\mathbb{R}^d)$ to the realization $\phi(T_x\omega)$.*

Definition 7 A measurable function $\phi(\omega)$ on Ω is called *invariant* if, for any $x \in \mathbb{R}^d$, $\phi(T_x\omega) = \phi(\omega)$ almost surely.

Definition 8 A d-dynamical system T_x is said to be *ergodic* if all its invariant functions are almost surely constant.

Definition 9 Let $\theta \in L_1^{loc}(\mathbb{R}^d)$. We say that the function θ has a spatial average if the limit

$$M(\theta) = \lim_{\varepsilon \to 0} \frac{1}{|B|} \int\limits_B \theta\left(\frac{x}{\varepsilon}\right) \, dx$$

exists for any bounded Borel set $B \in \mathcal{B}$ with $|B| > 0$, and moreover this limit does not depend on the choice of B. The quantity $M(\theta)$ is called the *spatial average* of the function θ.

The following results are proved in [14].

Proposition 2 *Let P be a measurable subset of $\mathbb{R}^d$ containing a neighbourhood of the origin. Let $q \geq 1$ or $q = \infty$. Suppose that a measurable function $\theta(x, \xi)$, $x \in P$, $\xi \in \mathbb{R}^d$, has a space mean value $M(\theta)(x)$ in $\mathbb{R}^d_\xi$ (that is, with respect to the variable ξ) for every $x \in P$ and the family $\{\theta\big(x, \frac{x}{\varepsilon}\big),\ 0 < \varepsilon \leqslant 1\}, x \in \mathcal{K}$, is bounded in $L_q(\mathcal{K})$, where $\mathcal{K}$ is an arbitrary bounded subset in P containing a neighbourhood of the origin.*

Then $M(\theta)(\cdot) \in L_q^{loc}(P)$ and, for $q \geq 1$, we have $\theta\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup M(\theta)(x)$ weakly in $L_q^{loc}(P)$ as $\varepsilon \to 0$,

while, for $q = \infty$, we have $\theta\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup M(\theta)(x)$ $*$-weakly in $L_\infty^{loc}(P)$ as $\varepsilon \to 0$.

From now on we use the following notation $\widehat{x} = (x_1, \dots, x_{d-1})$. For a given group $T_x, x \in \mathbb{R}^d$, we also consider its subgroup $T_{\widehat{x}} : \Omega \to \Omega$, $\widehat{x} = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$, $T_{\widehat{x}} = T_{(\widehat{x},0)}$.

Let T_x be a d-dynamical system in Ω. Wa assume that $T_{\widehat{x}}$ is also a $(d-1)$-dynamical system in Ω (see Definition 5 with the number d replaced with $(d-1)$).

All the above Definitions and Propositions hold true for the $(d-1)$-dynamical system $T_{\widehat{x}}$ with evident modifications. In particular, we have the following

Definition 10 A random field $\zeta(\widehat{x}, \omega)$ ($\widehat{x} \in \mathbb{R}^{d-1}$, $\omega \in \Omega$) is called *statistically homogeneous* if the following representation holds $\zeta(\widehat{x}, \omega) = \zeta(T_{\widehat{x}}\omega)$, where ζ is a random variable on $(\Omega, \mathcal{A}, \mu)$ and $T_{\widehat{x}}$ is a $(d-1)$-dynamical system on Ω.

All along the article, we make use of the Birkhoff ergodic theorem in the following particular form (see, for instance, [14] and [13] for more details).

Theorem 3 (Birkhoff ergodic theorem) *Let T_x, $x \in \mathbb{R}^d$, be a d-dynamical system and let $\psi(\omega) \in L_1(\Omega, \mu)$. Then, for almost all $\omega \in \Omega$, the realization $\psi(T_x\omega)$ has the space mean value $M(\psi(T_x\omega))$ in $\mathbb{R}^d$. Moreover, $M(\psi(T_x\omega))$ is an invariant function and*

$$\mathbb{E}(\psi) \equiv \int\limits_{\Omega} \psi(\omega)\, d\mu = \int\limits_{\Omega} M(\psi(T_x\omega))\, d\mu,$$

where $\mathbb{E}(\phi)$ is the mathematical expectation of ψ. In particular, if T_x is ergodic then, for almost all $\omega \in \Omega$, we have the identity

$$\mathbb{E}(\psi) = M(\psi(T_x\omega)).$$

We shall also apply Birkhoff ergodic theorem to the ergodic $(d-1)$-dynamical system $T_{\widehat{x}}$, $\widehat{x} \in \mathbb{R}^{d-1}$.

We are now ready to make assumptions on the random fields $F(\widehat{\xi}, \omega)$, $p(\widehat{\xi}, \omega)$ and $q(\widehat{\xi}, \omega)$ which we use in the definition of the stochastic geometry and coefficients in the Fourier boundary condition. First, we assume that these random fields are statistically homogeneous, that is

$$F(\widehat{\xi}, \omega) = \mathbf{F}(T_{\widehat{\xi}}\omega), \quad p(\widehat{\xi}, \omega) = \rho(T_{\widehat{\xi}}\omega), \quad q(\widehat{\xi}, \omega) = \varrho(T_{\widehat{\xi}}\omega), \quad \forall \widehat{\xi} \in \mathbb{R}^{d-1},$$

where $\mathbf{F}$, ρ and ϱ are random variables on $(\Omega, \mathcal{A}, \mu)$, and $T_{\widehat{x}}$ is an ergodic $(d-1)$-dynamical system on Ω.

Moreover, we assume that $\mathbf{F}$ has, almost surely, continuously differentiable or locally Lipschitz realizations. We denote

$$\partial^i_\omega \mathbf{F}(\omega) = \partial_{\xi_i} \mathbf{F}(T_{\widehat{\xi}}\omega)\big|_{\widehat{\xi}=0}, \quad \partial_\omega \mathbf{F}(\omega) = \nabla_{\widehat{\xi}} \mathbf{F}(T_{\widehat{\xi}}\omega)\big|_{\widehat{\xi}=0}.$$

We have $\nabla_{\widehat{\xi}} F(\widehat{\xi}, \omega) = \partial_\omega \mathbf{F}(T_{\widehat{\xi}}\omega)$ (see, for instance, [13]). Finally, we make the following assumptions on the functions $\mathbf{F}$, ρ and ϱ:

(h1) $\mathbf{F} \in L_\infty(\Omega)$, $\mathbf{F}(\omega) \le 0$ a.s.;

(h2) $\partial_\omega \mathbf{F} \in (L_2(\Omega))^{d-1}$;

(h3) $\rho \in L_\infty(\Omega)$, $\rho(\omega) \ge 0$ a.s., $\mu\{\omega : \rho(\omega) > 0\} > 0$.

(h4) $\varrho \in L_2(\Omega)$, $\varrho \partial_\omega \mathbf{F} \in (L_2(\Omega))^{d-1}$;

3 Homogenezation of attractors to the reaction-diffusion system in a domain with locally periodic oscillating boundary

3.1 Statement of the problem

Let D be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary $\partial D = \Gamma_1 \cup \Gamma_2$, where D lies in a half-space $x_d > 0$ and $\Gamma_1 \subset \{x : x_d = 0\}$. Given smooth nonpositive 1-periodic in the $\hat{\xi}$ function $F(\hat{x}, \hat{\xi})$, $\hat{x} = (x_1, ..., x_{d-1}), \hat{\xi} = (\xi_1, ..., \xi_{d-1})$, define the domain D_ε as follows: $\partial D_\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2$, where we set $\Gamma_1^\varepsilon = \{x = (\hat{x}, x_d) : (\hat{x}, 0) \in \Gamma_1, x_d = \varepsilon^\alpha F(\hat{x}, \hat{x}/\varepsilon)\}$, $0 < \alpha < 1$, i.e. we add thin oscillating layer $\Pi_\varepsilon = \{x = (\hat{x}, x_d) : (\hat{x}, 0) \in \Gamma_1, x_d \in [0, \varepsilon^\alpha F(\hat{x}, \hat{x}/\varepsilon))\}$ to the domain D. Usually, we assume $F(\hat{x}, \hat{\xi})$ to be compactly supported on Γ_1 uniformly in $\hat{\xi}$. Consider the following boundary-value problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \lambda \Delta u_\varepsilon - a\left(x, \frac{x}{\varepsilon}\right) f(u_\varepsilon) + h\left(x, \frac{x}{\varepsilon}\right), & x \in D_\varepsilon, t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^\beta p(\hat{x}, \frac{\hat{x}}{\varepsilon}) u_\varepsilon = \varepsilon^{1-\alpha} g(\hat{x}, \frac{\hat{x}}{\varepsilon}), & x = (\hat{x}, x_d) \in \Gamma_1^\varepsilon, t > 0, \\ u_\varepsilon = 0, & x \in \Gamma_2, t > 0, \\ u_\varepsilon = U(x), & x \in D_\varepsilon, t = 0, \end{cases} \tag{4}$$

where $u_\varepsilon = u_\varepsilon(x,t) = (u^1, \ldots, u^n)^\top$ is an unknown vector function, the nonlinear function $f = (f^1, \ldots, f^n)^\top$ is given, $h = (h^1, \ldots, h^n)^\top$ is the known right-hand side function, and λ is an $n \times n$-matrix with constant coefficients, having a positive symmetrical part: $\frac{1}{2}(\lambda + \lambda^\top) \geq \varpi I$ (where I is the unit matrix with dimension n). We assume that $\beta > 0, p\left(\hat{x}, \hat{\xi}\right) = \operatorname{diag}\{p^1, \ldots, p^n\}$, $g\left(\hat{x}, \hat{\xi}\right) = (g^1, \ldots, g^n)^\top$ are continuous, 1-periodic in $\hat{\xi}$ and $p^i\left(\hat{x}, \hat{\xi}\right)$, $i = 1, \ldots n$, are positive. Here $\dfrac{\partial u_\varepsilon}{\partial \nu} = \left(\dfrac{\partial u_\varepsilon^1}{\partial \nu}, \ldots, \dfrac{\partial u_\varepsilon^n}{\partial \nu}\right)^\top$ is the normal derivative of the vector function u_ε multiplied by the matrix λ, where $\dfrac{\partial u_\varepsilon^i}{\partial \nu} := \displaystyle\sum_{j=1}^{n} \sum_{k=1}^{d} \lambda_{ij} \frac{\partial u_\varepsilon^j}{\partial x_k} N_k$, $i = 1, \ldots, n$ and $N = (N_1, \ldots, N_d)$ is the unit outer normal to the boundary of the domain.

Function $a(x, \xi) \in C(\overline{D}_\varepsilon \times \mathbb{R}^d)$ such that $0 < a_0 \leq a(x, \xi) \leq A_0$ with some coefficient a_0, A_0. Assuming that function $a_\varepsilon(x) = a\left(x, \frac{x}{\varepsilon}\right)$ has average $\overline{a}(x)$ when $\varepsilon \to 0+$ in space $L_{\infty, *w}(D)$, that is

$$\int_D a\left(x, \frac{x}{\varepsilon}\right) \varphi(x) dx \to \int_D \overline{a}(x) \varphi(x) dx \quad (\varepsilon \to 0+) \tag{5}$$

for any function $\varphi \in L_1(D)$.

Denote by D^+ such a domain that $D_\varepsilon \subset D^+$ for any ε. For the vector function $h\left(\underline{x}, \xi\right)$, assume that for any $\varepsilon > 0$ the function $h_\varepsilon^i(x) = h^i\left(x, \frac{x}{\varepsilon}\right) \in L_2(D^+)$ and has the average $\overline{h^i}(x)$ in the space $L_2(D^+)$ for $\varepsilon \to 0+$, that is

$$h^i\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup \overline{h^i}(x) \quad (\varepsilon \to 0+) \text{ weakly in } L_2(D^+),$$

or

$$\int_{D^+} h^i\left(x, \frac{x}{\varepsilon}\right) \varphi(x) dx \to \int_{D^+} \overline{h^i}(x) \varphi(x) dx \quad (\varepsilon \to 0+) \tag{6}$$

for any function $\varphi \in L_2(D^+)$ and for all $i = 1, \dots, n$.

From the condition (6) it follows that the norm of the function $h^i_\varepsilon(x)$ are bounded uniformly in ε, in the space $L_2(D_\varepsilon)$, i.e. $\|h^i_\varepsilon(x)\|_{L_2(D_\varepsilon)} \leq M_0, \quad \forall \varepsilon \in (0,1]$.

It is assumed that the vector function $f(v) \in C(\mathbb{R}^n; \mathbb{R}^n)$ satisfies the following inequalities

$$\sum_{i=1}^{n} |f^i(v)|^{p_i/(p_i-1)} \leq C_0 \left(\sum_{i=1}^{n} |v^i|^{p_i} + 1 \right), \quad 2 \leq p_1 \leq \ldots \leq p_{n-1} \leq p_n, \tag{7}$$

$$\sum_{i=1}^{n} \gamma_i |v^i|^{p_i} - C \leq \sum_{i=1}^{n} f^i(v) v^i, \quad \forall v \in \mathbb{R}^n, \tag{8}$$

for $\gamma_i > 0$ for any $i = 1, \dots, n$. The inequality (7) is due to the fact that in real reaction-diffusion systems, the functions $f^i(u)$ are polynomials with possibly different degrees. Inequality (8) calls *dissipativity condition* for the reaction-diffusion system (4). In a simple model case $p_i \equiv p$ for any $i = 1, \dots, n$, condition (7) and (8) reduce to the following inequalities

$$|f(v)| \leq C_0 \left(|v|^{p-1} + 1 \right), \quad \gamma |v|^p - C \leq f(v) v, \quad \forall v \in \mathbb{R}^n.$$

Note that the fulfillment of the Lipschitz condition for the function $f(v)$ relative to the variable v *not expected.*

Remark 1 *Using the methods presented, it is also possible to study systems in which nonlinear terms have the form* $\sum_{j=1}^{m} a_j \left(x, \frac{x}{\varepsilon}\right) f_j(u)$, *where* a_j *are matrices whose elements allow averaging and* $f_j(u)$ *polynomial vectors of* u, *which satisfy conditions of the form* (7)–(8). *For brevity, we study the case* $m = 1$ *and* $a_1 \left(x, \frac{x}{\varepsilon}\right) = a \left(x, \frac{x}{\varepsilon}\right) I$, *where* I *is the identity matrix.*

Denote

$$G(\hat{x}) = \int\limits_{[0,1)^{d-1}} \sqrt{|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\, g(\hat{x}, \hat{\xi})\, d\hat{\xi}, \tag{9}$$

$$P(\hat{x}) = \int\limits_{[0,1)^{d-1}} \sqrt{|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\, p(\hat{x}, \hat{\xi})\, d\hat{\xi}. \tag{10}$$

and we have the following convergences (see [3]):

$$\varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon} g^i \left(\hat{x}, \frac{\hat{x}}{\varepsilon} \right) v \left(\hat{x}, \varepsilon^\alpha F \left(\hat{x}, \frac{\hat{x}}{\varepsilon} \right) \right) ds \to \int\limits_{\Gamma_1} G^i \left(\hat{x} \right) v \left(x \right) \, ds$$

and

$$\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon} p^i \left(\hat{x}, \frac{\hat{x}}{\varepsilon} \right) \, u \left(\hat{x}, \varepsilon F \left(\frac{\widehat{x}}{\varepsilon}, \omega \right) \right) \, v \left(\hat{x}, \varepsilon F \left(\frac{\widehat{x}}{\varepsilon}, \omega \right) \right) \, ds \to \int\limits_{\Gamma_1} P^i \left(\hat{x} \right) \, u(x) \, v \left(x \right) \, ds$$

for any $v \in H^1(D_\varepsilon)$ by $\varepsilon \to 0$, $i = 1, \ldots, n$. Here ds is the element of $(d-1)$-dimensional measure on the hypersurface.

Let us introduce the following notation for the spaces $\mathbf{H} := [L_2(D)]^n$, $\mathbf{H}_\varepsilon := [L_2(D_\varepsilon)]^n$, $\mathbf{V} := [H^1(D, \Gamma_2)]^n$, $\mathbf{V}_\varepsilon := [H^1(D_\varepsilon; \Gamma_2)]^n$. Here, $H^1(D, \Gamma_2)$ (respectively $H^1(D_\varepsilon, \Gamma_2)$), denotes the space of functions from the Sobolev space $H^1(D)$ (respectively $H^1(D_\varepsilon)$) with zero trace on Γ_2. The norms in these spaces are determined as follows

$$\|v\|^2 := \int\limits_D \sum_{i=1}^n |v^i(x)|^2 dx, \ \|v\|_\varepsilon^2 := \int\limits_{D_\varepsilon} \sum_{i=1}^n |v^i(x)|^2 dx,$$

$$\|v\|_1^2 := \int\limits_D \sum_{i=1}^n |\nabla v^i(x)|^2 dx, \ \|v\|_{1,\varepsilon}^2 := \int\limits_{D_\varepsilon} \sum_{i=1}^n |\nabla v^i(x)|^2 dx.$$

We denote by $\mathbf{V}'$ the dual space to the space $\mathbf{V}$, and by $\mathbf{V}'_\varepsilon$ the dual space to the space $\mathbf{V}_\varepsilon$.

Let $q_i = p_i/(p_i - 1)$ for any $i = 1, \ldots, n$. We will use the following vector notation $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$, and also define spaces

$$\mathbf{L}_\mathbf{p} := L_{p_1}(D) \times \ldots \times L_{p_n}(D), \ \mathbf{L}_{\mathbf{p},\varepsilon} := L_{p_1}(D_\varepsilon) \times \ldots \times L_{p_n}(D_\varepsilon),$$

$$\mathbf{L}_\mathbf{p}(\mathbb{R}_+; \mathbf{L}_\mathbf{p}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(D)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(D)),$$

$$\mathbf{L}_\mathbf{p}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(D_\varepsilon)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(D_\varepsilon)).$$

As in [17, 26] we study weak solutions of the initial boundary value problem (4), that is, functions

$$u_\varepsilon(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon})$$

which satisfy the equation (4) in the distributional sense (the sense of generalized functions), that is, the integral identity holds

$$-\int\limits_{D_\varepsilon \times \mathbb{R}_+} u_\varepsilon \cdot \frac{\partial \psi}{\partial t} \, dxdt + \int\limits_{D_\varepsilon \times \mathbb{R}_+} \lambda \nabla u_\varepsilon \cdot \nabla \psi \, dxdt + \int\limits_{D_\varepsilon \times \mathbb{R}_+} a_\varepsilon(x) f(u_\varepsilon) \cdot \psi \, dxdt +$$

$$+\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u_\varepsilon \cdot \psi \, dsdt = \int\limits_{D_\varepsilon \times \mathbb{R}_+} h_\varepsilon(x) \cdot \psi \, dxdt + \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot \psi \, dsdt$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V}_\varepsilon \cap \mathbf{L}_{\mathbf{p},\varepsilon})$. Here $y_1 \cdot y_2$ means scalar product of vectors $y_1, y_2 \in \mathbb{R}^n$.

If $u_\varepsilon(x,t) \in \mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_{\mathbf{p},\varepsilon})$, then from the condition (7) it follows that $f(u(x,t)) \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon})$. At the same time, if $u_\varepsilon(x,t) \in \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon)$, then $\lambda \Delta u_\varepsilon(x,t) + h_\varepsilon(x) \in \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon)$. Therefore, for an arbitrary weak solution $u_\varepsilon(x,s)$ to problem (4), satisfies

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon).$$

From the Sobolev embedding theorem follows that $\mathbf{L}_\mathbf{q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon) \subset \mathbf{L}_\mathbf{q}(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}})$, where space $\mathbf{H}_\varepsilon^{-\mathbf{r}} := H^{-r_1}(D_\varepsilon) \times \ldots \times H^{-r_n}(D_\varepsilon)$, $\mathbf{r} = (r_1, \ldots, r_n)$ and indexes

$r_i = \max\{1, d(1/q_i - 1/2)\}$ by $i = 1, \ldots, n$. Here $H^{-r}(D_\varepsilon)$ denotes the space conjugate to the Sobolev space $\overset{\circ}{W}{}_2^r(D_\varepsilon)$ with index $r > 0$ in the domain D_ε.

Therefore, for any weak solution $u_\varepsilon(x,t)$ to problem (4) it's time derivative $\frac{\partial u_\varepsilon(x,t)}{\partial t}$ belongs to $\mathbf{L_q}\left(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}}\right)$.

Remark 2 *Existence of a weak solution $u(x,t)$ to problem (4) for any initial data $U \in \mathbf{H}_\varepsilon$ and fixed ε, can be proved in the standard way (see, for example, [16], [26]). This solution may not be unique, since the function $f(v)$ satisfies only the conditions (7), (8) and it is not assumed that the Lipschitz condition is satisfied with respect to v.*

The following Lemma is proved in a similar way to the proposition XV.3.1 from [17].

Lemma 1 *Let $u_\varepsilon(x,t) \in \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon})$ be a weak solution of problem (4). Then*

(i) $u_\varepsilon \in \mathbf{C}(\mathbb{R}_+; \mathbf{H}_\varepsilon)$;

(ii) *function $\|u_\varepsilon(\cdot,t)\|^2$ is absolutely continuous on $\mathbb{R}_+$, and moreover*

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|^2 + \int\limits_{D_\varepsilon} \lambda \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) dx + \int\limits_{D_\varepsilon} a_\varepsilon(x) f(u_\varepsilon(x,t)) \cdot u_\varepsilon(x,t) dx + \quad (11)$$

$$+\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u_\varepsilon(x,t) \cdot u_\varepsilon(x,t)\, ds = \int\limits_{D_\varepsilon} h_\varepsilon(x) \cdot u_\varepsilon(x,t) dx + \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot u_\varepsilon(x,t)\, ds,$$

for almost all $t \in \mathbb{R}_+$.

To define the trajectory space $\mathcal{K}_\varepsilon^+$ for (4), we use the general approaches of Section 2.1 and for every $[t_1, t_2] \in \mathbb{R}$ we have the Banach spaces

$$\mathcal{F}_{t_1,t_2} := \mathbf{L_p}(t_1, t_2; \mathbf{L_p}) \cap \mathbf{L}_2(t_1, t_2; \mathbf{V}) \cap \mathbf{L}_\infty(t_1, t_2; \mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L_q}\left(t_1, t_2; \mathbf{H}^{-r}\right) \right\}$$

(sometimes we omit the parameter ε for brevity) with the following norm:

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{\mathbf{L_p}(t_1,t_2;\mathbf{L_p})} + \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(0,M;\mathbf{H})} + \left\|\frac{\partial v}{\partial t}\right\|_{\mathbf{L_q}(t_1,t_2;\mathbf{H}^{-r})}.$$

Setting $\mathcal{D}_{t_1,t_2} = \mathbf{L_q}\left(t_1, t_2; \mathbf{H}^{-r}\right)$ we obtain $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and for $u(t) \in \mathcal{F}_{t_1,t_2}$ we have $A(u(t)) \in \mathcal{D}_{t_1,t_2}$. One considers now weak solutions to (4) as solutions of an equation in the general scheme of Section 2.1.

Consider the spaces

$$\mathcal{F}_+^{loc} = \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L_p}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^{loc}(\mathbb{R}_+; \mathbf{H}^{-r}) \right\},$$

$$\mathcal{F}^{loc}_{\varepsilon,+} = \mathbf{L}^{loc}_{\mathbf{p}}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}^{loc}_2(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}^{loc}_\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}^{loc}_{\mathbf{q}}(\mathbb{R}_+;\mathbf{H}_\varepsilon{}^{-r}) \right\}.$$

We introduce the following notation. Let $\mathcal{K}^+_\varepsilon$ be the set of all weak solutions to (4). For any $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathcal{K}^+_\varepsilon$ such that $u(0) = U(x)$. Consequently, the space $\mathcal{K}^+_\varepsilon$ to (4) is not empty and is sufficiently large.

We define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ by means of the norms from $\mathbf{L}_2(t_1,t_2;\mathbf{H})$. We get

$$\rho_{t_1,t_2}(u,v) = \left(\int_{t_1}^{t_2} \|u(t) - v(t)\|^2_{\mathbf{H}} dt \right)^{1/2} \qquad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{t_1,t_2}.$$

The topology Θ^{loc}_+ in $\mathcal{F}^{loc}_+$ is generated by these metrics. Let us recall that $\{v_k\} \subset \mathcal{F}^{loc}_+$ converges to $v \in \mathcal{F}^{loc}_+$ as $k \to \infty$ in Θ^{loc}_+ if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(t_1,t_2;\mathbf{H})} \to 0$ $(k \to \infty)$ for all $[t_1,t_2] \subset \mathbb{R}_+$. The topology Θ^{loc}_+ is metrizable. We consider this topology in the trajectory space $\mathcal{K}^+_\varepsilon$ of (4). Similarly, we define the topology $\Theta^{loc}_{\varepsilon,+}$ in $\mathcal{F}^{loc}_{\varepsilon,+}$.

Denote by $S(\tau)$ the translation semigroup, i.e. $S(\tau)\,u(t) = u(t+\tau)$. The translation semigroup $S(\tau)$ acting on $\mathcal{K}^+_\varepsilon$, is continuous in the topology $\Theta^{loc}_{\varepsilon,+}$. It is easy to see that $\mathcal{K}^+_\varepsilon \subset \mathcal{F}^{loc}_{\varepsilon,+}$ and the space $\mathcal{K}^+_\varepsilon$ is translation invariant, i.e. $S(\tau)\mathcal{K}^+_\varepsilon \subseteq \mathcal{K}^+_\varepsilon$ for all $\tau \geq 0$.

Using the scheme of Section 2.1, one can define bounded sets in the space $\mathcal{K}^+_\varepsilon$ by means of the Banach space $\mathcal{F}^b_{\varepsilon,+}$. We naturally get

$$\mathcal{F}^b_{\varepsilon,+} = \mathbf{L}^b_{\mathbf{p}}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}^b_2(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}^b_{\mathbf{q}}(\mathbb{R}_+;\mathbf{H}^{-r}_\varepsilon) \right\}$$

and the space $\mathcal{F}^b_{\varepsilon,+}$ is a subspace of $\mathcal{F}^{loc}_{\varepsilon,+}$.

Suppose that $\mathcal{K}_\varepsilon$ is the kernel to (4), that consists of all weak complete solutions $u(t), t \in \mathbb{R}$, to our system, bounded in

$$\mathcal{F}^b_\varepsilon = \mathbf{L}^b_{\mathbf{p}}(\mathbb{R};\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}^b_2(\mathbb{R};\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R};\mathbf{H}_\varepsilon) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}^b_{\mathbf{q}}(\mathbb{R};\mathbf{H}^{-r}_\varepsilon) \right\}.$$

In analogous way we define the topology Θ^{loc}_ε in $\mathcal{F}^b_\varepsilon$.

Proposition 3 *Problem* (4) *has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space $\Theta^{loc}_{\varepsilon,+}$. The set $\mathfrak{A}_\varepsilon$ is bounded in $\mathcal{F}^b_{\varepsilon,+}$ and compact in $\Theta^{loc}_{\varepsilon,+}$. Moreover, $\mathfrak{A}_\varepsilon = \Pi_+\mathcal{K}_\varepsilon$, the kernel $\mathcal{K}_\varepsilon$ is non-empty and bounded in $\mathcal{F}^b_\varepsilon$ and compact in Θ^{loc}_ε.*

To prove this proposition we use the approach of the proof from [17]. To prove the existence of an absorbing set (bounded in $\mathcal{F}^b_{\varepsilon,+}$ and compact in $\Theta^{loc}_{\varepsilon,+}$) one can use Lemma 1 similar to [17].

3.2 Homogenized reaction-diffusion system and convergence of attractors in the critical case ($\beta = 1 - \alpha$)

Now we study the behaviour of the problem (4) as $\varepsilon \to 0$ in the critical case $\beta = 1 - \alpha$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{h}\,(x)\,, & x \in D, t > 0, \\ \frac{\partial u_0}{\partial \nu} + P(\hat{x}) u_0 = G(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1, t > 0, \\ u_0 = 0, & x \in \Gamma_2, t > 0, \\ u_0 = U(x), & x \in D, t = 0, \end{cases} \tag{12}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (5) and (6), respectively, $G(\hat{x})$ and $P(\hat{x})$ were defined in (9) and (10).

As before, we consider weak solutions of the problem (12), that is, functions

$$u_0(x,t) \in \mathbf{L}_{\infty}^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}^{loc}\,(\mathbb{R}_+; \mathbf{L}_{\mathbf{p}})\,,$$

which satisfy the following integral identity:

$$\begin{gathered} -\int\limits_{D\times\mathbb{R}_+} u_0 \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{D\times\mathbb{R}_+} \lambda \nabla u_0 \cdot \nabla \psi\, dxdt + \int\limits_{D\times\mathbb{R}_+} \bar{a}(x) f(u_0) \cdot \psi\, dxdt + \\ + \int\limits_{\Gamma_1\times\mathbb{R}_+} P\,(\hat{x})\, u_0 \cdot \psi\, dsdt = \int\limits_{D\times\mathbb{R}_+} \overline{h}(x) \cdot \psi\, dxdt + \int\limits_{\Gamma_1\times\mathbb{R}_+} G\,(\hat{x}) \cdot \psi\, dsdt \end{gathered} \tag{13}$$

for any function $\psi \in \mathbf{C}_0^{\infty}(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L}_{\mathbf{p}})$. For any weak solution $u(x,t)$ to problem (12), we have that $\frac{\partial u_0(x,t)}{\partial t} \in \mathbf{L}_{\mathbf{q}}\,(0, M; \mathbf{H}^{-\mathbf{r}})$ (see Section 3.1). Recall, that the "limit" domain D in (12) and (13) is independent of ε and it boundary contains the plain part Γ_1.

Similar to (4), for any initial data $U \in \mathbf{H}$, the problem (12) has at least one weak solution (see Remark 2). Lemma 1 also holds true for the problem (12) with replacing the ε-depending coefficients a, h, p and g by the corresponding averaged coefficients $\overline{a}(x), \overline{h}(x), P(\hat{x})$, and $G(\hat{x})$.

As usual, let $\overline{\mathcal{K}}^+$ be the the trajectory space for (12) (the set of all weak solutions), that belong to the corresponding spaces $\mathcal{F}_+^{loc}$ and $\mathcal{F}_+^{b}$ (see Section 2.1). Recall that $\overline{\mathcal{K}}^+ \subset \mathcal{F}_+^{loc}$ and the space $\overline{\mathcal{K}}^+$ is translation invariant with respect to translation semigroup $\{S(\tau)\}$, that is, $S(\tau)\overline{\mathcal{K}}^+ \subseteq \overline{\mathcal{K}}^+$ for all $\tau \geq 0$. We now construct the trajectory attractor in the topology Θ_+^{loc} for the problem (12) (see Sections 3.1 and 2.1).

Similar to Proposition 3 we have

Proposition 4 *Homogenized problem has the trajectory attractor $\overline{\mathfrak{A}}$ in the topological space Θ_+^{loc}. The set $\overline{\mathfrak{A}}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover, $\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}}$, the kernel $\overline{\mathcal{K}}$ of the homogenized problem is non-empty and bounded in $\mathcal{F}^b$.*

Here we formulate the main result concerning the limit behaviour of the trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (4) as $\varepsilon \to 0$ in the critical case $\beta = 1 - \alpha$.

Theorem 4 *The following limit holds in the topological space Θ_+^{loc}*

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \quad \text{as } \varepsilon \to 0+\,.$$

Moreover,

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \text{ as } \varepsilon \to 0+ \ \text{ in } \ \Theta^{loc}.$$

Finally, we consider the reaction–diffusion systems for which the uniqueness theorem is true for the Cauchy problem. It suffices to assume that the nonlinear term $f(u)$ in (4) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \geq -C|v_1 - v_2|^2 \ \text{ for any } v_1, v_2 \in \mathbb{R}^n. \tag{14}$$

(see [17, 26]). In [26] it was proved that if (14) is true, then (4) and (12) generate dynamical semigroups in $\mathbf{H}$, possessing global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ are bounded in $\mathbf{V}$ (see also [16], [15]). Moreover

$$\mathcal{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\}, \ \overline{\mathcal{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

Corollary 1 *Under the assumption of Theorem* 4 *the limit formula takes place*

$$\operatorname{dist}_{\mathbf{H}^{-\delta}}\left(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\right) \to 0 \ (\varepsilon \to 0+).$$

3.3 Homogenized reaction-diffusion system and convergence of attractors in the subcritical case ($\beta > 1 - \alpha$)

In the next sections, we study the behaviour of the problem (4) as $\varepsilon \to 0$ in the subcritical case $\beta > 1 - \alpha$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}(x) f(u_0) + \overline{h}(x), & x \in D, t > 0, \\ \frac{\partial u_0}{\partial \nu} = G(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1, t > 0, \\ u_0 = 0, & x \in \Gamma_2, t > 0, \\ u_0 = U(x), & x \in D, t = 0, \end{cases} \tag{15}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (5) and (6), respectively, $G(\hat{x})$ was defined in (9).

As before, we consider weak solutions of the problem (15), that is, functions

$$u(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_\mathbf{p}),$$

which satisfy the following integral identity:

$$\begin{aligned} -\int\limits_{D \times \mathbb{R}_+} u \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{D \times \mathbb{R}_+} \lambda \nabla u \cdot \nabla \psi \, dxdt + \int\limits_{D \times \mathbb{R}_+} \bar{a}(x) f(u) \cdot \psi \, dxdt = \\ = \int\limits_{D \times \mathbb{R}_+} \bar{h}(x) \cdot \psi \, dxdt + \int\limits_{\Gamma_1 \times \mathbb{R}_+} G(\hat{x}) \cdot \psi \, dsdt \end{aligned} \tag{16}$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L}_\mathbf{p})$. For any weak solution $u(x,t)$ to problem (15), we have that $\frac{\partial u(x,t)}{\partial t} \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{H}^{-\mathbf{r}})$ (see Section 3.1). Recall, that the "limit" domain D in (15) and (16) is independent of ε and it boundary contains the plain part Γ_1.

For homogenized problem (15) holds Proposition 4.

For trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (4) as $\varepsilon \to 0$ in the subcritical case $\beta > 1 - \alpha$ holds Theorem 4 and Corollary 1.

3.4 Homogenized reaction-diffusion system and convergence of attractors in the supercritical case ($\beta < 1-\alpha$)

In the next sections, we study the behaviour of the problem (4) as $\varepsilon \to 0$ in the supercritical case $\beta < 1-\alpha$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{h}\,(x)\,, & x \in D, t > 0, \\ u_0 = 0, & x \in \partial D, t > 0, \\ u_0 = U(x), & x \in D, t = 0, \end{cases} \tag{17}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (5) and (6), respectively.

We note that, in the supercritical case, the influence of the boundary layer on the part of the boundary Γ_1 completely disappears (compare with critical case [44] and subcritical case mentioned in Subsection 3.3).

As before, we consider weak solutions of the problem (17), that is, functions

$$u_0(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_\mathbf{p})\,,$$

which satisfy the following integral identity:

$$-\int\limits_{D\times\mathbb{R}_+} u_0 \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{D\times\mathbb{R}_+} \lambda \nabla u_0 \cdot \nabla \psi\, dxdt + \int\limits_{D\times\mathbb{R}_+} \bar{a}(x) f(u_0) \cdot \psi\, dxdt = \int\limits_{D\times\mathbb{R}_+} \bar{h}(x) \cdot \psi\, dxdt \tag{18}$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L}_\mathbf{p})$. For any weak solution $u(x,t)$ to problem (17), we have that $\frac{\partial u_0(x,t)}{\partial t} \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{H}^{-\mathbf{r}})$ (see Section 3.1). Recall, that the "limit" domain D in (17) and (18) is independent of ε and it boundary contains the plain part Γ_1.

For homogenized problem (17) holds Proposition 4.

For trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (4) as $\varepsilon \to 0$ in the supercritical case $\beta < 1-\alpha$ holds Theorem 4 and Corollary 1.

4 Homogenezation of attractors to the reaction-diffusion system in a domain with randomly oscillating boundary

4.1 Statement of the problem

Let $D \subset \mathbb{R}^d \cap \{x \,|\, x_d > 0\}$, $d \geq 2$, be a smooth bounded domain whose boundary has a nontrivial flat part $\Gamma_1 = \partial D \cap \{x \,|\, x_d = 0\}$ with a nonempty $(d-1)$-dimensional interior $\overset{\circ}{\Gamma}_1$. We perturb the flat part of the boundary in such a way that the perturbed domain has an oscillating boundary. To this end, we define a smooth nonnegative function $g(\widehat{x})$, $\widehat{x} = (x_1, \ldots, x_{d-1})$, such that $\operatorname{supp} g(\widehat{x}) \subset \Gamma_0 \Subset \overset{\circ}{\Gamma}_1$, and, given a statistically homogeneous non-positive random function $F(\widehat{\xi}, \omega)$, $\widehat{\xi} = (\xi_1, \ldots, \xi_{d-1})$, which has smooth realizations and is defined on a standard probability space $(\Omega, \mathcal{A}, \mu)$, we set, for $\varepsilon > 0$,

$$\Pi_\varepsilon = \{x \in \mathbb{R}^d \;:\; \widehat{x} \in \Gamma_1,\; \varepsilon g(\widehat{x}) F\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) < x_d \leq 0\}$$

and, finally, introduce the desired domain with random boundary as follows: $D_\varepsilon = D \cup \Pi_\varepsilon$. For more detailed definitions of randomness we refer to the next section. According to the above construction, the boundary ∂D_ε consists of the parts Γ_2 and $\Gamma_1^\varepsilon = \Big\{ x \in \partial D_\varepsilon \,:\, (\widehat{x}, 0) \in \Gamma_1,\, x_d = \varepsilon g(\widehat{x}) F\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) \Big\}$ forming together the domain boundary.

We consider the boundary-value problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \lambda \Delta u_\varepsilon - a\left(x, \frac{x}{\varepsilon}, \omega\right) f(u_\varepsilon) + r(x, \frac{x}{\varepsilon}, \omega), & x \in D_\varepsilon, t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} + g(\hat{x}) p\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) u_\varepsilon = g(\hat{x}) q\left(\frac{\widehat{x}}{\varepsilon}, \omega\right), & x = (\hat{x}, x_d) \in \Gamma_1^\varepsilon, t > 0, \\ u_\varepsilon = 0, & x \in \Gamma_2, t > 0, \\ u_\varepsilon = U(x), & x \in D_\varepsilon, t = 0, \end{cases} \tag{19}$$

where $u_\varepsilon = u_\varepsilon(x,t) = (u^1, \dots, u^n)^\top$ is an unknown vector function, the nonlinear function $f = (f^1, \dots, f^n)^\top$ is given, $r = (r^1, \dots, r^n)^\top$ is the known right-hand side function, and λ is an $n \times n$-matrix with constant coefficients, having a positive symmetrical part: $\frac{1}{2}(\lambda + \lambda^\top) \geq \varpi I$ (where I is the unit matrix with dimension n and $\varpi > 0$). We assume that $p\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) = \operatorname{diag}\{p^1, \dots, p^n\}$, $q\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) = (q^1, \dots, q^n)^\top$ are random statistically homogeneous functions and $p^i\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)$, $i = 1, \dots n$, are positive.

We assume that the random functions $a_\varepsilon(x, \omega) = a\Big(x, \frac{x}{\varepsilon}, \omega\Big)$ and $r_\varepsilon(x, \omega) = r\Big(x, \frac{x}{\varepsilon}, \omega\Big)$ are statistically homogeneous, that is $a(x, \xi, \omega) = \mathbf{A}(x, T_\xi \omega)$, $\quad r(x, \xi, \omega) = \mathbf{R}(x, T_\xi \omega)$, where $\mathbf{A} : D \times \Omega \to \mathbb{R}$ and $\mathbf{R} : D \times \Omega \to \mathbb{R}^n$ are measurable.

We also assume that $\mathbf{A}(x, \omega) \in C_b(\overline{D})$ for almost all $\omega \in \Omega$ and $0 < \alpha_0 \leq \mathbf{A}(x, \omega) \leq \alpha_1$, $\quad |\mathbf{R}(x, \omega)| \leq \phi(x)$, $\quad \forall x \in D$, where $\phi(x)$ is a positive function such that $\phi \in L_2(D)$.

Birkhoff ergodic theorem implies that the functions $a(x, \xi, \omega)$ and $r(x, \xi, \omega)$ have the space mean value

$$\overline{a}(x) := M(a)(x) = \mathbb{E}(\mathbf{A})(x), \qquad \overline{r}(x) := M(r)(x) = \mathbb{E}(\mathbf{R})(x)$$

for every $x \in D$. Note that the functions $\overline{a}(x)$ and $\overline{r}(x)$ also satisfy the inequality $\alpha_0 \leq \overline{a}(x) \leq \alpha_1$, $\quad |\overline{r}(x)| \leq \phi(x)$, $\quad \forall x \in D$. It follows from Proposition 2, that almost surely in $\omega \in \Omega$

$$\int\limits_D a_\varepsilon(x, \omega) \varphi(x) dx \to \int\limits_D \overline{a}(x) \varphi(x) dx \ (\varepsilon \to 0+) \quad \forall \varphi \in L_1(D), \tag{20}$$

$$\int\limits_{D^+} r_\varepsilon^i(x, \omega) \varphi(x) dx \to \int\limits_{D^+} \overline{r}^i(x) \varphi(x) dx \ (\varepsilon \to 0+) \quad \forall \varphi \in L_2(D^+),\ i = 1, \dots, n. \tag{21}$$

Here D^+ is such a domain that $D_\varepsilon \subset D^+$ for any ε.

We assume that the vector function $f(v) \in C(\mathbb{R}^n; \mathbb{R}^n)$ satisfies inequalities (7) and (8).

From (21) it follows that the norms of $r_\varepsilon^i(x, \omega)$ are almost surely uniformly bounded $\|r_\varepsilon^i\|_{L_2(D)} \leq M_0$, $\quad \forall \varepsilon \in (0, 1]$ in the space $L_2(D)$.

Denote

$$P(\widehat{x}) = \mathbb{E}\left(\rho(\omega)\ \sqrt{1 + (g(\widehat{x})\, \partial_\omega \mathbf{F}(\omega))^2}\right), \qquad Q(\widehat{x}) = \mathbb{E}\left(\varrho(\omega)\ \sqrt{1 + (g(\widehat{x})\, \partial_\omega \mathbf{F}(\omega))^2}\right).$$

(22)

and, due to Birkhoff ergodic theorem and Proposition 2, we have almost surely the following convergence (see [40]):

$$\int\limits_{\Gamma_1^\varepsilon} g(\widehat{x})q^i\left(\frac{\hat{x}}{\varepsilon},\omega\right)\,\upsilon\left(\hat{x},\varepsilon g(\widehat{x})F\left(\frac{\widehat{x}}{\varepsilon},\omega\right)\right)\,ds \to \int\limits_{\Gamma_1} g(\widehat{x})Q^i\left(\hat{x}\right)\,\upsilon\left(x\right)\,ds$$

and

$$\int\limits_{\Gamma_1^\varepsilon} g(\widehat{x})p^i\left(\frac{\hat{x}}{\varepsilon},\omega\right)\,u\left(\hat{x},\varepsilon g(\widehat{x})F\left(\frac{\widehat{x}}{\varepsilon},\omega\right)\right)\,\upsilon\left(\hat{x},\varepsilon g(\widehat{x})F\left(\frac{\widehat{x}}{\varepsilon},\omega\right)\right)\,ds \to \int\limits_{\Gamma_1} g(\widehat{x})P^i\left(\hat{x}\right)\,u(x)\,\upsilon\left(x\right)\,ds$$

for any $u,\upsilon \in H^1(D_\varepsilon)$ as $\varepsilon \to 0$, $i = 1,\ldots,n$. Here ds is the element of $(d-1)$-dimensional measure on the hypersurface.

As in [17,26] we study weak solutions of the initial boundary value problem (19), that is, functions

$$u_\varepsilon(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_{\mathbf{p}}^{loc}\left(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}\right)$$

which satisfy the equation (19) in the distributional sense (the sense of generalized functions), that is, the integral identity holds

$$-\int\limits_{D^\varepsilon\times\mathbb{R}_+} u_\varepsilon\cdot\frac{\partial\psi}{\partial t}\,dxdt + \int\limits_{D^\varepsilon\times\mathbb{R}_+} \lambda\nabla u_\varepsilon\cdot\nabla\psi\,dxdt + \int\limits_{D^\varepsilon\times\mathbb{R}_+} a_\varepsilon(x,\omega)f(u_\varepsilon)\cdot\psi\,dxdt +$$

$$+\int\limits_{\Gamma_1^\varepsilon\times\mathbb{R}_+} g(\hat{x})p\left(\frac{\hat{x}}{\varepsilon},\omega\right)u_\varepsilon\cdot\psi\,dsdt = \int\limits_{D^\varepsilon\times\mathbb{R}_+} r_\varepsilon(x,\omega)\cdot\psi\,dxdt + \int\limits_{\Gamma_1^\varepsilon\times\mathbb{R}_+} g(\hat{x})q\left(\frac{\hat{x}}{\varepsilon},\omega\right)\cdot\psi\,dsdt$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{V}_\varepsilon \cap \mathbf{L}_{\mathbf{p},\varepsilon})$. Here $y_1\cdot y_2$ means scalar product of vectors $y_1, y_2 \in \mathbb{R}^n$.

For any weak solution $u_\varepsilon(x,t)$ to problem (19) the time derivative $\frac{\partial u_\varepsilon(x,t)}{\partial t} \in \mathbf{L}_{\mathbf{q}}\left(0,M;\mathbf{H}_\varepsilon^{-\mathbf{r}}\right)$ (see Section 3.1).

Remark 3 *Existence of a weak solution $u(x,t)$ to problem* (19) *for any initial data $U \in \mathbf{H}_\varepsilon$ and fixed ε, can be proved in the standard way (see, for example,* [16], [26]*). This solution may not be unique, since the function $f(v)$ satisfies only the conditions* (8) *and it is not assumed that the Lipschitz condition is satisfied with respect to v.*

Proposition 5 *Under the hypotheses* (7) *and* (8) *the system* (19) *has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space $\Theta_{\varepsilon,+}^{loc}$. The set $\mathfrak{A}_\varepsilon$ is ω-almost surely bounded in $\mathcal{F}_{\varepsilon,+}^b$ and compact in $\Theta_{\varepsilon,+}^{loc}$. Moreover, $\mathfrak{A}_\varepsilon = \Pi_+\mathcal{K}_\varepsilon$, the kernel $\mathcal{K}_\varepsilon$ is non-empty, bounded in $\mathcal{F}_\varepsilon^b$ and compact in Θ_ε^{loc}.*

4.2 Homogenized reaction-diffusion system and convergence of attractors

In the next sections, we study the behaviour of the problem (19) as $\varepsilon \to 0$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{r}\,(x)\,, & x \in D, t > 0, \\ \frac{\partial u_0}{\partial \nu} + g(\hat{x}) P(\hat{x}) u_0 = g(\hat{x}) Q(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1, t > 0, \\ u_0 = 0, & x \in \Gamma_2, t > 0, \\ u_0 = U(x), & x \in D, t = 0, \end{cases} \tag{23}$$

Here $\overline{a}(x)$ and $\overline{r}(x)$ were defined in (20) and (21), respectively, $Q(\hat{x})$ and $P(\hat{x})$ were defined in (22).

As before, we consider weak solutions of the problem (23), that is, functions

$$u_0(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{loc}\left(\mathbb{R}_+; \mathbf{L}_\mathbf{p}\right),$$

which satisfy the following integral identity:

$$\begin{aligned} & - \int\limits_{D \times \mathbb{R}_+} u_0 \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{D \times \mathbb{R}_+} \lambda \nabla u_0 \cdot \nabla \psi\, dxdt + \int\limits_{D \times \mathbb{R}_+} \bar{a}(x) f(u_0) \cdot \psi\, dxdt + \\ & + \int\limits_{\Gamma_1 \times \mathbb{R}_+} g(\hat{x}) P\left(\hat{x}\right) u_0 \cdot \psi\, dsdt = \int\limits_{D \times \mathbb{R}_+} \bar{r}(x) \cdot \psi\, dxdt + \int\limits_{\Gamma_1 \times \mathbb{R}_+} g(\hat{x}) Q\left(\hat{x}\right) \cdot \psi\, dsdt \end{aligned} \tag{24}$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L}_\mathbf{p})$. For any weak solution $u(x,t)$ to problem (23), we have that $\frac{\partial u_0(x,t)}{\partial t} \in \mathbf{L}_\mathbf{q}\left(0, M; \mathbf{H}^{-\mathbf{r}}\right)$ (see Section 4.1). Recall, that the "limit" domain D in (23) and (24) is independent of ε and it boundary contains the plain part Γ_1.

For homogenized problem (23) holds Proposition 4.

Under assumptions (**h1**)–(**h4**), for trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (19) as $\varepsilon \to 0$, ω-almost surely holds Theorem 4 and Corollary 1.

5 Conclusion

In the paper we consider reaction–diffusion systems with rapidly oscillating terms in equations and in boundary conditions in domains with locally periodic or randomly oscillating boundary (rough surface) depending on a small parameter. We define the trajectory attractors of these systems and express that they converge (almost surely) in a weak sense to the trajectory attractors of the limit (homogenized) reaction–diffusion systems in domain independent of the small parameter.

Acknowledgment

The first part of this research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No.AP26199535), the second part of the research is supported by the Russian Scientific Foundation (grant number 25-11-00133) .

References

[1] Marchenko V.A., Khruslov E.Ya. Homogenization of partial differential equations, Boston (MA): Birkhäuser, 2006.

[2] Belyaev A.G., Piatnitski A.L., Chechkin G.A. Asymptotic Behavior of Solution for Boundary-Value Problem in a Perforated Domain with Oscillating Boundary, 1998; 39(4): 730–754. doi: 10.1007/BF02673049

[3] Chechkin G.A., Friedman A., Piatnitski A.L. The Boundary Value Problem in Domains with Very Rapidly Oscillating Boundary, 1999; 231(1): 213–234. doi: 10.1006/jmaa.1998.6226

[4] Chechkin G.A., Mel'nyk T.A. Homogenization of a Boundary-Value Problem in a Thick 3-Dimensional Multilevel Junction, 2009; 200(3): 357–383.

[5] Chechkin G.A., McMillan A., Jones R., Peng D. A computational study of the influence of surface roughness on material strength, 2018; 53(9): 2411–2436.

[6] Gaudiello A., Sili A. Homogenization of Highly Oscillating Boundaries with Strongly Contrasting Diffusivity, 2015; 47(3): 1671–1692.

[7] Amirat Y., Chechkin G.A., Gadyl'shin R.R. Asymptotics of Simple Eigenvalues and Eigenfunctions for the Laplace Operator in a Domain with Oscillating Boundary, 2006; 46(1): 97–110.

[8] Amirat Y., Chechkin G.A., Gadyl'shin R.R. Asymptotics for Eigenelements of Laplacian in Domain with Oscillating Boundary: Multiple Eigenvalues 2007; 86(7): 873–897.

[9] Amirat Y., Chechkin G.A., Gadyl'shin R.R. Asymptotics of the Solution of a Dirichlet Spectral Problem in a Junction with Highly Oscillating Boundary, 2008; 336(9): 693–698.

[10] Amirat Y., Chechkin G.A., Gadyl'shin R.R. Spectral Boundary Homogenization in Domains with Oscillating Boundaries 2010; 11(6): 4492–4499.

[11] Sanchez-Palencia E. Homogenization Techniques for Composite Media. Berlin: Springer–Verlag, 1987.

[12] Oleinik O.A., Shamaev A.S., Yosifian G.A. Mathematical Problems in Elasticity and Homogenization. Amsterdam: North–Holland, 1992.

[13] Jikov V.V., Kozlov S.M., Oleinik O.S. Homogenization of Differential Operators and Integral Functionals. Berlin: Springer–Verlag, 1994.

[14] Chechkin G.A., Piatnitski A.L., Shamaev A.S. Homogenization: Methods and Applications. Providence (RI): Am. Math. Soc. 2007.

[15] Temam R. Infinite-dimensional dynamical systems in mechanics and physics. New York (NY): Springer-Verlag 1998. doi: 10.1007/978-1-4684-0313-8

[16] Babin A.V., Vishik M.I. Attractors of evolution equations. Amsterdam: North–Holland 1992.

[17] Chepyzhov V.V., Vishik M.I. Attractors for equations of mathematical physics. Providence (RI): Amer. Math. Soc. 2002.

[18] Hale J.K., Verduyn Lunel S.M. Averaging in infinite dimensions. J. Integral Equations Applications, 1990; 2(4): 463–494. doi: 10.1216/jiea/1181075583

[19] Ilyin A.A. Averaging principle for dissipative dynamical systems with rapidly oscillating right-hand sides. Sb. Math., 1996; 187(5): 635–677. doi: 10.1070/SM1996v187n05ABEH000126

[20] Efendiev M., Zelik S. Attractors of the reaction-diffusion systems with rapidly oscillating coefficients and their homogenization. Ann. Inst. H. Poincaré Anal. Non Linéaire, 2002; 19(6): 961–989. doi: 10.1016/S0294-1449(02)00115-4

[21] Bekmaganbetov K.A., Chechkin G.A., Chepyzhov V.V. Homogenization of random attractors for reaction-diffusion systems. CR Mecanique, 2016; 344(11-12): 753–758. doi: 10.1016/j.crme.2016.10.015

[22] Bekmaganbetov K.A., Chechkin G.A., Chepyzhov V.V., Goritsky A.Yu. Homogenization of trajectory attractors of 3D Navier-Stokes system with randomly oscillating force. Discrete Contin. Dyn. Syst., 2017; 37(5): 2375–2393. doi: 10.3934/dcds.2017103

[23] Chechkin G.A., Chepyzhov V.V., Pankratov L.S. Homogenization of Trajectory Attractors of Ginzburg–Landau equations with Randomly Oscillating Terms. Discrete and Continuous Dynamical Systems. Series B, 2018; 23(3): 1133–1154. doi: 10.3934/dcdsb.2018058

[24] Bekmaganbetov K.A., Chechkin G.A., Chepyzhov V.V. Weak Convergence of Attractors of Reaction–Diffusion Systems with Randomly Oscillating Coefficients. Applicable Analysis 2019; 98(1-2): 256–271. doi: 10.1080/00036811.2017.1400538

[25] Boyer F., Fabrie P. Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models. Applied Mathematical Sciences. New York (NY): Springer, 2013.

[26] Chepizhov V.V.,. Vishik M.I. Trajectory attractors for reaction-diffusion systems. Top.Meth.Nonlin.Anal. J. Julius Schauder Center 1996; 7(1): 49–76.

[27] Maz'ya V.G. The S.L. Sobolev's Spaces. Leningrad: Leningrad State University Press, 1984.

[28] Chechkin G.A., Koroleva Yu.O.,Persson L.-E. On the Precise Asymptotics of the Constant in the Friedrich's Inequality for Functions, Vanishing on the Part of the Boundary with Microinhomogeneous Structure. Journal of Inequalities and Applications 2007.

[29] Chechkin G.A., Koroleva Yu.O., Meidell A., Persson L.-E. On the Friedrichs inequality in a domain perforated along the boundary. Homogenization procedure. Asymptotics in parabolic problems, Russian Journal of Mathematical Physics 2009; 16(1): 1–16.

[30] Mikhailov V.P. Partial differential equations, Moscow: Mir, 1978.

[31] Belyaev A.G., Piatnitski A.L., Chechkin G.A. Averaging in a Perforated Domain with an Oscillating Third Boundary Conditionю Sb. Math. 2001; 192(7): 933–949. doi: 10.4213/sm576

[32] Chechkin G.A., Piatnitski A.L. Homogenization of Boundary-Value Problem in a Locally Periodic Perforated Domain. Applicable Analysis 1998; 71(1): 215–235. doi: 10.1080/000368 19908840714

[33] Lions J.-L. Quelques méthodes de résolutions des problèmes aux limites non linéaires, Paris: Dunod, Gauthier-Villars, 1969.

[34] Ladyzhenskaya O.A. Boundary - Value Problems of Mathematical Physics, Moskow, 1973.

[35] Maz'ya V.G. Classes of spaces, measures, and capacities in the theory of spaces of differentiable functions, in Modern problems of Mathematics. Fundamental Investigations (Itogi Nauki i Techniki VINITI AN SSSR, Moskow: Nauka, 1987; 26.

[36] Lax P.D., Milgram A. Parabolic equations, in Contributions to the Theory of Partial Differential Equations. Ann. Math. Studies. Princeton: Princeton University Press 1954; 33: 167–190.

[37] Chechkin G.A., Chechkina T.P., D'Apice C., De Maio U., Mel'nyk T.A. Homogenization of 3D Thick Cascade Junction with the Random Transmission Zone Periodic in One direction. Russian Journal of Mathematical Physics 2010; 17(1): 35–55.

[38] Chechkin G.A., Chechkina T.P., D'Apice C., De Maio U., Mel'nyk T.A. Asymptotic Analysis of a Boundary Value Problem in a Cascade Thick Junction with a Random Transmission Zone. Applicable Analysis 2009; 88(10–11): 1543–1562.

[39] Chechkin G.A., Chechkina T.P., Ratiu T.S., Romanov M.S. Nematodynamics and Random Homogenization. Applicable Analysis 2016; 95(10): 2243–2253. doi: 10.1080/00036811.2015.1036241

[40] Amirat Y., Bodart O., Chechkin G.A.,Piatnitski A.L. Boundary homogenization in domains with randomly oscillating boundary. Stochastic Processes and their Applications 2011; 121(1): 1–23.

[41] Yosida K. Functional Analysis. Springer-Verlag, Berlin, 1995.

[42] Belyaev A.G. On Singular perturbations of boundary-value problems (Russian). Moscow State University,PhD Thesis, 1990.

[43] Sobolev S.L. Some applications of functional analysis in mathematical physics. Third Edition. Translations of Mathematical Monographs Serie. Providence, Rhode Island: AMS Press, 1991; 90.

[44] Azhmoldaev G.F., Bekmaganbetov K.A., Chechkin G.A., Chepyzhov V.V. Homogenization of attractors to reaction–diffusion equations in domains with rapidly oscillating boundary: Critical case. Networks and Heterogeneous Media, 2024; 19(3): 1381–1401.

Автор туралы мәлімет:

Ажмолдаев Газиз Файзуллаевич (корреспондент автор) – Л.Н.Гумилев атындағы Еуразия ұлттық университетінің докторанты (Астана, Қазақстан, электрондық пошта: azhmoldaevgf@gmail.com);

Бекмаганбетов Куаныш Абдрахманович – физика-математика ғылымдарының докторы, М.В. Ломоносов атындағы Мәскеу мемлекеттік университетінің Қазақстандағы филиалының профессоры (Астана, Қазақстан, электрондық пошта: bektaganbetov-ka@yandex.kz);

Чечкин Григорий Александрович – физика-математика ғылымдарының докторы, М.В. Ломоносов атындағы Мәскеу мемлекеттік университетінің профессоры (Мәскеу, Ресей, электрондық пошта: chechkin@mech.math.msu.su);

Чепыжов Владимир Викторович – физика-математика ғылымдарының докторы, А.А. Харкевич атындағы ақпаратты жіберу мәселелері институтының бас ғылым қызметкері (Мәскеу, Ресей, электрондық пошта: chep@iitp.ru).

Сведения об авторе:

Ажмолдаев Газиз Файзуллаевич (корреспондент автор)– PhD докторант Евразийского национального университета имени Л.Н. Гумилёва (Астана, Казахстан, электронная почта: azhmoldaevgf@gmail.com);

Бекмаганбетов Куаныш Абдрахманович – доктор физико-математиечких наук, профессор Казахстанского филиала МГУ имени М.В. Ломоносова (Астана, Казахстан, электронная почта: bektaganbetov-ka@yandex.kz);

Чечкин Григорий Александрович – доктор физико-математиечких наук, профессор Московского государственного университета имени М.В. Ломоносова (Москва, Россия, электронная почта: chechkin@mech.math.msu.su);

Чепыжов Владимир Викторович – доктор физико-математиечких наук, главный научный сотрудник Института проблем передачи информации им. А. А. Харкевича (Москва, Россия, электронная почта: chep@iitp.ru).

Information about author:

Gaziz Faizullaevich Azhmoldaev (corresponding author) – PHD student of L.N. Gumilev Eurasian National University (Astana, Kazakhstan, e-mail: azhmoldaevgf@gmail.com);

Kuanysh Abdrakhmanovich Bektaganbetov – Doctor of physical and mathematical sciences, Associate professor of M.V. Lomonosov Moscow State University, Kazakhstan Branch (Astana, Kazakhstan, e-mail: bektaganbetov-ka@yandex.kz);

Gregory Alexandrovich Chechkin – Doctor of physical and mathematical sciences, Full Professor of M.V. Lomonosov Moscow State University (Moscow, Russia, e-mail: chechkin@mech.math.msu.su);

Vladimir Victorovich Chepyzhov – Doctor of physical and mathematical sciences, leading researcher of Institute for Information Transmission Problems, Russian Academy of Sciences (Moscow, Russia, e-mail: chep@iitp.ru).

Received: April 16, 2025
Accepted: June 8, 2025