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SYMMETRY EQUIVALENCES OF BOUNDARY VALUE PROBLEMS FOR THE NON-UNIFORM BEAMS

In this paper, the models of Euler–Bernoulli non-uniform beams with the axial loads on the Winkler foundations are considered. The non-uniform beam in the model is described by three variable parameters/coefficients: bending stiffness, foundation and beam mass per unit length. The key finding of this study is the clear demonstration of how the agreed symmetry of variable parameters affects the spectral properties of a problem. The qualitative results for the symmetric equivalence (factorisation of sets of eigenvalues and eigenfunctions) of eigenvalues of non-uniform beams for two types of fixing at the ends (clamped-clamped and hinged-hinged) have been obtained. In order to demonstrate equivalence, a hybrid algorithm has been devised, based on the qualitative spectral properties of fourth-order ordinary differential equations and axial load calculations. The results have been validated using examples on the Maple computer package and compared with the experimental measurements.
Key words: Euler–Bernoulli beam, non-uniform beam, eigenvalue, symmetry, equivalence.

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Біркелкі емес бөренелер үшін шекаралық есептердің симметриялық эквиваленттігі

Бұл жұмыста іргесі Винклер бойынша осьтік жүктемелері бар Эйлер-Бернулли біркелкі емес бөренелердің модельдері қарастырылған. Модельдегі біркелкі емес бөрене үш айнымалы параметрмен/коэффициенттермен сипатталады: иілу қаттылығы, ұзындық бірлігіне қатысты бөрененің іргесі мен массасы. Бұл зерттеудің негізгі тұжырымы айнымалы параметрлердің келісілген симметриясының есептің спектрлік қасиеттеріне қалай әсер ететінін айқын көрсету болып табылады. Біркелкі емес бөренелердің меншікті мәндерінің симметриялық эквиваленттілігінің (меншікті мәндерінің және меншікті функциялар жиының көбейткіштерге жіктелінуі) екі ұштарында бекітудің (қатты-қатты және топсалы-топсалы бекітілген) түрлері үшін сапалы нәтижелер алынды. Эквиваленттілігін көрсету үшін төртінші ретті қарапайым дифференциалдық теңдеулердің сапалы спектрлік қасиеттеріне және осьтік жүктемені есептеуге негізделген гибридті алгоритм жасалды. Нәтижелер Maple компьютер пакетіндегі мысалдар арқылы расталды және эксперименттік өлшемдермен салыстырылды.
Түйін сөздер: Эйлер–Бернулли бөренесі, біркелкі емес бөренелер, меншікті мән, симметрия, эквиваленттік.

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Симметричная эквивалентность краевых задач для неоднородных балок

В данной работе рассматриваются модели неоднородных балок Эйлера–Бернулли с осевыми нагрузками на основание Винклера. Неоднородная балка в модели описывается тремя переменными параметрами/коэффициентами: жесткостью изгиба, основанием и массой балки на единицу длины. Ключевым выводом данного исследования является наглядная демонстрация того, как согласованная симметрия переменных параметров влияет на спектральные свойства задачи. Получены качественные результаты для симметричной эквивалентности (факторизации наборов собственных значений и собственных функций) собственных значений неоднородных балок для двух типов закрепления на концах (защемленно-защемленное и шарнирно-шарнирное). Для демонстрации эквивалентности разработан гибридный алгоритм, основанный на качественных спектральных свойствах обыкновенных дифференциальных уравнений четвертого порядка и расчетах осевой нагрузки. Результаты были проверены с использованием примеров в компьютерном пакете Maple и сравнены с экспериментальными измерениями.
Ключевые слова: балка Эйлера–Бернулли, неоднородная балка, собственное значение, симметрия, эквивалентность.

1 Introduction

The majority of mechanical systems comprising beam construction, as employed in technology and engineering, are defined by their geometric and physical variable parameters. Such structures include parabolic tapering and functionally graded beams [1–3], which can be adopted for a light-weight design or specific wave propagation effects [4, 5], as well as piezoelectric energy harvesting [6, 7]. In [3], a closed-form dynamic stiffness formulation for the analysis of transverse free vibration in non-uniform symmetric Euler–Bernoulli beams was proposed, and effects of boundary conditions were investigated. A beam with a heterogeneous temperature distribution exhibits variable physical properties. The presence of variable parameters introduces a significant degree of complexity into the dynamic analysis. The modelling of mechanical systems comprising non-uniform beam construction gives rise to the formation of fourth-order linear equations with variable coefficients. Consequently, both approximate analytical [8–10] and numerical methods [12–16] for solving differential equations with variable coefficients under different conditions are being actively developed. A thorough literature review on the solution methods for transverse vibration of non-uniform beams with variable cross-sections can be found in [9]. In [10], approximate analytical expressions for the natural frequencies of non-uniform beams were obtained in terms of asymptotic theory. The isospectral problems for non-uniform beams were studied in [11,12]. The isospectral problems between non-uniform and uniform beams were presented in [12]. The natural frequencies of free boundary value problems for beams with symmetric coefficient without an axial load were studied in [9]. In [15], a regular variation approach to finding natural frequencies and modes of vibration of non-homogeneous beams were studied.

In the modelling of mechanical systems, it is essential to have a closed analytical formula for natural frequencies [17, 18, 20, 23]. In [17], a closed-form solution for non-uniform beams was proposed using special functions. In [18], an asymptotic formula of natural frequencies for the non-uniform beams with different boundary conditions was derived based on perturbation method. Eigenvalue asymptotics of an even order differential ordinary operator with square integrable potential were obtained in [19]. In [20], a solution for the free vibrations of non-uniform beams on a non-uniform Winkler foundation was presented, employing the Laguerre collocation method. The influence of axial loads on the natural frequencies of uniform beams with various boundary conditions were investigated in [21, 22]. Additionally, the papers revealed critical values of axial loads. In [23], several results pertaining to the closed-form expression for the natural frequencies of uniform beams were modified. Additionally, the concept of symmetrical equivalence was demonstrated for a uniform Euler-Bernoulli beam subjected to an axial load. The spectral properties of hinged-hinged beams, both with and without axial loads on an elastic foundation, were investigated based on the characteristic determinant in [24, 25] and [26], respectively. The effects of a foundation coefficient for calculating of the critical load were presented in [27]. The symmetric equivalence of boundary value problems for the uniform beams without and with axial loads lying on a Winkler's type foundation were studied in [28] and [29], respectively. Nevertheless, the symmetric equivalence of boundary value problems for non-uniform beams remains an area of incomplete investigation. One of the methods for studying non-uniform beams is to represent them as stepped beams. The symmetric equivalence of stepped beams can be employed for identification problems pertaining to the physical properties of beams, as evidenced by the findings set forth in the paper by [30].

The goal of this research is to identify conditions for the variable coefficients of bending stiffness, foundation and mass of the beam per unit length and fixing types of a non-uniform beam under which it is possible to establish the equivalence of the eigenvalues and eigenfunctions. The novelty of the paper is the agreed symmetry of the variable coefficients (see Theorems 1). The results presented here extend several known results from the cited sources, namely [23, 28, 29].

The problem of transverse vibrations of a non-uniform beam of unit length

$$\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}+k(x)w(x,t)+T\frac{\partial^2 w(x,t)}{\partial x^2}+\frac{\partial^2}{\partial x^2}\left(EJ(x)\frac{\partial w(x,t)}{\partial x^2}\right)=0,$$

after replacement $w(x,t)=v(\lambda,x)\sin(\omega t)$ reduces to the following spectral problem:

$$(EJ(x)v''(\lambda,x))''+Tv''(\lambda,x)+k(x)v(\lambda,x)=\lambda\rho A(x)v(\lambda,x),\quad x\in I_p, p=1,2, \tag{1}$$

where $v(\lambda,x)$ are the eigenfunctions of the transverse static deflection of the beam; $EJ(x)$ is the bending stiffness; $\rho\, A(x)$ is mass of the beam per unit length; T is corresponding to a constant compressive force if $T>0$ or a constant tensile force if $T<0$; $\lambda=\rho\,\omega^2$ are the eigenvalues; ω is the circular frequency; ρ is the material density; $k(x)$ is the variable coefficient of foundation, $I_1=(0,\,1)$, $I_2=\left(\frac{1}{2},\,1\right)$. Notice that $J(x)$ and $A(x)$ are assumed twice continuously differentiable and strictly positive, $k(x)$ is the real-valued summable function.

In this study, two types of beams are considered. The first is the hinged-hinged beam on the interval I_1 with the boundary conditions (see Figure 1)

$$v(\lambda,0)=0,\ v''(\lambda,0)=0, v(\lambda,1)=0,\ v''(\lambda,1)=0, \tag{2}$$

and the second is the clamped-clamped beam on the interval I_1 with the boundary conditions (see Figure 2)

$$v(\lambda,0)=0,\ v'(\lambda,0)=0, v(\lambda,1)=0,\ v'(\lambda,1)=0. \tag{3}$$

In addition, we introduce the sliding-hinged boundary conditions

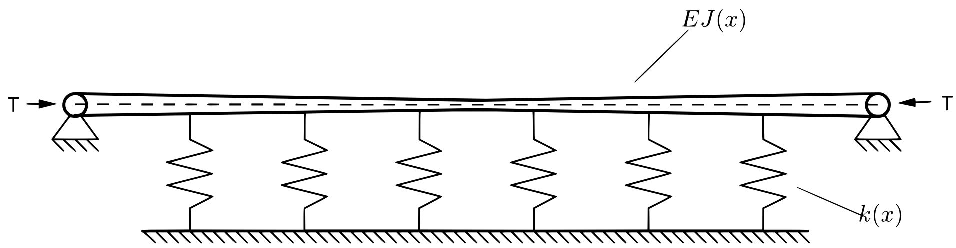

Figure 1: Hinged-hinged Euler–Bernoulli non-uniform beam.

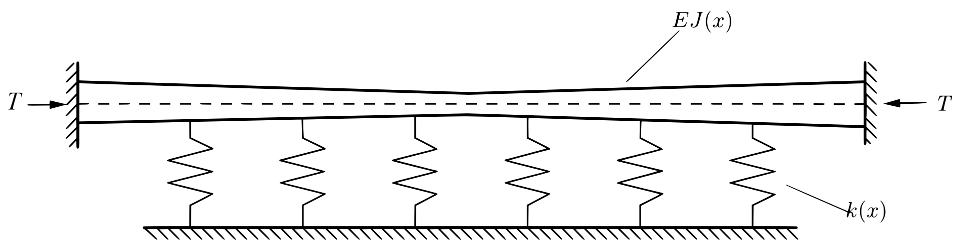

Figure 2: Clamped-clamped Euler–Bernoulli non-uniform beam.

$$v'\left(\lambda, \frac{1}{2}\right) = 0,\ v'''\left(\lambda, \frac{1}{2}\right) = 0,\ v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0, \tag{4}$$

and hinged-hinged boundary conditions

$$v\left(\lambda, \frac{1}{2}\right) = 0,\ v''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0 \tag{5}$$

which are connected with hinged-hinged fixing on the interval I_2. Furthermore, we introduce the sliding-clamped boundary conditions

$$v'\left(\lambda, \frac{1}{2}\right) = 0,\ v'''\left(\lambda, \frac{1}{2}\right) = 0,\ v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0, \tag{6}$$

and hinged-clamped boundary conditions

$$v\left(\lambda, \frac{1}{2}\right) = 0,\ v''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0 \tag{7}$$

which are connected with clamped-clamped fixing on the interval I_2.

2 Main results

Let $\sigma(A_1), \sigma(B_1), \sigma(C_1)$ be a set of eigenvalues of problems $A_1 - \lambda I,\ B_1 - \lambda I,\ C_1 - \lambda I$ generated by Equation (1) on finite intervals by boundary conditions (2), (4), (5), respectively.

Theorem 1 *Let $J(x)$, $k(x)$ and $A(x)$ be the symmetric functions with respect to the point $x = \frac{1}{2}$*

$$J(x) = J(1-x), \quad k(x) = k(1-x), \quad A(x) = A(1-x), \quad x \in \left[0; \frac{1}{2}\right] \tag{8}$$

and $T < T_{cr}$. The following statements are true:
1. $\sigma(A_1) \equiv \sigma(B_1) \cup \sigma(C_1)$
2. If $\lambda \in \sigma(B_1)$ or $\lambda \in \sigma(C_1)$, then the eigenfunctions of problems $A_1 - \lambda I$ corresponding to the eigenvalues λ are symmetric or asymmetric with respect to the middle of the beam at the point $x = \frac{1}{2}$ on the interval $(0,1)$, respectively.

Let $\sigma(A_2)$, $\sigma(B_2)$, $\sigma(C_2)$ be a set of eigenvalues of problems $A_2 - \lambda I$, $B_2 - \lambda I$, $C_2 - \lambda I$ generated by Equation (1) on finite intervals by boundary conditions (3), (6), (7), respectively.

Theorem 2 *Let $J(x)$, $k(x)$ and $A(x)$ be the symmetric functions with respect to the point $x = \frac{1}{2}$, i.e. the condition in Equation (8) holds and $T < T_{cr}$. The following statements are true:*
1. $\sigma(A_2) \equiv \sigma(B_2) \cup \sigma(C_2)$
2. If $\lambda \in \sigma(B_2)$ or $\lambda \in \sigma(C_2)$, then the eigenfunctions of problems $A_2 - \lambda I$ corresponding to the eigenvalues λ are symmetric or asymmetric with respect to the middle of the beam at the point $x = \frac{1}{2}$ on the interval $(0,1)$, respectively.

The proof of Theorems 1 and 2 is ideologically similar to that presented in work [28]. Nevertheless, there is a single discrepancy, which require is calculating of the critical value T_{cr}. Further will be described the scheme for proving Theorems 1 and 2.

First step. The following functions $J(x)$, $k(x)$ and $A(x)$ will be selected to satisfy condition (8).

Second step. The critical value of T_{cr} will be calculated that corresponding to the first step and the value of T will be selected such that $T < T_{cr}$. The calculation of T_{cr} will be conducted using well-known numerical method (see, [22]).

Third step. The final step will employ the same technique used to prove the result presented in [28].

Upon completion of the aforementioned three steps, the proofs of Theorems 1 and 2 will be obtained. In the third step, the analytical or numerical method may be employed. It should be noted that if the functions $J(x)$, $k(x)$ and $A(x)$ satisfy condition (8) and the additional condition from [12], then the non-uniform beam can be transformed into a uniform one.

Remark 1 *Results from Theorem 1 and Theorem 2 are preserved for stepped beams. Experimental and numerical simulations for the clamped-clamped stepped beam were carried out in [30, 31]. The symmetric equivalence of the clamped-clamped stepped beam was used for solving the inverse coefficients problems in [30].*

3 Examples and discussion

In this section, we calculate approximately the four or five eigenvalues of boundary value problems $A_n - \lambda I$, $B_n - \lambda I$, $C_n - \lambda I$ $(n = 1, 2, 3)$ generated by the Euler–Bernoulli equation for the various coefficients $J(x)$, $k(x)$, $A(x)$ and $p(x)$. The results of calculation of the eigenvalues are shown in the corresponding columns of Tables 1–4.

Example 1 *In this analysis, we examine three steps.*

First step. Let $J(x) = 1 + x(1-x)$, $k(x) = 4x(1-x)$, $A(x) = x(1-x)$ *and* $E = 1$.

Second step. In this example $T_{cr} \approx 12.09$ *and take* $T = 5$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 1 are shown in Table 1.*

Table 1: Numerical calculations of the first five eigenvalues from the Example 1

Hinged-hinged at the points $x = 0$, $x = 1$	Sliding at the point $x = \frac{1}{2}$, hinged at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, hinged at the point $x = 1$
(2)	(4)	(5)
17.95	17.95	95.65
95.65	229.93	420.31
229.93	666.79	969.34
420.31	1327.96	1742.63
666.79	2213.36	2740.14

The calculations presented in Example 1 provide corroboration for the validity of Statement 1 of Theorem 1 pertaining to the factorization of the set of eigenvalues.

Example 2 *In this analysis, we consider three steps.*

First step. Let $J(x) = x(1-x)$, $k(x) = 5(1+x)^3$, $A(x) = x(1-x)$ *and* $E = 1$.

Second step. In this example $T_{cr} \approx 3.73$ *and take* $T = 1$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 2 are shown in Table 2.*

Table 2: Numerical calculations of the first five eigenvalues from the example 2.

Hinged-hinged at the points $x = 0$, $x = 1$	Sliding at the point $x = \frac{1}{2}$, hinged at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, hinged at the point $x = 1$
(2)	(4)	(5)
11.31	12.37	37.09
36.38	84.70	152.81
84.31	240.98	349.07
152.55	476.99	624.72
240.81	792.23	979.52

The violation of the regularity of factorization of eigenvalues in Example 2 is due to the failure to satisfy the symmetry condition for the function $k(x)$. The aforementioned calculations in Example 2 confirm the validity of Statement 1 of Theorem 1.

Example 3 *We consider three steps.*

First step. Let $J(x) = 1 + x(1-x)$, $k(x) = 4$, $A(x) = x^2(1-x)^2$ *and* $E = 1$.

Second step. In this example $T_{cr} \approx 45.71$ *and take* $T = 30$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 3 are shown in Table 3.*

Table 3: Numerical calculations of the first five eigenvalues from the Example 3

Clamped-clamped at the points $x = 0, \quad x = 1$	Sliding at the point $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
(3)	(6)	(7)
27.4	27.4	121.79
121.79	285.1	511.35
285.1	800.52	1152.37
511.35	1566.79	2043.71
800.52	2583.07	3184.87

The calculations which represent in Example 3 confirm the validity of Statement 1 of Theorem 2 on the factorization of the set of eigenvalues.

Example 4 *We consider three steps.*

First step. Let $J(x) = 1 + x(1+x)$, $k(x) = 4$, $A(x) = x^2(1-x)^2$ *and* $E = 1$.

Second step. In this example $T_{cr} \approx 67.4$ *and take* $T = 30$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 4 are shown in Table 4.*

Table 4: Numerical calculations of the first four eigenvalues from the Example 4

Clamped-clamped at the points $x = 0, \quad x = 1$	Sliding at the point $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
(3)	(6)	(7)
43.21	62.46	191.41
159.65	420.72	725.72
357.61	1126.92	1602.73
631.47	3562.85	2820.71

The violation of the regularity of factorization of eigenvalues in Example 4 is is due to the failure to satisfy the symmetry condition for the function $J(x)$. The aforementioned calculations in Example 4 confirm the validity of Statement 1 of Theorem 2.

Example 5 *In this study, we consider a two-stepped beam with clamped-clamped boundary conditions. The geometric dimensions of the composite beam are as follows:* $L_1 = L_3 =$

247.6mm $L_2 = 508mm$, $h_1 = h_2 = h_3 = 3.6mm$, $b_1 = b_3 = 38.2mm$ *and* $b_2 = 25.7mm$ *and these are illustrated in in Fig.3. Young's modulus and the density are* $28.3GPa$ *and* $\rho = 1800kg/m^3$, *respectively. In this example,* $T = 0$, $k(x) = 0$. *The natural frequencies*

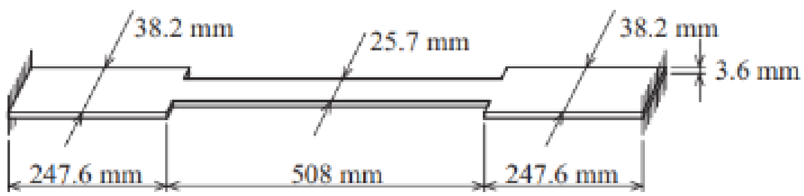

Figure 3: Two-stepped composite beam with clamped-clamped boundary conditions [30,31].

are computed by transcendental eigenvalue problem (TEP) and the results compared with the experimental measurements [30] in Table 5.

Table 5: The first five natural frequencies (Hz) from the Example 5

Clamped-clamped at the points $x = 0$, $x = 1$. Experiment [30]	Sliding at the point $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
(3)	(6)	(7)
16.1 ± 0.16	16.12	41.01
41.3 ± 0.16	78.67	130.55
79.3 ± 0.16	195.01	270.97
134.0 ± 0.16	360.78	465.47
196.5 ± 0.16	581.72	708.17

The calculations presented in Example 5 confirm the validity of Statement 1 of Theorem 2 on the factorization of the set of eigenvalues.

A numerical method was employed for the calculation of the eigenvalues at the variable coefficients of $J(x)$, $A(x)$ and $k(x)$ with the polynomial expansion and integral techniques as outlined in [14]. The degree of the polynomial was selected as $N = 25$, which ensures accuracy of calculations. The numerical calculations were carried out using the Maple computer mathematics system [32].

The results obtained in this work permits to study the qualitative spectral properties of a non-uniform beam. Symmetrical equivalence permits to calculate the natural frequencies of a full beam using the natural frequencies of two short beams with different lengths and fixing methods. This paper presents examples of partial factorization of the eigenvalues of a full-length beam in the case of an asymmetric foundation coefficient. Furthermore, the length of short beams is also contingent upon the agreed symmetry. To illustrate, when the parameters are symmetric about the $x = 1/2$, the length of the short beams is equal to half the length of the full beam. The aforementioned capabilities are play a significant role in computer calculations and the modeling of mechanical systems with complex structures. It is therefore anticipated that future research will focus on the behavior of mechanical systems of complex structure, with the star graph serving as a case in point [33] .

4 Conclusions

In this paper, the problems for determining the eigenvalues of the non-uniform Euler–Bernoulli beam with the axial load lying on the Winkler's type foundation at two types of fixings at the ends have been solved: clamped-clamped and hinged-hinged. A sufficient condition has been found for the variable coefficients of the differential equation of the beam, which a symmetrical equivalence of the eigenvalues and eigenfunctions is satisfied.

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