IRSTI 27.25.19 DOI: https://doi.org/10.26577/JMMCS202512732

A. Arystangalikyzy [1*], **A.Zh.Zhubanysheva** [2]
[1]K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan
[2]L. N. Gumilyov Eurasian National University, Astana, Kazakhstan
*e-mail: arystangalikyzya@gmail.com

OPTIMAL APPROXIMATION OF SOLUTIONS OF POISSON EQUATION BY INITIAL DATA IN THE FORM OF ACCURATE AND INACCURATE INFORMATION OF TRIGONOMETRIC FOURIER COEFFICIENTS

Partial differential equations along with a function, derivative, and integral are basic mathematical models. Therefore, the problem of their approximation by accurate and inaccurate information with the construction of optimal computational aggregates (approximation methods) of approximation is relevant and many articles are devoted to this issue.
In the article is considered the problem of approximation of solutions of Poisson equation with the right-hand side f from the Nikol'skii classes $H_2^r(0,1)^s$ in the Lebesgue metrics $L^2(0,1)^s$ and $L^\infty(0,1)^s$. The orders of error of approximation of solutions of the Poisson equation by accurate and inaccurate information in the form of trigonometric Fourier coefficients of f are obtained. Namely, a lower bound for the approximation error based on accurate information is found for all possible computational agregates using an arbitrary finite set of trigonometric Fourier coefficients. A computational agregate (approximation method) by the trigonometric Fourier coefficients of the right-hand side f of the equation is constructed that confirms this lower bound. The boundaries of $\tilde{\varepsilon}_N$ of inaccurate information preserving the order of error of approximation by accurate information are established.
Key words: Poisson equation, approximation by accurate and inaccurate information, Nikol'skii classes, optimal computational aggregate, boundaries of inaccurate information.

А. Арыстанғалиқызы[1*], А.Ж. Жубанышева[2]
[1]Қ. Жұбанов атындағы Ақтөбе өңірлік университеті, Ақтөбе, Қазақстан
[2]Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Астана, Қазақстан
*e-mail: arystangalikyzya@gmail.com

Тригонометриялық Фурье коэффициенттерінен алынған дәл және дәл емес бастапқы ақпарат бойынша Пуассон теңдеуінің шешімдерін оптималды жуықтау

Дербес туындылы дифференциалдық теңдеулер функция, туынды және интегралмен қатар негізгі математикалық модельдер қатарына жатады. Сондықтан, дәл және дәл емес ақпарат бойынша оларды жуықтаудың оптималды есептеу агрегаттарын (жуықтау әдістерін) құру мәселесі өзекті болып табылады және осы мәселеге көптеген мақалалар арналған.
Мақалада f оң жағы $H_2^r(0,1)^s$ Никольский класында жататын Пуассон теңдеуінің шешімдерін $L^2(0,1)^s$ және $L^\infty(0,1)^s$ Лебег метрикаларында жуықтау есебі қарастырылады. f функциясының тригонометриялық Фурье коэффициенттері түрінде берілген дәл және дәл емес ақпарат бойынша Пуассон теңдеуінің шешімдерін жуықтау қателігінің реті алынды. Атап айтқанда, тригонометриялық Фурье коэффициенттерінің кез келген ақырлы жиынын қолданып, барлық мүмкін есептеу агрегаттары үшін дәл ақпараттарға негізделген жуықтау қателігінің төменнен бағалауы алынды. Төменнен бағалауды растайтын есептеуіш агрегат (жуықтау әдісі) теңдеудің оң жақ тригонометриялық Фурье коэффициенттері бойынша құрылды. Дәл ақпарат бойынша жуықтау қателігінің ретін сақтайтын дәл емес ақпараттың $\tilde{\varepsilon}_N$ шекаралары анықталды.
Түйін сөздер: Пуассон теңдеуі, дәл және дәл емес ақпарат бойынша жуықтау, Никольский класстары, тиімді есептеу агрегат, дәл емес ақпарат шекаралары.

А. Арыстангаликызы[1*], А.Ж. Жубанышева[2]
[1]Актюбинский региональный университет им. К. Жубанова, Актобе, Казахстан
[2]Евразийский национальный университет им. Л.Н. Гумилева, Астана, Казахстан
*e-mail: arystangalikyzya@gmail.com

Оптимальное приближение решений уравнений Пуассона по исходным данным в виде точных и приближенных значений тригонометрических коэффициентов Фурье

Дифференциальные уравнения в частных производных наряду с функцией, производной, интегралом относятся к основным математическим моделям.
Следовательно задача их приближения по точным и неточным данным с построением оптимальных вычислительных агрегатов (методов приближения) является актуальной и данному вопросу посвещено множество статей. В статье изучается задача приближения решений уравнения Пуассона с правой частью f из классов Никольского $H_2^r(0,1)^s$ в Лебеговой метриках $L^2(0,1)^s$ и $L^\infty(0,1)^s$. Получены порядки погрешности приближения решений уравнения Пуассона по точным и неточным данным в виде тригонометрических коэффициентов Фурье функции f. Именно, найдена оценка снизу погрешности приближения по точным данным по всем возможным вычислительным агрегатам, использующим конечный набор тригонометрических коэффициентов Фурье. Построен вычислительный агрегат (метод приближения) по тригонометрическим коэффициентам Фурье правой части f уравнения, подтверждающий данную оценку снизу. Установлены границы $\tilde{\varepsilon}_N$ неточной информации, сохраняющие порядок убывание по точной информации.
Ключевые слова: уравнение Пуассона, приближение по точным и неточным данным, классы Никольского, оптимальный вычислительный агрегат, границы неточной информации.

1 Introduction

Solutions of partial differential equations, even when expressed explicitly by means of Fourier series in the eigenfunctions of the corresponding differential operator or convolution with the corresponding kernels, being represented by series or integrals, in fact again infinite objects. Therefore, the problem of approximating them with finite objects again arises. In the article is considered the problem of approximation of solutions of Poisson equations in the sence of computational (numerical) diameter (denoted by C(N)D). Poisson equation has an various applications. One of them is that it describes the distribution of an electrostatistics, potential theory, scalar field, such as an electric potential or gravitational potential, in space. Thus, its physical meaning is that it relates the distribution of field sources to the field itself. Therefore, it is important to take this equation into account. Let at first consider the definition of computational (numerical) diameter problem.

In computational (numerical) diameter the initial definition is (see, for example, [1]- [2])

$$\delta_N(\varepsilon_N; D_N)_Y \equiv \delta_N(\varepsilon_N; T; F; D_N)_Y \equiv \inf_{(l^{(N)};\varphi_N)\in D_N} \delta_N(\varepsilon_N; (l^{(N)}; \varphi_N))_Y$$

where

$$\delta_N(\varepsilon_N; (l^{(N)}; \varphi_N))_Y \equiv \delta_N(\varepsilon_N; T; F; (l^{(N)}; \varphi_N))_Y \equiv$$
$$\equiv \sup_{\substack{f\in F, \\ \left\{\gamma_N^{(\tau)}\right\}_{\tau=1}^N, \left|\gamma_N^{(\tau)}\right|\le 1, \\ (\tau=1,\dots,N)}} \|Tf(\cdot) - \varphi_N(l_N^{(1)}(f) + \gamma_N^{(1)}\varepsilon_N^{(1)}; \dots, l_N^{(N)}(f) + \gamma_N^{(N)}\varepsilon_N^{(N)}; \cdot)\|_Y.$$

Here, a *mathematical model* is given by the operator $T : F \to Y$. X and Y are the normalized spaces of functions defined on Ω_X and Ω_Y respectively, $F \subset Y$ is a class of functions. Numerical information $l^{(N)} \equiv l^{(N)}(f) = (l_N^{(1)}(f), ..., l_N^{(N)}(f))$ of volume $N(N = 1, 2, ...)$ about f from class F is taken by linear functionals $l_N^{(1)}(f), ..., l_N^{(N)}(f)$ (in general, not necessarily linear). An *information processing algorithm* $\varphi_N(z_1, ..., z_N; \cdot) : C^N \times \Omega_X \to C$ is a correspondence, which for every fixed $(z_1, ..., z_N) \in C^N$ as a function of $(\cdot)$ is an element of Y and $\varphi_N(0, ..., 0; \cdot) = 0$. If the class of functions under consideration is centrally symmetric, then the last condition $\varphi_N(0, ..., 0; \cdot) = 0$ can be ignored. The record $\varphi_N \in Y$ means that φ_N satisfies all the conditions listed above, and $\{\varphi_N\}_Y$ is a set composed of all $\varphi_N \in Y$. Further, $(l^{(N)}; \varphi_N)$ is a *computational aggregate* of recovery from accurate information for the function $f \in F$ acting according to the rule $\varphi_N(l_N^{(1)}, ..., l_N^{(N)}; \cdot)$. The recovery of Tf by inaccurate information is proceeding as follows. At first the boundaries of the inaccuracy are set: a vector $\varepsilon_N = (\varepsilon_N^{(1)}, ..., \varepsilon_N^{(N)})$ with non-negative components. Then, the accurate values $l_N^{(\tau)}(f)$ are replaced with a given accuracy $\varepsilon_N^{(\tau)} \geq 0$ by approximate values $z_\tau \equiv z_\tau(f)$, $|z_\tau - l_N^{(\tau)}(f)| \leq \varepsilon_N^{(\tau)}$ $(\tau = 1, ..., N)$, numbers $z_\tau \equiv z_\tau(f)$ $(\tau = 1, ..., N)$ are processed using the algorithm φ_N up to the function $\varphi_N(z_1(f), ..., z_N(f); \cdot)$, which will constitute the computational aggregate $(l^{(N)}; \varphi_N) \equiv \varphi_N(z_1(f), ..., z_N(f); \cdot)$ constructed according to information of the precision $\varepsilon_N = (\varepsilon_N^{(1)}, ..., \varepsilon_N^{(N)})$.

Let $D_N \equiv D_N(F)_Y$ be a given set of complexes $(l_N^{(1)}, ..., l_N^{(N)}; \varphi_N) \equiv (l^{(N)}, \varphi_N)$, we emphasize, operators of recovery by accurate information.

For nonnegative sequences $\{A_N\}$ and $\{B_N\}$, we write $A_N \ll B_N$ (or, equivalently $A_N = O(B_N)$) if there exists a positive constant $c > 0$ such that, for all $N(N = 1, 2, \dots)$ hold $A_N \leq cB_N$. Furthermore, we write $A_N \asymp B_N$ if both $A_N \ll B_N$ and $B_N \ll A_N$ hold simultaneously.

Within the framework of given notations and definitions, the problem of optimal recovery by inaccurate information, framed under the name *computational (numerical) diameter*, according to the [1]- [2], consists in a collective sense in sequential solution of the following three problems: C(N)D-1, C(N)D-2 and C(N)D-3.

For given $T; F; Y; D_N$:

C(N)D-1: an order of $\asymp \delta_N(0; D_N)_Y \equiv \delta_N(0; T; F; D_N)_Y$ is found with the construction of a specific computional aggregate $(\bar{l}^{(N)}, \overline{\varphi}_N)$ from $D_N \equiv D_N(F)_Y$ supporting ordering

$$\asymp \delta_N(0; D_N)_Y;$$

C(N)D-2: for $(\bar{l}^{(N)}, \overline{\varphi}_N)$ is considered the problem of existence and finding a sequence $\widetilde{\varepsilon}_N \equiv \widetilde{\varepsilon}_N(D_N; (\bar{l}^{(N)}; \overline{\varphi}_N))_Y$ with non-negative components such that

$$\delta_N(0; D_N)_Y \asymp \delta_N(\widetilde{\varepsilon}_N; (\bar{l}^{(N)}; \overline{\varphi}_N))_Y \equiv$$

$$\equiv \sup\{\|Tf(\cdot) - \overline{\varphi}_N(z_1, ..., z_N; \cdot)\|_Y : f \in F, |\bar{l}_\tau(f) - z_\tau| \leq \widetilde{\varepsilon}_N^{(\tau)}(\tau \in \{1, ..., N\})\}$$

with simultaneous satisfying the following expression

$$\forall \eta_N \uparrow +\infty (0 < \eta_N < \eta_{N+1}, \eta_N \to +\infty) :$$

$$\overline{\lim_{N \to +\infty}} \delta_N(\eta_N \widetilde{\varepsilon}_N; (\bar{l}^{(N)}, \overline{\varphi}_N))_Y / \delta_N(0; D_N)_Y = +\infty;$$

C(N)D-3: *massiveness* of limiting error $\widetilde{\varepsilon}_N$ is set: as huge as possible set $M_N(\bar{l}^{(N)};\overline{\varphi}_N)$ from D_N (usually associated with the structure of the $(\bar{l}^{(N)};\overline{\varphi}_N)$) of computional aggregates $(l^{(N)},\varphi_N)$ is found, such that for each of them the following relation holds

$$\forall \eta_N \uparrow +\infty (0<\eta_N<\eta_{N+1}, \eta_N \to +\infty):$$

$$\overline{\lim_{N\to+\infty}} \delta_N(\eta_N\widetilde{\varepsilon}_N;(l^{(N)},\varphi_N))_Y/\delta_N(0;D_N)_Y = +\infty.$$

In the article is considered the following concretization of computational (numerical) diameter problem. $Tf = u(x,f)$ – the solution of Dirichlet problem of Poisson equations

$$\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_s^2} = f(x), \qquad (1)$$

on a unit cube $[0,1]^s$, where $f(x) = f(x_1,...,x_s) \in F = H_2^r$ – Nikol'skii class, Y are Lebesgue metrics L^2 and L^∞ and recovery is performed over all computational aggregates, in which numerical information is specified by trigonometric Fourier coefficients with an arbitrary spectrum:

$$D_N = \Phi_N = \{l_N^{(1)}(f) = \widehat{f}(m^{(1)}),...,l_N^{(N)}(f) = \widehat{f}(m^{(N)}) : m^{(j)} \in Z^s (j=1,...,N)\} \times \{\varphi_N\}_Y,$$

where Y is L^2 or L^∞,

$$\widehat{f}(m) = \int_{[0,1]^s} f(x) e^{-2\pi i (m,x)} dx$$

are trigonometric Fourier coeficients, $(m,x) = m_1x_1 + ... + m_sx_s$, $m = (m_1,...,m_s)$, $x = (x_1,...,x_s)$.

In this article, the computational (numerical) diameter problem in the specified concretization is solved in parts C(N)D-1 and the first part of C(N)D-2. Let's move on to a brief overview of the issue.

One of the first result, when f is odd, approximation of solution to Poisson equation is considered by N.M.Korobov in [3, p. 187-189]. There are approximation operator is constructed on the value of the function f (initial condition) at the points $\left(\left\{\frac{a_1k}{N}\right\},...,\left\{\frac{a_sk}{N}\right\}\right), k \in 1,...,N$, ($\{b\}$ – fractional part of b). If $a_1,...,a_s$ are the optimal coefficients (see definition of optimal coeffitients in [3, p. 96]) modulo N and β index, then the approximation of error is $O\left(\frac{(lnN)^{\frac{r\beta}{2}+s}}{N^{\frac{r}{2}-\frac{1}{2}+\frac{1}{s}}}\right)$.

The authors of [4] were achieved sharp estimates in the power scale for the approximation error, which is almost square times better in comparison with previous result of Korobov. More precisely, with accuracy $O\left(\frac{(lnN)^{(r+2/s)(s-1)}}{N^{r-(1-1/p-2/s)}}\right)$ and $O\left(\frac{(lnN)^{r(\beta+s)+s}}{N^r}\right)$ in cases $1-\frac{1}{p}-\frac{2}{s}>0$ and $1-\frac{1}{p}-\frac{2}{s}\le 0$ respectively.

For practical purposes, however, in [5] got the result about sampling on sparse grids by the Smolyak's algorithm. In [6] considered the approximation of a function in the Besov class and used it to approximate solutions of Laplace equation. As well as, approximate the solution of 2D and 3D Poisson's equations by the Haar wavelet method is considered in [7]. Research on the problem of approximating solutions of the Poisson equation with accurate information in anisotropic Korobov classes $E^{r_1,...,r_s}(0,1)^s$ has been studied recently in the papers [8]-[9]. The problem of approximation of solutions to Poisson equation with right-hand side from Nikol'skii-Besov classes $B^r_{2,\theta}(0,1)^s$ and anisotropic Korobov classes $E^{r_1,...,r_s}(0,1)^s$ by the value of the function at the points $\left(\left\{\frac{a_1k}{N}\right\},...,\left\{\frac{a_sk}{N}\right\}\right), k\in 1,...,N$ is considered in [10]. Approximation by inaccurate information of solutions of Poisson equations with right-hand side $f\in E^{r_1,...,r_s}$ is considered in [11] and $f\in E^r_s$ and W^r_2 cases are considered in [12]-[13] respectively. There are obtained upper bound of error of approximation by innacurate information from values at the points of f in uniform metric. In [12], the author approximates the solutions of the Poisson equation in the L^2 metric using an approximation operator constructed from a finite set of Fourier coefficients of the function with right hand side $f\in E^r_s$. Here is given a complete solution for C(N)D problem.

2 Necessary definitions and statements

Definition 1 (see [14], p. 75-76). *The Nikol'skii class* $H^r_q(0,1)^s$ $(s=1,2,...;r>0;1<q<+\infty)$ is the set of all functions $f(x)\in L^q(0,1)^s$ that 1-periodic in each of their variable satisfying the inequality

$$\sup_{j=0,1,...} 2^{jr}\left\|\sum_{[2^{j-1}]\le\|m\|<2^j}\widehat{f}(m)\cdot e^{2\pi i(m,x)}\right\|_{L^q(0,1)^s}\le 1, \tag{2}$$

where the square bracket [...] means the integer part. For everywhere below for $m=(m_1,...,m_s)$ we set $\|m\|=\max_{j=1,...,s}|m_j|$.

Let F be some class of $f(x)=f(x_1,...,x_s)$ functions 1-periodic in each variable whose trigonometric Fourier series converges absolutely.

Assume that $\widehat{f}(0)\neq 0$. It is easy to verify that, for any boundary condition there exists a function $\omega(x)$ depending on this condition such that $\omega(x)$ is continuous on $[0,1]^s$ and $\Delta\omega\equiv 1$ on $[0,1]^s$. Moreover, solution of (1) has the form

$$u_\omega(x,f)=\omega(x)\cdot\widehat{f}(0)-\frac{1}{4\pi^2}\sum_{m\in Z^s}{}^{*}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}. \tag{3}$$

If $\widehat{f}(0)=0$, then for a solution of (1) to exist, it is necessary that the boundary condition $u|_G=h(x)$ on the boundary of G satisfies

$$h(x)=-\frac{1}{4\pi^2}\sum_{m\in Z^s}{}^{*}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}(x\in G).$$

If $f(x_1,...,x_s)$ is odd in each of the variables $x_1,...,x_s$ then the function (see, [3], p.187-189)

$$u(x,f)=-\frac{1}{4\pi^2}\sum_{m\in Z^s}{}^{*}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}$$

is a solution of (1) with zero boundary condition on $[0,1]^s$. Here and everywhere below the asterisk "*" over the sum means that $m=(0,...,0)$ is dropped in the summation.

3 Main result and its proof

Theorem 1. Let are given positive integer s and $r > s/2$. Then the following statements hold ($N = (2^{n+1}+1)^s$, $n = 1, 2, ...$)

C(N)D-1:

$$\delta_N(0; D_N)_{L^2} \equiv$$

$$\equiv \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s;\\ \varphi_N}} \sup_{f\in H_2^r} \left\| u_\omega(\cdot, f) - \varphi_N(\widehat{f}(m^{(1)}), ..., \widehat{f}(m^{(N)}); \cdot)\right\|_{L^2(0,1)^s} \asymp N^{-\frac{r+2}{s}}, \quad (4)$$

upper bound is sharps on computational aggregate

$$\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x) = \omega(x)\widehat{f}(0) - \frac{1}{4\pi^2} {\sum_{m\in I_{2^n}}}^* \frac{\widehat{f}(m)}{(m,m)} e^{2\pi i (m,x)}, \quad (5)$$

here in (5) the set $\{\overline{m}^{(1)} = 0, \overline{m}^{(2)}, ..., \overline{m}^{(N)}\}$ is some ordering of the set I_{2^n}, i.e.

$$I_{2^n} = \{m = (m_1, ..., m_s) \in Z^s : |m_j| \le 2^n (j = 1, 2, ..., s)\} = \{\overline{m}^{(1)} = 0, \overline{m}^{(2)} ..., \overline{m}^{(N)}\}. \quad (6)$$

C(N)D-2 (first part): For computational aggregates $\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}), x)$ from (5) and for the numerical sequence

$$\widetilde{\varepsilon}_N \asymp \begin{cases} N^{-\frac{r+2}{s}}, if s < 4, \\ (lnN)^{-\frac{1}{2}} \cdot N^{-\frac{r+2}{4}}, if s = 4, \\ N^{-\frac{r}{s}-\frac{1}{2}}, if s > 4. \end{cases} \quad (7)$$

satisfy

$$\delta_N(0; D_N)_{L^2} \asymp \delta_N\left(\tilde{\varepsilon}_N; D_N\right)_{L^2} \asymp$$

$$\asymp \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s,\\ \varphi_N}} \sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N, |\gamma_N^{(\tau)}|\le 1,\\ (\tau=1,...,N)}} \Big\| u_\omega(x, f) - \varphi_N(\widehat{f}(m^{(1)}) + \widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)}, ..., \widehat{f}(m^{(N)}) +$$

$$+ \widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)}; x)\Big\|_{L^2} \asymp \sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N, |\gamma_N^{(\tau)}|\le 1,\\ (\tau=1,...,N)}} \Big\| u_\omega(x, f) - \varphi_N(\widehat{f}(\overline{m}^{(1)}) + \widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)}, ..., \widehat{f}(\overline{m}^{(N)}) +$$

$$+ \widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)}; x)\Big\|_{L^2} \asymp N^{-\frac{r+2}{s}}. \quad (8)$$

Proof. Let f belongs to Nikol'skii classes H_2^r. Then since $r > s/2$ from the definition of class H_2^r follows $u_\omega(x, f) \in L^2(0,1)^s$. Let n be a given positive integer, we set $N = |I_{2^n}| = (2^{n+1}+1)^s$. According to the definition of D_N, we set

$$B_N = \{\overline{m}^{(1)} = 0; \overline{m}^{(2)}; ...; \overline{m}^{(N)}\}, B_N = I_{2^n},$$

$$l_N^{(\overline{m}^{(1)})}(f) = \widehat{f}(\overline{m}^{(1)}) = \widehat{f}(0), l_N^{(\overline{m}^{j})}(f) = \widehat{f}(\overline{m}^{(j)}), j = 2, 3, ..., N.$$

Let's start with an upper bound for the value of $\delta_N\left(\tilde{\varepsilon}_N; D_N\right)_{L^2}$ from C(N)D-2. Let are given $\{\gamma_N^{(\tau)}\}_{\tau=1}^N \equiv \{\gamma_N^{(m)}\}_{m\in I_{2^n}}$, $|\gamma_N^{(\tau)}| \leq 1(\tau = 1, ..., N)$. By (3) and (5), we have ($L^2 \equiv L^2(0,1)^s$)

$$\left\|u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}) + \widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)}, ..., \widehat{f}(\overline{m}^{(N)}) + \widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)}; x)\right\|_{L^2} \leq$$

$$\leq \left\|u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x)\right\|_{L^2} + \left\|\omega(x)\widetilde{\varepsilon}_N\gamma_N^{(0)} - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}}{}^{*}\frac{\widetilde{\varepsilon}_N\gamma_N^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2} \leq$$

$$\leq \left\|-\frac{1}{4\pi^2}\sum_{m\in Z^s/I_{2^n}}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2} + \left\|\omega(x)\widetilde{\varepsilon}_N\gamma_N^{(0)} - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}}{}^{*}\frac{\widetilde{\varepsilon}_N\gamma_N^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2} \equiv$$

$$\equiv \|I_1\|_{L^2} + \|I_2\|_{L^2}.$$

Estimating from above for $\|I_1\|_{L^2}$ gives upper bound for $\delta_N(0; D_N)_{L^2}$ in C(N)D-1. We will evaluate upper bound of the error of approximation in L^2 metric by using (2), (3), (5) and Parseval's equality:

$$\|I_1\|_{L^2}^2 \equiv \left\|u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x)\right\|_{L^2(0,1)^s}^2 =$$

$$= \left\|-\frac{1}{4\pi^2}\sum_{m\in Z^s/I_{2^n}}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2}^2 = \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|^2}{16\pi^4(m,m)^2} \ll$$

$$\ll \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|^2}{(m_1^2 + ... + m_s^2)^2} \ll \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|^2}{(\max_{j=1,...,s}|m_j|^2)^2} \ll$$

$$\ll \sum_{j=n+1}^{+\infty}\frac{1}{2^{4j}}\left(\sum_{2^j\leq\|m\|<2^{j+1}}|\widehat{f}(m)|^2\right)\cdot 2^{2(j+1)r}\cdot 2^{-2(j+1)r} \ll \frac{1}{2^{4n}}\sum_{j=n+1}^{+\infty}2^{-2(j+1)r} \ll$$

$$\ll 2^{-4n-2nr} \asymp N^{-\frac{2(r+2)}{s}}.$$

Further,

$$\sup_{f\in H_2^r}\left\|u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x)\right\|_{L^2} \ll N^{-\frac{r+2}{s}}$$

and

$$\delta_N(0; D_N)_{L^2} \equiv \inf_{\substack{m^{(1)}\in Z^s, ..., m^{(N)}\in Z^s,\\ \varphi_N}}\sup_{f\in H_2^r}\left\|u_\omega(x,f) - \varphi_N(\widehat{f}(m^{(1)}), ..., \widehat{f}(m^{(N)}); x)\right\|_{L^2} \ll N^{-\frac{r+2}{s}}.$$

which is the upper bound in (4).

Then let's evaluate $\|I_2\|_{L^2}$ (see(7))

$$\|I_2\|_{L^2} \equiv \left\|\omega(x)\widetilde{\varepsilon}_N\gamma_N^{(0)} - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}}{}^{*}\frac{\widetilde{\varepsilon}_N\gamma_N^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2} \ll \widetilde{\varepsilon}_N + \left(\sum_{m\in I_{2^n}}{}^{*}\widetilde{\varepsilon}_N^2\frac{1}{(m,m)^2}\right)^{\frac{1}{2}} \ll$$

$$\ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}\sum_{2^j\leq\|m\|<2^{j+1}}\frac{1}{(m_1^2+...+m_s^2)^2}\right)^{\frac{1}{2}}\right) \ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}\frac{1}{2^{4j}}\sum_{2^j\leq\|m\|<2^{j+1}}1\right)^{\frac{1}{2}}\right) \ll$$

$$\ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}2^{-4j}\cdot 2^{js}\right)^{\frac{1}{2}}\right) \asymp \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}2^{j(s-4)}\right)^{\frac{1}{2}}\right).$$

If $s<4$, then

$$\|I_2\|_{L^2} \ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}2^{j(s-4)}\right)^{\frac{1}{2}}\right) \asymp \widetilde{\varepsilon}_N \asymp N^{-\frac{r+2}{s}}.$$

If $s=4$, then

$$\|I_2\|_{L^2} \ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}1\right)^{\frac{1}{2}}\right) \asymp \widetilde{\varepsilon}_N\cdot n^{\frac{1}{2}} \asymp \widetilde{\varepsilon}_N\cdot(lnN)^{\frac{1}{2}} \asymp (lnN)^{-\frac{1}{2}}\cdot N^{-\frac{r+2}{4}}\cdot(lnN)^{\frac{1}{2}} \asymp N^{-\frac{r+2}{4}}.$$

If $s>4$, then

$$\|I_2\|_{L^2} \ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}2^{j(s-4)}\right)^{\frac{1}{2}}\right) \asymp \widetilde{\varepsilon}_N\cdot 2^{\frac{n(s-4)}{2}} \asymp \widetilde{\varepsilon}_N\cdot N^{\frac{1}{2}-\frac{2}{s}} \asymp N^{-\frac{r}{s}-\frac{1}{2}}\cdot N^{\frac{1}{2}-\frac{2}{s}} \asymp N^{-\frac{r+2}{s}}.$$

Then, for $f\in H_2^r$ and $\{\gamma_N^{(\tau)}\}_{\tau=1}^N$, such that $|\gamma_N^{(\tau)}|\leq 1$ $(\tau=1,...,N)$ satisfies

$$\left\|u_\omega(x,f)-\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)})+\widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)},...,\widehat{f}(\overline{m}^{(N)})+\widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)};x)\right\|_{L^2} \ll \|I_1\|_{L^2}+\|I_2\|_{L^2} \ll N^{-\frac{r+2}{s}}.$$

Further, by the arbitrariness of the function $f\in H_2^r$ and$\{\gamma_N^{(\tau)}\}_{\tau=1}^N$, $|\gamma_N^{(\tau)}|\leq 1$ $(\tau=1,...,N)$

$$\sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N,|\gamma_N^{(\tau)}|\leq 1,\\ (\tau=1,...,N)}} \left\|u_\omega(x,f)-\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)})+\widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)},...,\widehat{f}(\overline{m}^{(N)})+\widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)};x)\right\|_{L^2} \ll N^{-\frac{r+2}{s}}.$$

In the end, we obtain the required upper bound in C(N)D-2

$$\delta_N(\widetilde{\varepsilon}_N;D_N)_{L^2} \equiv \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s;\\ \varphi_N}} \sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N,|\gamma_N^{(\tau)}|\leq 1,\\ (\tau=1,...,N)}} \left\|u_\omega(x,f)-\varphi_N(\widehat{f}(m^{(1)})+\widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)},...,\right.$$

$$\left.\widehat{f}(m^{(N)})+\widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)};x)\right\|_{L^2} \ll N^{-\frac{r+2}{s}}$$

and, by the definition of $\delta_N(0;D_N)_{L^2}$ and $\delta_N(\tilde{\varepsilon}_N;D_N)_{L^2}$,

$$\delta_N(0;D_N)_{L^2} \ll \delta_N(\tilde{\varepsilon}_N;D_N)_{L^2} \ll N^{-\frac{r+2}{s}}. \tag{9}$$

Let's evaluate lower bound for $\delta_N(0,D_N)_{L^2}$. Now, let us prove the lower bound in the case of approximation from accurate information. Let are given an integer $N \geq 1$ and set $A_N = \{m^{(1)},...,m^{(N)} : m^{(j)} \in Z^s (j=1,...,N)\}$. According to the choice of D_N, we define the functionals $l_N^{(1)}(f) = \widehat{f}(m^{(1)}),...,l_N^{(N)}(f) = \widehat{f}(m^{(N)})$. Let $\varphi_N(\tau_1,...,\tau_N;x)$ also be an arbitrary algorithm for processing information, such that $\varphi_N(0,...,0;x) = 0$. We define an integer $n = n(s,N) \geq 1$ from the conditions $|I_{2^n}| \geq 2N$ and $|I_{2^n}| \asymp N$.

Let consider the function

$$\overline{g}(x) = \sum_{m\in I_{2^n}\setminus A_N}{}^{*} \overline{a}_n^{(m)} e^{2\pi i(m,x)}, \tag{10}$$

where $\overline{a}_n^{(m)} = k_j(m) \equiv k(j,n,s)$ when $[2^{j(m)-1}] \leq \|m\| < 2^{j(m)}$, $m \in I_{2^n}\setminus A_N$, $j=0,1,...,n$. For the number of points of $I_{2^n}\setminus A_N$:

$$N \asymp |I_{2^n}| \geq |I_{2^n}\setminus A_N| \geq |I_{2^n}| - |A_N| \geq 2N - N = N,$$

therefore

$$|I_{2^n}\setminus A_N| \asymp N.$$

By using Parseval's equality, let define the norm of $\overline{g}$

$$\|\overline{g}\|_{H_2^r} = \sup_{j=0,1,...,n} 2^{jr} \| \sum_{\substack{[2^{j-1}]\leq\|m\|<2^j,\\ m\notin A_N}}{}^{*} k_j e^{2\pi i(m,x)}\|_{L^2} =$$

$$= \sup_{j=0,1,...,n} 2^{jr} \left(\sum_{\substack{[2^{j-1}]\leq\|m\|<2^j,\\ m\notin A_N}}{}^{*} |k_j|^2 \right)^{\frac{1}{2}} = \sup_{j=0,1,...,n} 2^{jr}\cdot k_j \left(\sum_{\substack{[2^{j-1}]\leq\|m\|<2^j,\\ m\notin A_N}}{}^{*} 1 \right)^{\frac{1}{2}} \ll$$

$$\ll \sup_{j=0,1,...,n} 2^{jr}\cdot k_j \cdot 2^{\frac{sj}{2}} = \sup_{j=0,1,...,n} 2^{j(r+\frac{s}{2})}\cdot k_j.$$

k_j is defined from the condition $\|\overline{g}\|_{H_2^r} \ll \sup_{j=0,1,...,n} 2^{j(r+\frac{s}{2})}\cdot k_j \asymp 1$ (in that case $\overline{g}$ belong to H_2^r class)

$$k_j = 2^{-j(r+\frac{s}{2})}, j = 0,1,...,n. \tag{11}$$

By putting (11) into (10), there are exist a positive constant $c(s)$ such that

$$g(x) = c(s)\overline{g}(x) = c(s) \sum_{m\in I_{2^n}\setminus A_N}{}^{*} \overline{a}_n^{(m)} e^{2\pi i(m,x)} = c(s) \sum_{j=0}^{n} \sum_{\substack{m\in I_{2^n}\setminus A_N,\\ [2^{j-1}]\leq\|m\|<2^j}}{}^{*} 2^{-j(r+\frac{s}{2})} e^{2\pi i(m,x)}. \tag{12}$$

Then, according to definition of $g(x)$ satisfies $l_N^{(1)}(g) = \widehat{g}(m^{(1)}) = 0,...,l_N^{(N)}(g) = \widehat{g}(m^{(N)}) = 0$, so it should be $\varphi_N(\widehat{g}(m^{(1)}),...,\widehat{g}(m^{(N)});\cdot) = 0$. Then for the lower bound of error of approximation by accurate information we have

$$\sup_{f\in H_2^r(0,1)^s} \left\| u_\omega(x,f) - \varphi_N\left(\widehat{f}(m^{(1)}),...,\widehat{f}(m^{(N)});x\right) \right\|_{L^2} \geq$$

$$\geq \left\| u_\omega(x,g) - \varphi_N\left(\widehat{g}(m^{(1)}),...,\widehat{g}(m^{(N)});x\right)\right\|_{L^2} = \|u_\omega(x,g)\|_{L^2}.$$

By definition of function g satisfies $\widehat{g}(0)=0$. Let calculate the error by using Parseval's equality.

$$\|u_\omega(x,g)\|_{L^2}^2 = \left\| \omega(x)\cdot\widehat{g}(0) - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}\setminus A_N}{}^{*}\frac{\overline{a}_n^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2(0,1)^s}^2 \asymp$$

$$\asymp \left\| -\frac{1}{4\pi^2}\sum_{m\in I_{2^n}\setminus A_N}{}^{*}\frac{\overline{a}_n^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2(0,1)^s}^2 \asymp \sum_{m\in I_{2^n}\setminus A_N}{}^{*}\frac{|\overline{a}_n^{(m)}|^2}{(m,m)^2}\asymp$$

$$\asymp \sum_{j=0}^{n-1}\sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*}\left(2^{-2j(r+\frac{s}{2})}\frac{1}{m_1^2+...+m_s^2}\right)^2 \gg$$

$$\gg 2^{-2n(r+\frac{s}{2})}\sum_{j=0}^{n-1}\sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*}\left(\frac{1}{\max_{j=1,...,s}|m_j|^2}\right)^2 \gg$$

$$\gg 2^{-2n(r+\frac{s}{2})}\sum_{j=0}^{n-1}\frac{1}{2^{4(j+1)}}\sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*}1 \gg$$

$$\gg 2^{-2n(r+\frac{s}{2})-4n-4}\cdot\sum_{j=0}^{n-1}\sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*}1 \asymp 2^{-2n(r+\frac{s}{2})-4n-4}\cdot|I_{2^n}\setminus A_N| \asymp$$

$$\asymp 2^{-2nr-ns-4n}\cdot 2^{ns}\asymp 2^{-2nr-4n}\asymp N^{-\frac{2r}{s}-\frac{4}{s}}.$$

Finally, for (4) we have

$$\sup_{f\in H_2^r}\left\| u_\omega(x,f) - \varphi_N(\widehat{f}(m^{(1)}),...,\widehat{f}(m^{(N)});x)\right\|_{L^2} \gg N^{-\frac{r+2}{s}}. \tag{13}$$

Then, due to the arbitrariness of $m^{(1)},...,m^{(N)}$ from Z^s and the information processing algorithm φ_N, satisfies

$$\delta_N(0,D_N)_{L^2}\equiv \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s;\\ \varphi_N}}\sup_{f\in H_2^r}\left\| u_\omega(x,f) - \varphi_N(\widehat{f}(m^{(1)}),...,\widehat{f}(m^{(N)});x)\right\|_{L^2} \gg N^{-\frac{r+2}{s}}. \tag{14}$$

As a result, by (9) and (14) we have (8)

$$\delta_N(0,D_N)_{L^2}\asymp\delta_N(\widetilde{\varepsilon}_N,D_N)_{L^2}\asymp N^{-\frac{r+2}{s}}.$$

Theorem 1 is proven.

Theorem 2. Let are given positive integer s and $r>s/2$. Then the following statements hold($N=(2^{n+1}+1)^s$, $n=1,2,...$)

C(N)D-1:

$$\delta_N(0; D_N)_{L^\infty} \equiv$$

$$\equiv \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s;\\ \varphi_N}} \sup_{f\in H_2^r} \left\| u_\omega(\cdot, f) - \varphi_N(\widehat{f}(m^{(1)}), ..., \widehat{f}(m^{(N)}); \cdot)\right\|_{L^\infty(0,1)^s} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \quad (15)$$

C(N)D-2 (first part): For the computational aggregates $\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}), x)$ from (5) and for the numerical sequence

$$\widetilde{\varepsilon}_N \asymp \begin{cases} N^{-r-\frac{3}{2}}, if s = 1, \\ (lnN)^{-1} \cdot N^{-\frac{r+1}{2}}, if s = 2, \\ N^{-\frac{r}{s}-\frac{1}{2}}, if s > 2. \end{cases} \quad (16)$$

satisfy

$$\delta_N(0; D_N)_{L^\infty(0,1)^s} \asymp \delta_N\left(\tilde{\varepsilon}_N; D_N\right)_{L^\infty(0,1)^s} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \quad (17)$$

Proof. The proof will be carried out similarly by Theorem 1. Let are given $f \in H_2^r$ positive integer n, $N = |I_{2^n}| = (2^{n+1}+1)^s$ and $\{\gamma_N^{(\tau)}\}_{\tau=1}^N \equiv \{\gamma_N^{(m)}\}_{m\in I_{2^n}}$, such that $|\gamma_N^{(\tau)}| \leq 1$. Then for the error of approximation by computational aggregates (5)-(6) by inaccurate information ($L^\infty \equiv L^\infty(0,1)^s$)

$$\left\| u_\omega(x, f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}) + \widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)}, ..., \widehat{f}(\overline{m}^{(N)}) + \widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)}; x)\right\|_{L^\infty} \leq$$

$$\leq \left\| u_\omega(x, f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x)\right\|_{L^\infty} +$$

$$+ \left\| \omega(x)\widetilde{\varepsilon}_N\gamma_N^{(0)} - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}}{}^{*} \frac{\widetilde{\varepsilon}_N\gamma_N^{(m)}}{(m,m)} e^{2\pi i(m,x)}\right\|_{L^\infty} \equiv \|I_3\|_{L^\infty} + \|I_4\|_{L^\infty}.$$

Let's estimate from above $\|I_3\|_{L^\infty}$ and $\|I_4\|_{L^\infty}$

$$\|I_3\|_{L^\infty} \equiv \left\| u_\omega(x, f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x)\right\|_{L^\infty} =$$

$$= \left\| -\frac{1}{4\pi^2}\sum_{m\in Z^s\setminus I_{2^n}} \frac{\widehat{f}(m)}{(m,m)} e^{2\pi i(m,x)}\right\|_{L^\infty} \ll \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}} \frac{|\widehat{f}(m)|}{|(m,m)|} \ll$$

$$\ll \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}} \frac{|\widehat{f}(m)|}{m_1^2+...+m_s^2} \ll \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}} \frac{|\widehat{f}(m)|}{\max_{j=1,...,s}|m_j|^2} \ll$$

$$\ll \sum_{j=n+1}^{+\infty} 2^{-2j} \sum_{2^j\leq\|m\|<2^{j+1}} |\widehat{f}(m)|.$$

Applying Holder's inequality and (2), we will get required upper bound

$$\|I_3\|_{L^\infty} \ll 2^{-2n}\sum_{j=n+1}^{+\infty}\left(\sum_{2^j\leq\|m\|<2^{j+1}} |\widehat{f}(m)|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{2^j\leq\|m\|<2^{j+1}} 1\right)^{\frac{1}{2}} \ll$$

$$\ll 2^{-2n} \sum_{j=n+1}^{+\infty} 2^{-j\left(r-\frac{s}{2}\right)} \cdot 2^{jr} \left\| \sum_{2^j \le \|m\| < 2^{j+1}} \widehat{f}(m) e^{2\pi i(m,x)} \right\|_{L^2} \ll 2^{-2n-nr+\frac{ns}{2}} = N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.$$

From the upper bound for $\|I_3\|_{L^\infty}$, we obtain the upper bound for (15) the approximation by the accurate information

$$\delta_N(0; D_N)_{L^\infty} \equiv$$

$$\equiv \inf_{\substack{m^{(1)} \in Z^s, \dots, m^{(N)} \in Z^s; \\ \varphi_N}} \sup_{f \in H_2^r} \left\| u_\omega(\cdot, f) - \varphi_N(\widehat{f}(m^{(1)}), \dots, \widehat{f}(m^{(N)}); \cdot) \right\|_{L^\infty(0,1)^s} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \quad (18)$$

Then evaluate of $\|I_4\|_{L^\infty}$ (see also (16))

$$\|I_4\|_{L^\infty} \ll \widetilde{\varepsilon}_N + {\sum_{m \in I_{2^n}}}^{*} \widetilde{\varepsilon}_N \frac{1}{4\pi^2(m,m)} \ll \widetilde{\varepsilon}_N \left(1 + \sum_{j=0}^{n-1} \frac{1}{2^{2j}} \sum_{2^j \le \|m\| < 2^{j+1}} 1 \right) \ll \widetilde{\varepsilon}_N \left(1 + \sum_{j=0}^{n-1} 2^{j(s-2)} \right).$$

If $s = 1$, then

$$\|I_4\|_{L^\infty} \ll \widetilde{\varepsilon}_N \asymp N^{-r-\frac{3}{2}}.$$

If $s = 2$, then

$$\|I_4\|_{L^\infty} \ll \widetilde{\varepsilon}_N \left(1 + \sum_{j=0}^{n-1} 1 \right) \asymp \widetilde{\varepsilon}_N \cdot n \asymp \widetilde{\varepsilon}_N \cdot lnN \asymp (lnN)^{-1} \cdot N^{-\frac{r}{2}-\frac{1}{2}} \cdot lnN \asymp N^{-\frac{r}{2}-\frac{1}{2}}.$$

If $s > 2$, then

$$\|I_4\|_{L^\infty} \ll \widetilde{\varepsilon}_N \left(1 + \sum_{j=0}^{n-1} 2^{j(s-2)} \right) \asymp \widetilde{\varepsilon}_N \cdot 2^{n(s-2)} \asymp \widetilde{\varepsilon}_N \cdot N^{1-\frac{2}{s}} \asymp N^{-\frac{r}{s}-\frac{1}{2}} \cdot N^{1-\frac{2}{s}} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.$$

Finally, by estimation from above $\|I_3\|_{L^\infty}$ and $\|I_4\|_{L^\infty}$ we obtain the required upper bounds in (17)

$$\delta_N(0, D_N)_{L^\infty} \ll \delta_N(\widetilde{\varepsilon}_N, D_N)_{L^\infty} \equiv$$

$$\equiv \inf_{\substack{m^{(1)} \in Z^s, \dots, m^{(N)} \in Z^s; \\ \varphi_N}} \sup_{\substack{f \in H_2^r, \\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N, |\gamma_N^{(\tau)}| \le 1, \\ (\tau=1,\dots,N)}} \Big\| u_\omega(x, f) - \varphi_N(\widehat{f}(m^{(1)}) + \widetilde{\varepsilon}_N^{(1)} \gamma_N^{(1)}, \dots, \widehat{f}(m^{(N)}) +$$

$$+ \widetilde{\varepsilon}_N^{(N)} \gamma_N^{(N)}; x) \Big\|_{L^\infty} \ll N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \quad (19)$$

A lower bound in the case of approximation from accurate information gives the desired relation. Suppose we are given an integer $N \ge 1$, N linear functionals $l_N^{(1)}(f) = \widehat{f}(m^{(1)}), \dots, l_N^{(N)}(f) = \widehat{f}(m^{(N)}), \{m^{(1)}, \dots, m^{(N)}\} \in Z^s$ and a function $\varphi_N(\tau_1, \dots, \tau_N; x)$, $\varphi_N(0, \dots, 0; x) = 0$. We define an integer $n = n(s, N) \ge 1$ from the conditions $|I_{2^n}| \ge 2N$ and $|I_{2^n}| \asymp N$.

Let consider the function

$$g(x) = c(s) N^{-\frac{r}{s}-\frac{1}{2}} \sum_{m \in I_{2^n} \setminus A_N}{}^{*} e^{2\pi i (m,x)} \in H_2^r.$$

where $c(s)$ is a positive constant, defined so that $g(x) \in H_2^r$.

Then, for the lower bound of error of approximation by accurate information

$$\sup_{f \in H_2^r} \left\| u_\omega(x,f) - \varphi_N\left(\widehat{f}(m^{(1)}), ..., \widehat{f}(m^{(N)}); x\right) \right\|_{L^\infty} \geq$$

$$\geq \sup_{f \in H_2^r} \left\| u_\omega(x,g) - \varphi_N\left(\widehat{g}(m^{(1)}), ..., \widehat{g}(m^{(N)}); x\right) \right\|_{L^\infty} \geq$$

$$\geq \left\| u_\omega(x,g) - \varphi_N(0,...,0;x) \right\|_{L^\infty} = \left\| u_\omega(x,g) \right\|_{L^\infty}.$$

Let estimate from below the norm of the solution.

$$\left\| u_\omega(x,g) \right\|_{L^\infty} = \left\| -\frac{1}{4\pi^2} \sum_{m \in I_{2^n} \setminus A_N}{}^{*} \frac{N^{-\frac{r}{s}-\frac{1}{2}}}{(m,m)} e^{2\pi i (m,x)} \right\|_{L^\infty} =$$

$$= \sup_{x \in [0,1]^s} \left| -\frac{1}{4\pi^2} \sum_{m \in I_{2^n} \setminus A_N}{}^{*} \frac{N^{-\frac{r}{s}-\frac{1}{2}}}{(m,m)} e^{2\pi i (m,x)} \right| \geq \left| -\frac{1}{4\pi^2} \sum_{m \in I_{2^n} \setminus A_N}{}^{*} \frac{N^{-\frac{r}{s}-\frac{1}{2}}}{(m,m)} \right| \gg$$

$$\gg 2^{-n(r+\frac{s}{2})} \sum_{j=0}^{n-1} \frac{1}{2^{2(j+1)}} \sum_{\substack{2^j \leq \|m\| < 2^{j+1}, \\ m \notin A_N}}{}^{*} 1 \asymp$$

$$\asymp 2^{-n(r+\frac{s}{2})-2n-2} \cdot \sum_{j=0}^{n-1} \sum_{\substack{2^j \leq \|m\| < 2^{j+1}, \\ m \notin A_N}}{}^{*} 1 \asymp 2^{-n(r+\frac{s}{2})-2n-4} \cdot |I_{2^n} \setminus A_N| \asymp$$

$$\asymp 2^{-nr-2n+\frac{ns}{2}} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.$$

As a result,

$$\delta_N(0, D_N)_{L^\infty} \gg N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \tag{20}$$

Then by (19) and (20) we have

$$\delta_N(0, D_N)_{L^\infty} \asymp \delta_N(\widetilde{\varepsilon}_N, D_N)_{L^\infty} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.$$

Theorem 2 is proven.

4 Conclusion

In the present paper, the problem of the approximation of solutions of the Poisson equation with right-hand side from the Nikol'skii classes $H_2^r(0,1)^s$ by accurate and inaccurate information of the trigonometric Fourier coefficients in the sense of C(N)D-1 and the first part of C(N)D-2 is considered.

Firstly, two-sided estimates for the error $\delta_N(0; D_N)_Y$ $(Y = L^(0,1)^s$ and $Y = L^\infty(0,1)^s)$ of approximation by accurate informmation were obtained (C(N)D-1 problem) with indicating a computational aggregate that confirms the lower bound. For this computational aggregate, bounds arises of inaccurate information that preserve the order of the error of approximation by accurate information were found—the first part of problem C(N)D-2.

References

[1] Temirgaliev N., Zhubanysheva A.Zh. Computational (Numerical) Diameter in a Context of General Theory of a Recovery, Iz.Vuz. 63(1), (2019), p. 79-86.

[2] Temirgaliev N., Zhubanysheva A.Zh. Informative Cardinality of Trigonometric Fourier Coefficients and Their Limiting Error in the Discretization of a Differentiation Operator in Multidimensional Sobolev Classes, Computational Mathematics and Mathematical Physics. 55(9), (2015), p. 1432-1443.

[3] Korobov N.M., Numerical-theoretic methods in approximate analysis [in Russian], Fizmatlit, Moscow (1963), p. 187-189.

[4] Bailov E.A., Temirgaliev N. Discretzation of the solutions Poisson's equation, Computational Mathematics and Mathematical Physics. 46(9), (2006), p. 1515-1525.

[5] Sickel W. and Ullrich T. The Smolyak's algorithm, sampling on sparse grids and function spaces of dominating mixed smoothness, East J. Approx. 73(4), (2007), p.287-425.

[6] Vybiral J. Sampling numbers and function spaces, journal of complexity. 23, (2007), p. 773-792.

[7] Zhi Shi, Yong-yan Cao, Qing-jiang Chen. Solving 2D and 3D Poisson equations and biharmonic equations by the Haar wavelet method, Applied Mathematical Modelling. 36, (2012), p. 5143-5161.

[8] Kudaibergenov S.S., Sabitova S.G. Discretization of Solutions to Poisson's equation in the Korobov class, Computational Mathematics and Mathematical Physics. 57(3), (2013), p. 896-907.

[9] Tashatov N.N. Approximate recovery of functions and solutions of the Poisson equation with initial condition from anisotropic classes E and SW [in Russian]. diss. ... kand. fiz.-mat. nauk. Karaganda. (2002)

[10] Bailov E.A. Approximative integration and restoration of the functions from anisotropic classes and restoration of the solutions of the Poisson equation [in Russian]. diss. ... kand. fiz.-mat. nauk. Almaty. (1998)

[11] Arystangalikyzy A. Discretization of solutions of Poisson equation by inaccurate information, Bulletin of the L.N.Gumilov Eurasian National University. Mathematics. Computer Science. Mechanics Series. 144(3), (2023), p. 39-44.

[12] Utesov, A.B. On Error Estimates for Discretization Operators for the Solution of the Poisson Equation, Differential Equations. 60(1), (2024), p. 136-143.

[13] Utesov A.D., Shanauov R.A. On the optimal discretisation of the solution Poisson's equation, Journal of Mathematics, Mechanics and Computer Science. 2(122), (2024), p. 67-76.

[14] Kudryavtsev L.D., Nikol'skii S.M. Spaces of differentiable functions of several variables and imbedding theorems, Contemporary Problems in Mathematics: Fundamental Directions. 26, (1988), p. 5-157.

Авторлар туралы мәлімет:

Ақмарал Арыстангалиқызы (корреспондент автор) – Қ.Жұбанов атындағы Ақтөбе өңірлік университетінің PhD студенті (Ақтөбе, Қазақстан, email: arystangalikyzya@gmail.com);

Аксауле Жубанышева Жанбыршиевна – PhD, Л.Н.Гумилев атындағы Еуразия ұлттық университетінің доцент м.а (Астана, Қазақстан, email: zhubanysheva_azh@enu.kz);

Информация об авторах:

Акмарал Арыстангаликызы (корреспондент автор) – PhD студент Актюбинского регионального университета имени К. Жубанова (Актобе, Казахстан, email: arystangalikyzya@gmail.com);

Аксауле Жубанышева Жанбыршиевна – PhD, и.о. доцента Евразийского национального университета им. Л.Н. Гумилева (Астана, Казахстан, email: zhubanysheva_azh@enu.kz);

Information about authors:

Akmaral Arystangalikyzy (corresponding author) – PhD student of K. Zhubanov Aktobe Regional University (Aktobe, Kazakhstan, email: arystangalikyzya@gmail.com);

Aksaule Zhubanysheva – PhD, Acting associate professor of L.N.Gumilyov Eurasian National University (Astana, Kazakhstan, email: zhubanysheva_azh@enu.kz);

Received: 6 May, 2025
Accepted: 17 September, 2025