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RECOVERING A SURFACE IN ISOTROPIC SPACE USING DUAL MAPPING ACCORDING TO CURVATURE INVARIANTS

The problem of recovering a surface according to its curvature is one of the fundamental problems of differential geometry. Problems of recovering surfaces in various spaces by their total or mean curvature have been widely studied in many works.Recovering of a surface by its total curvature is equivalent to solving the Monge-Ampere equation of elliptic type; such problems are solved in special cases. When the right part is given concretely.The Monge-Ampere equation is solved using a dual mapping of isotropic space, in which the dual surface is a transfer surface. Also, some special cases are used to find the surface equation.The connection between dual mean curvature and amalgamatic curvature is studied.The equivalence of the problem of recovering by dual mean and amalgamatic curvature is shown. In particular, the problem of recovering surfaces with total negative constant curvature, the mean curvature of which is a function of one variable, is solved. Furthermore, the problems of the recovering surfaces are solved according to their dual mean curvature, amalgamatic and Casorati curvatures.
Key words: isotropic space, Monge-Ampere equation, dual mapping, amalgamatic curvature, Casorati curvature.

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Қисықтық инварианттары бойынша дуаль бейнелеуді қолдану аркылы изотроптық кеңістікте бетті қалпына келтіру

Беткейді оның қисығына қарап қалпына келтіру мәселесі – дифференциалдық геометриядағы негізги міндеттердің бірі болып табылады.Әртүрлі кеңістіктерде беткейлерді толық немесе орташа қисығына қарап қалпына келтіру мәселелері көптеген еңбектерде кеңінен зерттелген. Беткейді оның толық қисығына қарап қалпына келтіру Монж-Ампердің эллиптикалық типтегі теңдеуін шешуге тең. Мұндай есептер кейбір жекелеген жағдайларда, оң жақ бөлігі нақты берілген кезде шешілген. Монж–Ампер теңдеуі изотропты кеңістіктегі дуальды бейнелеу арқылы шешіледі, мұнда дуальды беткей бұл көшу беткейі болып табылады. Сондайақ беткей теңдеуін табу үшін кейбір жеке жағдайлар қолданылған.Дуальды орташа қисығымен және амальгамалық қисығымен байланыс зерттелген. Дуальды орташа қисық пен амальгамалық қисық бойынша қалпына келтіру мәселесінің эквиваленттілігі көрсетілген. Атап айтқанда, толық теріс тұрақты қисыққа ие беткейлерді, олардың орташа қисығы бір айнымалыға тәуелді болған жағдайда, қалпына келтіру есебі шешілген. Сонымен қатар, беткейлерді дуальды орташа қисығына, амальгамалық және Касорати қисығына сәйкес қалпына келтіру есептері де қарастырылған.

Түйін сөздер: Изотропты кеңістік, Монж–Ампер теңдеуі, дуаль бейнелеу, амальгаматик қисықтық, Касорати қисықтығы.

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Восстановление поверхности в изотропном пространстве с использованием двойственного отображения по инвариантам кривизны

Задача восстановления поверхности по ее кривизне является одной из основных задач дифференциальной геометрии. Задачи восстановления поверхностей в различных пространствах по их полной или средней кривизне широко изучались во многих работах. Восстановление поверхности по ее полной кривизне эквивалентно решению уравнения Монжа-Ампера эллиптического типа, такие задачи решена в частных случаях, когда правая част дана конкретно. Уравнение Монжа-Ампера решается с помощью двойственного отображения изотропного пространства, в котором двойственная поверхность является поверхностью переноса. Также для нахождение уравнению поверхности использована некоторые частные случае. Изучается связь между дуальной средней кривизной и амальгаматической кривизной. Показана эквивалентность задачи восстановления по двойственному среднему и амальгаматической кривизне. В частности, решена задача восстановления поверхностей с полной отрицательной постоянной кривизной, средняя кривизна которых является функцией одной переменной. Кроме того, задачи восстановления поверхностей решаются в соответствии с их двойной средней кривизной, амальгаматической и кривизной Касорати.

Ключевые слова: Изотропное пространство, уравнение Монжа-Ампера, дуальное отображение, амальгаматическая кривизна, кривизна Касорати.

1 Introduction

K.Strubecker studied the basic concepts related to isotropic geometry [1, 2]. Currently, many mathematicians are conducting scientific research on isotropic space. M.E. Aydin studied the types of transfer surfaces by a given constant curvature in isotropic space [3, 4]. Z.M. Sipus found equations of transfer surfaces by a given constant Gaussian and mean curvature in 3-dimensional isotropic space.Also she studied transfer Wiengarten surfaces in this space [5]. M.Karacan ,B.Bukcu, D.Yoon and N.Yuksel investigated transfer and ruled surfaces satisfying [6, 7]:

$$\Delta^J x_i = \lambda_i x_i$$

A.Cakmak, S.Kiziltug, M.Karacan found dual surface for the surface $z = f(u)+g(v)$ satisfying the condition

$$\Delta^J x_i^* = \lambda_i x_i^*$$

in 3-dimensional isotropic space. Besides that they solved the recovering problem of the transfer dual surface by given non-zero total and mean curvatures [8]. Several mathematicians solved the Monge-Ampere equation for transfer surfaces in some special cases. In the article [9], M.S.Lone,M.K.Karacan solved the problem of recovering a given dual transfer surface with total curvature being constant. Sh. Ismoilov solved this problem by given total curvature being the product of two functions with separate variables [10]. Moreover, in the article of A.Artykbaev and Sh. Ismoilov [11, 12], the connection of total curvatures between the given surface and dual surface is proved. In Euclidean space, A.D.Alexandrov solved the problem of existence and uniqueness of a surface by a given external curvature [13]. I.Y.Bakelman presented a solution to the Dirichlet problem for the elliptic Monge-Ampere equation related to this geometric problem [14].

In addition to the problem of recovering a surface from its total curvature, one of the important problems of differential geometry is also the problem of recovering it from its mean curvature. In many works, the problem of recovering a surface by its total or mean curvature was solved in different special cases. However, in addition to these geometric characteristics, the problem of surface recovering can be considered by other curvature invariants. In surface theory, there are the amalgamatic and Casorati curvatures, which are associated with principal normal curvatures that differ from the total and mean curvatures. Amalgamatic curvature in Euclidean space was studied by Suceava and a calculation formula was found [15]. Decu and Verstraelen investigated isotropic Casorati curvature [16]. The problem of recovering a surface in isotropic space by amalgamatic and Casorati curvatures was solved, where these curvatures are equal to zero and constant for surfaces with a total curvature of -1 [17]. In this work, we find the surface equation by solving the Monge-Ampere equation, in the case of that the dual surface is a transfer surface. Also, by studying the connection between the amalgamatic curvature of a surface and the mean curvature of its dual surface, we find the equation of the surface for surfaces with total curvature of -1 in isotropic space, where the mean curvature is a differentiable function of one variable.

2 Preliminaries

2.1 Geometry of isotropic space and duality

Let there be given an affine space A_3 with the coordinate system $Oxyz$. We consider $\vec{X}\{x_1, y_1, z_1\}$ and $\vec{Y}\{x_2, y_2, z_2\}$ vectors in A_3.

Definition 1 *If the scalar product of the two vectors* $\vec{X}\{x_1, y_1, z_1\}$ *and* $\vec{Y}\{x_2, y_2, z_2\}$ *is defined by the following formula:*

$$\begin{cases} (X,Y)_1 = x_1x_2 + y_1y_2 & if \quad (X,Y)_1 \neq 0 \\ (X,Y)_2 = z_1z_2 & if \quad (X,Y)_1 = 0 \end{cases}$$

then, the affine space A_3 *is the isotropic space and denoted by* R_3^2.

Two types of spheres are defined in isotropic space [18]. The first is the metric sphere, which is given by the following formula:

$$x^2 + y^2 = r^2 \tag{1}$$

where $(0, 0, z)$ is the center, r is radius.

The second sphere in isotropic space is defined as follows [18]:

$$x^2 + y^2 = 2z \tag{2}$$

It is called the isotropic sphere. Consider a plane Π in this space. Let this plane not be parallel to the axis Oz. The section of this sphere by the plane Π, forms a closed curve .This curve is an ellipse and denote it by γ [11]. Pass tangent planes to the isotropic sphere (2) through the points $P \in \gamma$. We denote the set of these planes to points Φ by $\{\Pi\}$. We get the following:

Theorem 1 *All planes belonging to the set* $\{\Pi\}$ *intersect at one point [11].*

If the plane Π_0 is as follows:

$$z = A_0x + B_0y + C_0 \tag{3}$$

then the intersection point of these planes belonging to the set $\{\Pi\}$ is $(A_0, B_0, -C_0)$.

Definition 2 *The point* $(A_0, B_0, -C_0)$ *is a dual point to the plane* (3) *with respect to the isotropic sphere* (2) *in the isotropic space [11].*

Let there be given a plane $z = T$. And γ be the section of this plane on the isotropic sphere. Consider a surface Φ that is given by the following:

$$\Phi : \{ z = f(x, y) | (x, y) \in D \} \tag{4}$$

And the curve γ be the boundary of the surface (4). The surface (4) is convex and it is located inside the part of the isotropic sphere bounded by the plane.

Let us pass a tangent plane Π_P to the given surface Φ at a point $P(x_0, y_0, z_0)$. Let us denote by P^* the dual image of the tangent plane Π_P with respect to the isotropic sphere (2). If the given point $P \in \Phi$ changes on the surface Φ, the dual image of this point forms a surface Φ^*.

Definition 3 *The surface* Φ^* *is called the dual surface to the given surface* Φ *in the isotropic space. If* Φ *has the following form, i.e.:*

$$z = f(x, y)$$

then the parametric equations for the dual surface Φ^* *are:*

$$\begin{cases} x^*(u, v) = f_u{}'(u, v) \\ y^*(u, v) = f_v{}'(u, v) \\ z^*(u, v) = u \cdot f_u{}'(u, v) + v \cdot f_v{}'(u, v) - f(u, v) \end{cases} \tag{5}$$

The above equation (5)is the dual mapping in isotropic space [10]. Following connection is valid between the total curvatures for the given surface Φ and its dual surface Φ^*

Theorem 2 *For the product of total curvatures* K *and* K^**, the following holds [11]:*

$$K \cdot K^* = 1 \tag{6}$$

From this, the total curvature K^* is equal to the following:

$$K^* = \frac{1}{K} \tag{7}$$

The following equality holds for the mean curvature of a given surface and the mean curvature of its dual surface:

$$H^* = \frac{H}{K} \tag{8}$$

The question "Can the result obtained from the problem solved for the dual surface be applied to the Φ surface?"is considered important. If we apply a dual mapping to the dual surface again, then we have the following theorem that the dual image of a dual surface is equal to the given surface, that is:

Theorem 3 *The dual image of the dual surface* Φ^* *coincides with the given surface* Φ *[19]:*

$$\Phi^{**} = \Phi \tag{9}$$

2.2 Transfer surfaces

M.E.Aydin classified transfer surfaces and found equations for these surfaces in the case that their total and mean curvatures are constant [3]. M.S.Lone, M.K.Karacan found a dual surface by a given constant total and mean curvatures of this surface [9]. Sh.Sh.Ismoilov solved in the case that the total curvature for the transfer surface is the product of two functions with separate variables for this class [12]. In general, the vector equation of the transfer surface can be expressed as the sum of two isotropic planar curves in isotropic space:

$$\overline{r}\,(u,v)=\overline{\rho}\,(u)+\overline{\sigma}\,(v)$$

where, $\overline{\rho}\,(u)$ and $\overline{\sigma}\,(v)$ are the vector forms of these curves. The surface is one-valued projected onto the Oxy plane. Let this surface not be parallel to the axis Oz in the isotropic space, then we obtain the following:

$$\overline{r}\,(u,v)=u\overline{i}+v\overline{j}+(f(u)+g(v))\,\overline{k}$$

where, $\overline{\rho}\,(u)=(u,0,f(u))$ and $\overline{\sigma}\,(v)=(0,v,g(v))$.

3 Solving the Monge-Ampere equation by using duality

The Monge-Ampere equation is generally as follows:

$$z_{xx}z_{yy}-z_{xy}^2=\varphi(x,y,z,z_x,z_y) \tag{10}$$

where the function $\varphi(x,y,z,z_x,z_y)-$ is the given function. In this paper, for

$$z_{xx}z_{yy}-z_{xy}^2=\varphi(z_x,z_y) \tag{11}$$

we will find the solution. If a regular surface is given by the following form

$$z=z(x,y),\quad (x,y)\in D\subset R_2$$

in the isotropic space R_3^2, then the total curvature of this surface is expressed by the following formula:

$$z_{xx}z_{yy}-z_{xy}^2=K \tag{12}$$

Where, K is the total curvature for the surface, the left side of the formula (12) is the Monge-Ampere operator. The problem of recovering the surface is equivalent to solve the Monge-Ampere equation in isotropic space [3]. Equation (11) can be solved for transfer surfaces if the dual mapping of isotropic space is used.

Theorem 4 *In the isotropic space, the Monge-Ampere equation is in the form* (11)*, and the function on the right side can be written in the form* $\varphi(z_x,z_y)=\frac{1}{\psi_1(z_x)\psi_2(z_y)}$*, then the general solution of the transfer surface is equal to:*

$$z(x,y)=x\cdot\psi_1^{-1}\left(\frac{dx}{\tau}\right)+y\cdot\psi_2^{-1}\left(\tau dy\right)-\int xd\left(\psi_1^{-1}\left(\frac{dx}{\tau}\right)\right)-\int yd\left(\psi_2^{-1}\left(\tau dy\right)\right) \tag{13}$$

where, $\tau-$ *is const.* z_x, z_y *are first-order derivatives of* $z(x,y)$.

Proof of the Theorem 4. Let us assume that the regular surface Φ be given by the

$$z = z(x, y), \quad (x, y) \in D \subset R_2$$

in the space R_3^2. The Monge-Ampere equation for this surface is as follows:

$$z_{xx} z_{yy} - z_{xy}^2 = \varphi(z_x, z_y)$$

Let the function on the right side be given in the form $\varphi(z_x, z_y) = \frac{1}{\psi_1(z_x)\psi_2(z_y)}$. We write the Monge-Ampere equation of the dual surface respect to the given surface using a dual mapping (5) of the isotropic space, that is:

$$z^*_{x^*x^*} z^*_{y^*y^*} - \left(z^*_{x^*y^*}\right)^2 = \frac{1}{\varphi\left(x^*, y^*\right)} \tag{14}$$

where, the dual mapping is as follows:

$$\begin{cases} x^* = z_x \\ y^* = z_y \\ z^* = x \cdot z_x + y \cdot z_y - z \end{cases} \tag{15}$$

We solve the Monge-Ampere equation by the given formula (14)for transfer surfaces for the case where the total curvature of these surfaces is a product of two separate variable functions. The vector form of the transfer surface is as follows:

$$\overline{r}\left(x^*, y^*\right) = x^* \overline{i} + y^* \overline{j} + \left(f\left(x^*\right) + g\left(y^*\right)\right) \overline{k}$$

If we put it in the formula (14), we obtain the following:

$$f_{x^*x^*} \cdot g_{y^*y^*} = \psi_1\left(x^*\right) \psi_2\left(y^*\right)$$

From this,

$$\frac{f_{x^*x^*}}{\psi_1\left(x^*\right)} = \frac{\psi_2\left(y^*\right)}{g_{y^*y^*}} = \tau$$

$\tau-$ const.

The above equations are second-order differential equations. By solving these differential equations, we recover the given dual transfer surface according to its given total curvature.

1) $\frac{f_{x^*x^*}}{\psi_1(x^*)} = \tau$

$$\frac{f_{x^*x^*}}{\psi_1\left(x^*\right)} = \tau \quad \Rightarrow f_{x^*x^*} = \tau \cdot \psi_1\left(x^*\right) \quad \Rightarrow f_{x^*} = \tau \int \psi_1\left(x^*\right) dx^*$$

Integrating again, we get the following:

$$f\left(x^*\right) = \int \left[\tau \int \psi_1\left(x^*\right) dx^*\right] dx^*$$

2) $\frac{\psi_2(y^*)}{g_{y^*y^*}} = \tau$

$$\frac{\psi_2\left(y^*\right)}{g_{y^*y^*}} = \tau \quad \Rightarrow g_{y^*y^*} = \frac{\psi_2\left(y^*\right)}{\tau} \quad \Rightarrow g_{y^*} = \frac{1}{\tau} \int \psi_2\left(y^*\right) dy^*$$

From this,

$$g(y^*) = \int \left[\frac{1}{\tau} \int \psi_2(y^*)\, dy^* \right] dy^*$$

From this, we obtain the following equation for the transfer surface:

$$\Phi^* : \ z^*(x^*, y^*) = f(x^*) + g(y^*) = \int \left[\tau \int \psi_1(x^*)\, dx^* \right] dx^* + \int \left[\frac{1}{\tau} \int \psi_2(y^*)\, dy^* \right] dy^* \quad (16)$$

If we apply the dual mapping (5) for the points of the surface Φ^*, we get the following:

$$\begin{cases} x^{**} = z^*_{x^*} \\ y^{**} = z^*_{y^*} \\ z^{**} = x^* \cdot z^*_{x^*} + y^* \cdot z^*_{y^*} - z^* \end{cases}$$

The parametric equations of the surface Φ^{**} are as follows:

$$\begin{gathered} x^{**} = \tau \int \psi_1(x^*)\, dx^* \\ y^{**} = \tfrac{1}{\tau} \int \psi_2(y^*)\, dy^* \\ z^{**} = x^* \cdot \tau \int \psi_1(x^*)\, dx^* + y^* \cdot \tfrac{1}{\tau} \int \psi_2(y^*)\, dy^* - \int \left[\tau \int \psi_1(x^*)\, dx^* \right] dx^* - \int \left[\tfrac{1}{\tau} \int \psi_2(y^*)\, dy^* \right] dy^* \end{gathered} \quad (17)$$

Finding the following expressions from the first and second equalities of the system (17) above,

$$\begin{gathered} x^* = \psi_1^{-1}\left(\tfrac{dx^{**}}{\tau}\right) \\ y^* = \psi_2^{-1}(\tau dy^{**}) \end{gathered}$$

if we put it in the third equation, we get the following equation of the surface Φ^{**}, i.e:

$$z^{**} = x^{**} \cdot \psi_1^{-1}\left(\frac{dx^{**}}{\tau}\right) + y^{**} \cdot \psi_2^{-1}(\tau dy^{**}) - \int x^{**} d\left(\psi_1^{-1}\left(\frac{dx^{**}}{\tau}\right)\right) - \int y^{**} d\left(\psi_2^{-1}(\tau dy^{**})\right) \quad (18)$$

From the Theorem 3 above, the following holds for this surface:

$$\Phi^{**} = \Phi \quad (19)$$

From this, the surface equation Φ is also calculated according to the formula (18) and we get the following for this surface:

$$z(x, y) = x \cdot \psi_1^{-1}\left(\frac{dx}{\tau}\right) + y \cdot \psi_2^{-1}(\tau dy) - \int x d\left(\psi_1^{-1}\left(\frac{dx}{\tau}\right)\right) - \int y d\left(\psi_2^{-1}(\tau dy)\right)$$

The theorem is completely proved.

Theorem 5 *For the special case of the Monge-Ampere equation*

$$z_{xx} z_{yy} - z_{xy}^2 = z_x \quad (20)$$

there is a solution in the family of transfer surfaces and it is as follows:

$$z(x, y) = \frac{\mu}{2} y^2 - C_2 \cdot \mu y + \mu e^{\frac{x - C_1}{\mu}} + C$$

Where, $\mu, C_1, C_2, C-$ *const.*

Proof of the Theorem 5. Let there be given a regular surface Φ and its equation is $z = z(x,y), \quad (x,y) \in D \subset R_2$. Assume that this surface satisfies the special case of the Monge-Ampere equation (20). We find the dual surface for the given surface by dual mapping (15):

$$z^*_{x^*x^*} z^*_{y^*y^*} - \left(z^*_{x^*y^*}\right)^2 = \frac{1}{x^*} \tag{21}$$

The vector form of the transfer surface Φ^* is as follows:

$$\overline{r}\left(x^*, y^*\right) = x^* \overline{i} + y^* \overline{j} + \left(f\left(x^*\right) + g\left(y^*\right)\right) \overline{k}$$

From this, $f_{x^*x^*} \cdot g_{y^*y^*} = \frac{1}{x^*} \quad \Rightarrow f_{x^*x^*} \cdot x^* = \frac{1}{g_{y^*y^*}} = \mu$, $\mu-$const.

1)$f_{x^*x^*} \cdot x^* = \mu$

$$f_{x^*x^*} = \frac{\mu}{x^*} \quad \Rightarrow f_{x^*} = \mu \ln\left|x^*\right| + C_1$$

We obtain the following:

$$f\left(x^*\right) = \mu \cdot x^* \left(\ln\left|x^*\right| - 1\right) + C_1 x^* + C_3$$

2) $\frac{1}{g_{y^*y^*}} = \mu$

$$g_{y^*y^*} = \frac{1}{\mu} \quad \Rightarrow g_{y^*} = \frac{y^*}{\mu} + C_2$$

From this:

$$g\left(y^*\right) = \frac{\left(y^*\right)^2}{2\mu} + C_2 y^* + C_4$$

$$F^*: \; z^*\left(x^*, y^*\right) = f\left(x^*\right) + g\left(y^*\right) = \mu \cdot x^* \left(\ln\left|x^*\right| - 1\right) + C_1 x^* + C_3 + \frac{\left(y^*\right)^2}{2\mu} + C_2 y^* + C_4 =$$

$$= \mu \cdot x^* \left(\ln\left|x^*\right| - 1\right) + C_1 x^* + \frac{\left(y^*\right)^2}{2\mu} + C_2 y^* + C$$

where, $C_3 + C_4 = C-$ const. If we also apply dual mapping for the transfer surface Φ^*, the following is valid:

$$x^{**} = \mu \cdot \ln\left|x^*\right| + C_1 \quad \Rightarrow x^* = e^{\frac{x^{**} - C_1}{\mu}}$$
$$y^{**} = \frac{y^*}{\mu} + C_2 \quad \Rightarrow y^* = \mu\left(y^{**} - C_2\right)$$
$$z^{**} = x^* \cdot x^{**} + y^* \cdot y^{**} - z^*$$

$$z^{**}\left(x^{**}, y^{**}\right) = \mu \cdot e^{\frac{x^{**} - C_1}{\mu}} + \frac{\mu}{2}\left(y^{**}\right)^2 - C_2 \mu y^{**} + \frac{\mu}{2} C_2^2 - C \tag{22}$$

According to the Theorem 3 above:

$$\Phi^{**} = \Phi \quad \Rightarrow z^{**}\left(x^{**}, y^{**}\right) = z\left(x, y\right)$$

From this, there exists a solution to the equation (20) and we get the following by simplifying the constant numbers:

$$z\left(x, y\right) = \frac{\mu}{2} y^2 - C_2 \cdot \mu y + \mu e^{\frac{x - C_1}{\mu}} + C \tag{23}$$

Theorem is proved.

4 Amalgamatic and Casorati curvatures in the isotropic space

For investigating of the theory of surfaces, studying the connection between their total and mean curvatures is important in solving many geometric problems. We know that in surface theory, the problems of recovering surfaces with respect to their total curvature K and mean curvature H were studied in many works [3–5, 9, 10, 12]. In addition these characteristics, studying the $\frac{K}{H}$, $\frac{K}{H^2}$ ratios also reveals some new features of the geometry of surfaces. The original idea can be found in the works of Weingarten [20, 21]. This ratio $\frac{K}{H}$ was later called amalgamatic curvature. The amalgamatic curvature of the surface and the information about it are given by B. Suceava [22]. The aim of studying amalgamatic curvature is to study surfaces by analogue the ratio $\frac{\tau}{k}$ of the torsion to the curvature of curves in higher-dimensional geometric objects.

Now let us define amalgamatic curvature:

Definition 4 *Let $\xi : G \subset R^2 \to R_3^2$ be a surface given by the smooth mapping ξ. Then the amalgamatic curvature at point p is:*

$$A = \frac{2k_1k_2}{k_1 + k_2}$$

To study surfaces through a certain connection between the total curvature K and the mean curvature H, the concept of Casorati curvature is presented in the following works [15, 16, 23, 24]. This curvature was introduced by Feliz Casorati in 1890 and is defined as follows [23]:

$$C = \frac{k_1^2 + k_2^2}{2}$$

In isotropic space, the amalgamatic and Casorati curvatures of a surface are respectively as follows [17]:

$$A = \frac{2k_1k_2}{k_1 + k_2} = \frac{K}{H} \quad C = \frac{k_1^2 + k_2^2}{2} = 2H^2 - K$$

Where, k_1, k_2- are principal curvatures. We know that the mean curvature of the dual surface to the given surface is determined by (8) [10]. The problems of surface recovering using the dual mean curvature are discussed in detail by the authors in the following works [8–10]. As can be seen from the formula for finding the amalgamatic curvature, it is inversely proportional to the dual mean curvature. So, from this, we can conclude that the problem of recovering a surface according to its amalgamatic curvature is equivalent to the problem of recovering the surface according to its dual mean curvature. The solutions to all problems when $H^* = \chi(u, v)$ are also the solutions to the problem of recovering a surface according to amalgamatic curvature. From this, the following theorem holds:

Theorem 6 *The following connection holds for the amalgamatic curvature of a given surface and the mean curvature of the its dual surface:*

$$A = \frac{1}{H^*} \tag{24}$$

Proof of the Theorem 6. We are given a surface F and its dual surface F^*. The amalgamatic curvature of the surface F is as follows:

$$A = \frac{K}{H}$$

For the mean curvature of the surface F^*:

$$H^* = \frac{H}{K}$$

From this, equality (24) follows. The isotropic Casorati curvature of a dual surface to the given surface is equal to:

$$C^* = 2(H^*)^2 - K^* = 2\left(\frac{H}{K}\right)^2 - \frac{1}{K} = \frac{C}{K^2}$$

From this,

$$C^* = \frac{C}{K^2} \tag{25}$$

Thus, this equality shows the connection between the Casorati curvatures of a surface and its dual surface.

5 Recovering surfaces with constant negative curvature according to their curvature invariants

Now we will present some properties of the curvatures of surfaces with a given negative curvature in isotropic space: Let the surface F be given as:

$$r\,(u,v) = (r_1\,(u,v)\,, r_2\,(u,v)\,, r_3\,(u,v))$$

In this,

$$r_1\,(u,v) = f_1\,(u) + g_1\,(v)$$
$$r_2\,(u,v) = f_2\,(u) + g_2\,(v)$$

Let the condition

$$f_1{}'g_2{}' - f_2{}'g_1{}' \neq 0 \tag{26}$$

be fulfilled and $f_i, g_i \in C^2, \quad i = 1,2$, also

$$\begin{vmatrix} f_1{}' & f_2{}' & r_{3u} \\ g_1{}' & g_2{}' & r_{3v} \\ f_1{}'' & f_2{}'' & r_{3uu} \end{vmatrix} = 0 \quad \begin{vmatrix} f_1{}' & f_2{}' & r_{3u} \\ g_1{}' & g_2{}' & r_{3v} \\ g_1{}'' & g_2{}'' & r_{3vv} \end{vmatrix} = 0 \tag{27}$$

If conditions (27) are valid, then the parametric curves will be asymptotic. From conditions (26) and (27), the functions $a\,(u,v)$ and $b\,(u,v)$ are found one-valued through the $r_3\,(u,v)$

$$r_{3u} = af_1{}' + bf_2{}' \quad r_{3v} = ag_1{}' + bg_2{}' \quad r_{3uu} = af_1{}'' + bf_2{}'' \quad r_{3vv} = ag_1{}'' + bg_2{}'' \tag{28}$$

From solving equations (28), $a_{uv} = b_{uv} = 0$ is valid. From this, $a = \alpha_1(u) + \beta_1(v)$, $b = \alpha_2(u) + \beta_2(v)$ are found. From the previous equation, for arbitrary functions $\alpha_i, \beta_i, \quad i = 1, 2$, we get:

$$\alpha_1{}'f_1{}' + \alpha_2{}'f_2{}' = 0 \quad \beta_1{}'g_1{}' + \beta_2{}'g_2{}' = 0$$

From this,

$$\alpha_1{}' = \lambda(u) f_2{}' \quad \alpha_2{}' = -\lambda(u) f_1{}' \quad \beta_1{}' = \delta(v) g_2{}' \quad \beta_2{}' = -\delta(v) g_1{}'$$

Then,

$$r_{3uv} = \delta(v) (g_2{}'f_1{}' - g_1{}'f_2{}') \qquad r_{3vu} = \lambda(u) (g_1{}'f_2{}' - g_2{}'f_1{}')$$

If, we get $\lambda(u) = -\delta(v) = const. = \kappa_1$

$$a = \kappa_1 (f_2 - g_2) + \kappa_2, \quad b = -\kappa_1 (f_1 - g_1) + \kappa_3$$

As a result, from these expressions

$$r_{3u} = \kappa_1 (f_1{}'f_2 - f_1 f_2{}') + \kappa_1 (f_2{}'g_1 - f_1{}'g_2) + \kappa_2 f_1{}' + \kappa_3 f_2{}'$$

$$r_{3v} = \kappa_1 (g_2{}'g_1 - g_2 g_1{}') + \kappa_1 (g_1{}'f_2 - g_2{}'f_1) + \kappa_2 g_1{}' + \kappa_3 g_2{}'$$

$r_{3uv} = r_{3vu}$ is valid. By integrating, we obtain:

$$r_3(u,v) = \kappa_1 \left\{ (f_2{}'g_1 - f_1{}'g_2) + \int (f_1{}'f_2 - f_1 f_2{}') \, du + \int (g_2{}'g_1 - g_2 g_1{}') \, dv \right\} + \kappa_2 (f_1 + g_1) + \kappa_3 (f_2 + g_2) + \kappa_4$$

So, if $\kappa_1 = 1, \ \kappa_2 = \kappa_3 = \kappa_4 = 0$, the surface equation is:

$$r(u,v) = \left(f_1 + g_1, f_2 + g_2, (f_2 g_1 - f_1 g_2) + \int (f_1{}'f_2 - f_1 f_2{}') \, du + \int (g_2{}'g_1 - g_2 g_1{}') \, dv \right) \quad (29)$$

In equation (29),isotropic total and mean curvatures of the surface are as follows [17]:

$$K = -1 \quad H = \frac{f_1{}'g_1{}' + f_2{}'g_2{}'}{f_2{}'g_1{}' - f_1{}'g_2{}'} \qquad (30)$$

Let the following conditions be satisfied in equality (29), that is:

$$f_1 = u \quad f_2 = f' \quad g_1 = v \quad g_2 = g' \qquad (31)$$

In this case, the equation of the surface is:

$$r(u,v) = (u + v, f' + g', 2(f - g) + (v - u)(f' + g')) \qquad (32)$$

By calculating the fundamental forms of this surface, the isotropic total, mean, Casorati, and amalgamatic curvatures are as follows:

$$K = -1 \quad H = \frac{1 + f''g''}{f'' - g''} \quad C = 1 + \frac{2(1 + f''g'')^2}{(f'' - g'')^2} \quad A = \frac{g'' - f''}{1 + f''g''}$$

From here, for the total and mean curvatures of the dual surface:

$$K^* = -1 \quad H^* = \frac{1 + f''g''}{g'' - f''}$$

Now, let us consider the problem of recovering surfaces with total curvature -1 according to their different curvature invariants:

5.1 The problem of recovering the surface by the mean curvature

If $H = \phi_1(u)$ or $H = \phi_2(v)$ is an arbitrary continuously differentiable function, we consider the problem of recovering of the surface given by formula (32) by the mean curvature. In this case, by simplifying the equation $\frac{1+f''g''}{f''-g''} = \phi_1(u)$, we get:

$$\frac{f''\phi_1(u) - 1}{f'' + \phi_1(u)} = g'' = \eta \quad \eta = const$$

We get two ordinary differential equations with separate variables. Solving these equations, we find the following:

$$\begin{cases} f(u) = \int \left[\int \frac{1+\eta\phi_1(u)}{\phi_1(u)-\eta} du\right] du + c_0 u + c_1 \\ g(v) = \frac{\eta v^2}{2} + d_0 v + d_1 \end{cases} \tag{33}$$

Even if $H = \phi_2(v)$, by using the same method we obtain the following expressions:

$$\begin{cases} f(u) = \frac{\eta u^2}{2} + c_0 u + c_1 \\ g(v) = \int \left[\int \frac{\eta\phi_2(v)-1}{\phi_2(v)+\eta} dv\right] dv + d_0 v + d_1 \end{cases} \tag{34}$$

Theorem 7 *If the parameterization of the surface F is defined by formula* (32) *and the mean curvature is given by $H = \phi_1(u)$ or $H = \phi_2(v)$ arbitrary continuous differentiable functions, then the functions $f(u)$ and $g(v)$ are found by expressions* (33) *and* (34)*, respectively.*

5.2 Problems of surface recovering from amalgamatic curvature and dual mean curvature

The amalgamatic curvature of a surface F with total curvature -1 is:

$$A = -\frac{1}{H} \tag{35}$$

and for the mean curvature of the dual surface F^* is:

$$H^* = \frac{1}{A} = -H \tag{36}$$

Therefore, it can be concluded that if the mean curvature of the surface F is given by the arbitrary continuous differentiable functions in Theorem 7, then from formulas (35) and (36) we will have solved the problems of recovering surface by the amalgamatic curvature and by the mean curvature of the its dual surface.

5.3 The problem of recovering the surface according to the Casorati curvature

For the Casorati curvature of the surface given by (32),

$$C = 1 + \frac{2(1 + f''g'')^2}{(f'' - g'')^2} \tag{37}$$

is valid. If the Casorati curvature is given by positive continuous differentiable functions $C = \theta_1(u)$ or $C = \theta_2(v)$, then for the functions $f(u)$ and $g(v)$ of the surface given by equation (32), we find the following equalities: In the case $C = \theta_1(u)$, $\theta_1(u) > 1$

$$\begin{cases} f(u) = \int \left[\int \frac{1+\eta\sqrt{\frac{\theta_1(u)-1}{2}}}{\sqrt{\frac{\theta_1(u)-1}{2}}-\eta} du \right] du + c_0 u + c_1 \\ g(v) = \frac{\eta v^2}{2} + d_0 v + d_1 \end{cases}$$

or $C = \theta_2(v)$, $\theta_2(v) > 1$:

$$\begin{cases} f(u) = \frac{\eta u^2}{2} + c_0 u + c_1 \\ g(v) = \int \left[\int \frac{\eta\sqrt{\frac{\theta_2(v)-1}{2}}-1}{\sqrt{\frac{\theta_2(v)-1}{2}}+\eta} dv \right] dv + d_0 v + d_1 \end{cases}$$

Corollary 1 *The following equality holds for the Casorati curvatures of a surface with total curvature* -1 *and its dual surface:*

$$C^* = C$$

Because, from equality (25), the problem of reconstructing surfaces with total curvature -1 according to the Casorati curvature is equivalent to the problem of recovering its dual surface by this curvature.

6 Conclusion

In this paper, in the first part of the main results, the application of the dual mapping of isotropic space to the theory of surfaces makes it possible to solve the Monge-Ampere equation in a special case. We know that this equation has applications in various fields. Namely, in the theory of surfaces in differential geometry, the recovering of a surface according to its total curvature coincides with the solution of this equation. Moreover, the problems of recovering surfaces according to other curvature invariants are also important in the study of surfaces. Therefore, in the second part of the results obtained in this work, problems of surface recovering from curvature invariants are considered. In addition to mathematical problems, in physics the connection between the Hamiltonian and Lagrange functions is studied using the dual mapping mentioned above. Putting the energy to the Lagrangian function can be used to solve extremal problems [25].

The problems solved in the article are a generalization of the problems considered in the works of M.E.Aydin [3], M. S. Lone and M. K. Karasen [9], A. Artikbaev and Sh. Ismoilov [10].

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