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# MATHEMATICAL MODEL WITH NON-LOCAL BOUNDARY CONDITION OF INCOMPRESSIBLE FLUID FILTRATION

This work is devoted to an actual problem today – the creation of cost-effective technology of combined development of several reservoirs. Joint development of oil reservoirs combining two or more oil reservoirs into one production facility by simultaneous extraction of reservoir fluids from them by a single network of wells [1, 2]. Oil fields, as a rule, are multilayer, and the productive formations are heterogeneous, first of all, by reservoir properties: first of all, they have different permeability and thickness. It is economically unprofitable to drill a different production grid for each of the productive formations. One of the primary tasks of putting an oil field into commercial development is to combine productive formations into single production facilities and to carry out joint development of these formations. After the reservoirs are combined into a single production facility, they are drilled using a single grid of production and injection wells [3, 4]. This paper considers a two-layer reservoir with different permeability and thickness. Numerical solution of the model is proposed to determine the pressure field of incompressible fluid at known total flow rate. The technology of combined development of several reservoirs isolated from each other is used. We construct special difference equations in the neighborhood of internal boundaries that allow us to apply the integro-interpolation method in a two-connected domain. Special differences equations in the vicinity of internal borders, allowing overcome the difficulties arising from the borders of the domain are constructed. The necessity of combined development in order to reduce the economic costs is revealed and justified. Based on the numerical investigations of the problem, obtained numerical results in programming language Fortran, and graphics in Tecplot for double-layer reservoirs. The article also found an analytical solution of this problem for the two reservoirs and made a comparative analysis of the results. Given conclusions about the quality and accuracy of used iterative method. The scientific novelty of this work is to research several layers by simultaneous selection of the reservoir fluids single well.The method of solving and analysis of the results will be of interest to those skilled in the development the oil fields.

**Key words**: combined development, doubly connected domain, reservoir thickness, mesh of wells, numerical solution, finite difference method, analytical solution.

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**Шекаралық шарты локалды емес сығылмайтын сұйықтың фильтрленуінің математикалық моделі**

Зерттеу жұмысы қазіргі кездегі өзекті – бірнеше пласты бірге өндірудің экономикалық тиімді технологиясын құру мәселесіне арналған. Екі немесе көпқатпарша пластарды бірмезгілде ұңғыма жүйесін енгізіп өндіру жолдарына [1, 2] байланысты математикалық модельдердің шешімін қарастыру бірмезгілде мұнайды өндірудің қажеттілігін талдау мүмкіндіктерін береді. Мұнай қоры жерасты қабаттарының әртүрлі коллекторлық қасиеттеріне және пластқабатшалардың қалыңдығына байланысты орналасқан. Соған байланысты мұнай қоры бар қабатшаларға жеке ұңғыма жүйесін пайдаланудың тиімділігі шамалы. Сондықтан мұнай қоры бар пластқабатшаларды бірге өндіру жолдарын қарастрып зерттеу мұнай механикасында жиі қолданылады [3, 4]. Өзара байланыспаған екі пластағы сұйық қорын өндіру есебі зерттелген. Мұнда ұңғыманың ықпалдық радиусымен шектелген аймақта арнайы ақырлы-айырымдық схема алынып, барлық аймақ үшін интегро-интерполяциялық әдісті қолдану жолы қойылған. Осымен бірге екі пласт үін шекаралық шарты локальды емес сығылмайтын сұйықтық фильтрлену есебінің аналитикалық шешімі алынып, жуық шешіміне талдау жасалады. Экономикалық шығындарды азайтуда бірге өндірудің қажеттігі көрсетіліп, дәлелденеді. Жүргізілген сандық зерттеулердің нәтижесінде Fortran бағдарламалау тілінде екі пласт үшін есептің сандық мәндері және Tecplot бағдарламалық пакетінде сызбалары алынған. Қолданылған итерациалық әдістің сапасы дәлелденген. Жұмыстың басқа қарастырылған жұмыстармен салыстырғандағы ғылыми жаңалығы – өзара байланыспаған пластардағы сұйық қорын бір мезгілде бір ұңғымалар торымен өндіру есебін зерттеу болып табылады. Модельді зерттеу әдістері мен алынған қорытынды нәтижелері мұнай кен орындары мамандарының қызығушылығын тудыруы мүмкін.

**Түйін сөздер**: бірге өндіру, екі байланысқан аймақ, пластың қуаты, ұңғымалар торы, сандық шешім, ақырлы-айырымдық әдіс, аналитикалық шешім.

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**Математическая модель с нелокальным граничным условием фильтрации несжимаемой жидкости**

Данная работа посвящена актуальной на сегодняшний день проблеме – созданию экономически эффективной технологии совместной разработки нескольких пластов. Совместная разработка нефтяных пластов – объединение двух и более нефтяных пластов в один эксплуатационный объект путём одновременного отбора из них пластовой жидкости единой сеткой скважин рассматривались многими авторами [1, 2]. Нефтяные месторождения, как правило, являются многопластовыми, причём продуктивные пласты неоднородны, прежде всего по коллекторским свойствам имеют первую очередь различную проницаемость и толщину. На каждый из продуктивных пластов бурить свою сетку добывающих экономически убыточно. Одной из первоочередных задач ввода нефтяного месторождения в промышленную разработку является объединение продуктивных пластов в единые эксплуатационные объекты и проведение совместной разработки этих пластов. После объединения пластов в единый эксплуатационный объект их разбуривают по единой сетке добывающих скважин [3, 4]. В работе рассматривается двухслойный пласт различной проницаемости и толщины. Предложено численное решение модели определения давления несжимаемой жидкости, когда известен суммарный дебит при одновременной совместной разработке нескольких изолированных между собой пластов методом конечных разностей. Построены специальные разностные уравнения в окрестности внутренних границ, позволяющие применить интегро-интерполяционный метод в двухсвязной области. На основе проведенного исследования поставленной задачи получены численные результаты на языке программирования Fortran и графики модели на Tecplot для двухслойного пласта. Также найдено аналитическое решение данной задачи для двух пластов и сделан сравнительный анализ полученных результатов. Получено аналитическое и численное решение задачи с нелокальным граничным условием при совместной разработке двухслойных пластов с заданным суммарным расходом. Научная новизна работы заключается в исследовании нескольких пластов путем одновременного отбора пластовой жидкости единой сеткой скважины. Метод решения и анализ полученных результатов будут интересны специалистам в области разработки нефтяных месторождений.

**Ключевые слова**: совместная разработка, двухсвязная область, мощность пласта, сетка скважин, численное решение, метод конечных разностей, аналитическое решение

## 1 Introduction

The isothermal filtration of a homogeneous liquid in two formations isolated from each other, but penetrated by one well, is considered. Thus, the problem of planned filtration of fluid into a well comes down to finding a solution to Laplace's equation in a doubly connected region, the outer boundary of which is the contour of the filtration area, and the inner boundary is the contour of the well.

Due to the fact that the size of the filtration area, as a rule, is much larger than the size of the well contour, when solving the problem using the grid method, approximating the filtration area by the grid area so as to take into account the size and shape of the well presents certain difficulties [5]– [7].

When the well is replaced by a material point – the well in which the source (sink) is located, the function $p$ at point $O_0$ becomes unlimited, and the flow rate q is defined as the limit

$$\lim_{l\to 0}\oint_{l}\sigma\frac{\partial p}{\partial n}dl=q, \tag{1}$$

where $l$ is some closed contour covering the well, $n$ is the external normal lose to $l$.

At filtration of homogeneous fluid the condition (1) is quite justified.If we keep the boundary $\partial\omega_{\varepsilon}$, associated with the control well, then specifying only the well flow rate for them is not sufficient; additional conditions are needed on the well contour, i.e.

$$\int_{\partial\omega_{\varepsilon}}\sigma\frac{\partial p}{\partial n}d\gamma=q, \tag{2}$$

$$p(x,y)=C,\ \text{at}\ (x,y)\in\partial\omega_{\varepsilon}. \tag{3}$$

where $C$ is some unknown constants. In this case it follows from relations (2), (3) that in $\varepsilon$ neighborhood of the well the function $p(x,y)$ is represented as

$$p=u+\alpha ln\frac{r}{r_c}, \tag{4}$$

where $\frac{q}{2\pi\sigma_c}$, $r_c$ is the radius of the well.

Then from (2) we have

$$\sigma_c=\frac{1}{2\pi}\int_{0}^{2\pi}\sigma(r_c,\varphi)d\varphi \tag{5}$$

Apparently, using relation (5) and hydraulic conductivity, it is necessary to continue to the inside of the well so that at the well point it takes $\sigma_c$, and require the fulfillment of condition (1) instead of condition (2). In this case, we can apply the substitution method [7]. However, as a result of such a transition, condition (3) will be fulfilled only approximately. Following the work [4, 8] given conditions (2), (3) on the well and taking into account the logarithmic dependence (4), the pressure function in $\varepsilon$ neighborhoods of the well, a finite-difference method of solving the problem is constructed.

When studying the issue of fluid flow to production wells in a multi-layer system or in layers with a permeable roof and bottom, it is necessary to take into account its possible flows from one horizon to another, which greatly complicates theoretical studies and mathematical solutions of practical problems. We will investigate the plane-radial motion of liquid in two-layer formations isolated from each other, but opened by a single well. Considering that the thickness of the formation $H$ is small compared to its dimensions in the horizontal plane, that the roof and the sole of the layers are impermeable, it is possible to pre-carry out all the necessary averaging of the parameters by power and thus move from spatial tasks to flat ones.Let's direct the $OZ$ axis against gravity and introduce the reduced pressure function $p^* = p + \rho g h$. Then we write the filtration rate in the form $\vec{v} = -\frac{kH}{\mu}\mathrm{grad}p^*$. In the following we will omit the asterisk at $p^*$ and by the function $p$ we will understand the reduced pressure.

## 2 Mathematical model of fluid filtration in two-layer formations

Let us introduce the notations: $\omega_\varepsilon$-area, enclosed within the contour $\partial\omega_\varepsilon$, $\Omega$-area enclosed within $\partial\Omega$, $Q_0$ is total flow rate of the well, selection from two layers, $k = 1, 2$ is layer number. Then $\Omega_k$ is a flat doubly connected region $\partial\Omega_k$, and $\omega_\varepsilon \in \Omega_k$ is a circle of radius $r_c = \varepsilon \ll \mathrm{diam}\Omega_k$. Assume that the center of the circle coincides with the origin. Let us pose the problem of finding pressures in the region $\Omega_{\varepsilon,k} = \Omega_k/\bar{\omega}_\varepsilon$ that satisfy the equation

$$\mathrm{div}\,\sigma_k\,\mathrm{grad}p_k = 0, \quad k = 1, 2 \quad (x, y) \in \Omega_{\varepsilon,k}, \tag{6}$$

where $\varepsilon$ is the radius of the well hereafter for convenience we will assume $r_c = \varepsilon$. On the contour $\partial\Omega_k$ takes the given values

$$p_k(x, y) = \varphi_0(x, y), \quad (x, y) \in \partial\Omega_k, \tag{7}$$

on $\partial\omega_\varepsilon$ satisfy the following conditions

$$\sum_{k=1}^{2} \oint_{\partial\omega_\varepsilon} \sigma_k \frac{\partial p_k}{\partial n} d\gamma = Q_0 \quad (x, y) \in \partial\omega_\varepsilon \tag{8}$$

$$p_2(x, y) = p_1(x, y) + \rho g z_c \quad \mathrm{at}(x, y) \in \partial\omega_\varepsilon \tag{9}$$

where $\sigma_k = \frac{k_k H_k}{\mu} > 0$ is coefficient of hydraulic conductivity, $p_1(x, y) = C$ is some unknown constant value, $z_c$ = const is trance between the center surfaces (horizontal planes) of the two layers, $\rho$ is the density of liquids, $g$ is the acceleration of free fall.

## 3 Analytical solution of the problem (6)–(9)

Let's pass to polar coordinates in two-dimensional space

$$r^2 = x^2 + y^2, \quad x = r\cos\varphi, \quad y = r\sin\varphi, \quad \tan\varphi = y/x.$$

In polar coordinates, the desired function $p(r,\varphi)$ must be periodic with period $2\pi:\ p(r,\varphi+2\pi)=p(r,\varphi)$. Let us write the isobaric pressure field for a circular reservoir with constant hydraulic conductivity coefficient ($\sigma_k=\text{const}$):

$$\frac{d^2p_i}{dr^2}+\frac{1}{r}\frac{dp_i}{dr}=0,\quad r_c<r<R. \tag{10}$$

On the contour of two-layer strata

$$p_i(R)=p_0. \tag{11}$$

We set the total flow rate $Q_0$ on the well

$$\sum_{i=1}^{2}\oint_{\partial\omega_\varepsilon}\sigma_i\frac{\partial p_i}{\partial r}d\gamma=Q_0, \tag{12}$$

and unknown pressure on the well contour

$$p_2(r_c)=p_1(r_c)+\rho gH. \tag{13}$$

Let us represent the general solution in layers in the following form

$$p_i(r)=A_i\ln r+B_i,\quad i=1,2, \tag{14}$$

Let us take a sector of a circular layer (Figure 1), where $\angle AOB=d\varphi$, $d\gamma=AB$. Then for small $\left(\frac{d\varphi}{2}\right)$ we have $\frac{d\gamma}{2}=r_c\sin\left(\frac{d\varphi}{2}\right)$ or write $d\gamma\approx r_cd\varphi$.

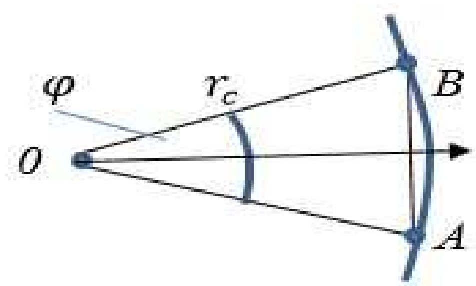


**Figure 1.** Sector of the circle $d\varphi=\angle AOB$.

Now let us write the condition (14) with period $2\pi$ in the form $\sum\limits_{i=1}^{2}\int\limits_{0}^{2\pi}\sigma_i\frac{\partial p_i}{\partial r}r_cd\varphi=Q_0$. Then after integration on the basis of the general solution we obtain

$$2\pi\sigma_1A_1+2\pi\sigma_2A_2=Q_0. \tag{15}$$

Taking into account the boundary conditions on the well contour, we determine the integral constants for the first and second reservoirs:

$$A_1=\left(P_0-p_1(r_c)\right)/\ln\left(\frac{R}{r_c}\right), \tag{16}$$

$$A_2 = (p_0 - p_1(r_c) - \rho g H) / \ln\left(\frac{R}{r_c}\right). \tag{17}$$

From equations (15)–(17) we find the pressure on the well contour of the first and second layers:

$$p_1(r_c) = p_0 - \frac{Q_0}{2\pi}\ln\left(\frac{R}{r_c}\right)/(\sigma_1+\sigma_2) - \sigma_2\cdot\rho g H/(\sigma_1+\sigma_2). \tag{18}$$

$$p_2(r_c) = p_0 - \frac{Q_0}{2\pi}\ln\left(\frac{R}{r_c}\right)/(\sigma_1+\sigma_2) + (1-\sigma_2)/(\sigma_1+\sigma_2)\cdot\rho g H. \tag{19}$$

Knowing the boundary conditions on the contour of two layers $p_i(R) = p_0$ and conditions on the contour of the well (18) and (19), we plot the field of pressure changes in the first and second layers:

$$p_1(r) = p_0 + \left[\frac{\frac{Q_0}{2\pi}}{\sigma_1+\sigma_2} + \frac{\sigma_2\cdot\rho g H/(\sigma_1+\sigma_2)}{\ln\left(\frac{R}{r_c}\right)}\right]\cdot\ln\left(\frac{r}{R}\right), \tag{20}$$

$$p_2(r) = p_0 + \left[\frac{\frac{Q_0}{2\pi}}{\sigma_1+\sigma_2} - \frac{(1-\sigma_2/(\sigma_1+\sigma_2))\cdot\rho g H}{\ln\left(\frac{R}{r_c}\right)}\right]\cdot\ln\frac{r}{R}. \tag{21}$$

If $\rho g H = 0$, we get a pressure field where the downhiller pressure at the well's same for the two layers, $P_1(r_c) = P_2(r_c) = \text{const}$. It should be noted that problems with nonlocal boundary condition for elliptic equation are considered by many authors. For example, in [9, 10] the asymptotes of solutions of nonlocal elliptic equations are considered in flat bounded domains.

## 4 Numerical solution of the problem by finite difference method

In the case of joint reservoir development, the finite element method is proposed in [11]. It is shown here that when using the finite difference method, the influence of the well radius on the filtration process presents certain difficulties. Following the work [8], we construct the solution by the finite difference method. First, we will construct a solution method for one layer (for the prostate we will take $p = p_1 = p_2$, $\sigma = \sigma_1 = \sigma_2$, $\Omega = \Omega_1 = \Omega_2$, $q_1 = \text{const}$). Let the area $\Omega$ is covered by a grid $\Omega_h (h \gg r_c)$. We will place the well point $O_0$ at node $(i_0, j_0)$. The point $O_0$ does not belong to the area $\Omega$, so the node $(i_0, j_0)$ does not belong to $\Omega_h$ either. We include all points formed by the intersection of grid lines with the boundary $\partial\omega$. We denote this set of nodes by $\partial\Omega_h$, and we denote an area (cell) by $\Omega_{i,j}$ and its boundary by $\partial\omega_{i,j}$. Then the cell $\partial\Omega_{i_0,j_0}$ is doubly connected, has an internal boundary $\omega_0$ and, unlike other elementary areas, contains not one grid point, but a set of nodes $\partial\omega_{0,h}$ (see Fig. 1). From the generalized Green's formula and applying the condition (8), we obtain

$$\iint_{\partial\omega_{i_0,j_0}} \text{div}\sigma\text{grad}p dV = \oint_{\partial\omega_{i_0,j_0}} \sigma\frac{\partial p}{\partial n}d\gamma - \oint_{\partial\omega_0}\sigma\frac{\partial p}{\partial n}d\gamma = \oint_{\partial\omega_{i_0,j_0}}\sigma\frac{\partial p}{\partial n}d\gamma - q_1.$$

The grid cell $\Omega_{i_0,j_0}$ is bi-connected because the cell contains a well with radius $r_c$ (Figure 2).

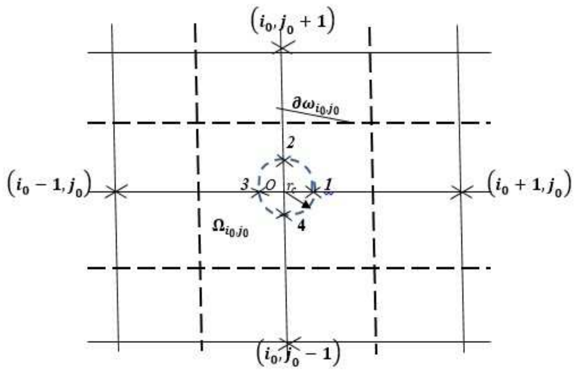


**Figure 2.** The well is located at a grid node.

Let the grid be square $x = ih$, $y = jh$, $i, j = \pm 1, \pm 2, \ldots$ then

$$\overline{\Omega}_{i,j} = \left\{ x_{i-1/2} \le x \le x_{i+1/2},\ y_{j-1/2} \le y \le y_{j+1/2} \right\}.$$

Thus, as a result of integration of equation (6) over the cells $\Omega_{i,j}$ we obtain

$$\oint_{\partial\omega_\varepsilon} \sigma \frac{\partial p}{\partial n} d\gamma = \Phi_{i,j}, \tag{22}$$

where $\Phi_{i,j} = \begin{cases} 0, \text{ at } (i,j) = (i_0, j_0), \\ q_1, \text{ at } (i,j) = (i_0, j_0). \end{cases}$

The radius ($\rho$) of influence of the well is comparable to the grid spacing ($h$). Then in the neighborhoods of $\rho$ point the function $p$ has a logarithmic dependence. Therefore, in $\rho$ neighborhoods we introduce an auxiliary function

$$u = p - \alpha_0 \ln r, \quad r^2 = (x - x_0)^2 + (y - y_0)^2 \tag{23}$$

where $\alpha$ is an undetermined constant.

Then we write (22) in the following form

$$\oint_{\partial\omega_\varepsilon} \sigma \frac{\partial p}{\partial n} d\gamma = \Phi_{i,j} - \alpha_0 \oint_{\partial\omega_\varepsilon} \sigma \frac{\partial lnr}{\partial n} d\gamma, \tag{24}$$

where $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \cos(\widehat{n,x}) + \frac{\partial u}{\partial y} \cos(\widehat{n,y})$ and taking into account the difference derivative along the normal $\nabla n, m$ to the boundary $\gamma_m$ (Figure 2). Thus $\gamma_m$ is orthogonal to the $m$-th grid line leaving node $(i,j)$ of the grid. As a result of numerical differentiation and integration of the left side of equation (24), we obtain

$$\sum_m \sigma_m \nabla_{n,m} u \nabla\gamma_m = \Phi_{i,j} - \alpha_0 \sum_m \sigma_m \int\limits_{\gamma_m} \sigma \frac{\partial \ln r}{\partial n} d\gamma. \tag{25}$$

Here $\nabla\gamma_m$ is the length of the boundary $\gamma_m$.

Consider the cell $\Omega_{i_0,j_0}$ (Figure 2). Its outer boundary is formed by line segments $x = x_0 \pm \frac{1}{2}(h + r_c)$, $y = y_0 \pm \frac{1}{2}(h + r_c)$. In this cell, four points belong to the grid $\Omega_h$. Obviously, the distance from all points to the well point O is equal to $r_c$. When $m = 1, 3$ and $m = 2, 4$, we write down the difference approximation $\nabla_{x,m} u\,(h + r_c)$ and $\nabla_{y,m} u\,(h + r_c)$. Then we make the reverse transition on the grid, i.e. we exclude the grid values of the auxiliary function $u(x, y)$ using equality (23) connecting the values of the functions $p(x, y)$ and $u(x, y)$. Since in our case $h \gg r_c$, we can get:

$$\nabla_{x,m} u = \left( p_{i_0\pm1,j_0} - p_{i_0,j_0} - \alpha ln \frac{h}{r_c} \right), \quad \nabla_{y,m} u = \left( p_{i_0,j_0\pm1} - p_{i_0,j_0} - \alpha ln \frac{h}{r_c} \right),$$

It is not difficult to make sure that for all m we have

$$\int_{\gamma_m} \frac{\partial ln r}{\partial n} d\gamma = \int_{-1/2}^{1/2} \frac{1/2 h dy}{1/4h^2 + (x - i_0 h)} = \frac{\pi}{2}.$$

Here $\alpha_0 = \frac{q_1}{2\pi\overline{\sigma}_{cp}}$, where $\overline{\sigma}_{cp} = \frac{1}{4} \left( \sigma_{i_0+1/2,j_0} + \sigma_{i_0-1/2,j_0} + \sigma_{i_0,j_0+1/2} + \sigma_{i_0,j_0-1/2} \right)$.

Finally, for the cell $\Omega_{i_0,j_0}$, the difference equations with a given flow rate $q_1$ for one layer will be written in the form

$$h^2 L(\sigma, p)_{i_0,j_0} = \sigma_{i_0+1/2,j_0} \left( p_{i_0+1,j_0} - p_{i_0,j_0} \right) + \sigma_{i_0-1/2,j_0} \left( p_{i_0-1,j_0} - p_{i_0,j_0} \right)$$

$$+\sigma_{i_0,j_0+1/2} \left( p_{i_0,j_0+1} - p_{i_0,j_0} \right) + \sigma_{i_0,j_0-1/2} \left( p_{i_0,j_0-1} - p_{i_0,j_0} \right) = q_1 \ln \frac{h}{r_c}.$$

Thus, when the radius of influence of the well ($\rho$) is equal to the grid step ($h$), for one reservoir the condition (22) has the following difference approximation

$$h^2 L(\sigma, p)_{i_0,j_0} = \frac{2}{\pi} q_1 \ln \frac{h}{r_c}. \tag{26}$$

Accordingly, expression (26) gives a difference approximation at the point-well and for the second layer. Let us assume that the grid spacing for the two layers is the same. In this case, the following expression can be written for the two layers

$$h^2 L_1(\sigma_1, p)_{i_0,j_0} + h^2 L_2(\sigma_2, p)_{i_0,j_0} = \frac{2}{\pi} q_1 \ln \frac{h}{r_c} + \frac{2}{\pi} q_2 \ln \frac{h}{r_c}. \tag{27}$$

Then, taking into account the total flow rate ($Q_0 = q_1 + q_2$) from two layers, we obtain

$$h^2 L_1(\sigma_1, p)_{i_0,j_0} + h^2 L_2(\sigma_2, p)_{i_0,j_0} = \frac{2}{\pi} Q_0 \ln \frac{h}{r_c}, \tag{28}$$

We construct numerical solutions of the problem (6)–(9) by the longitudinal-transverse scheme proposed by Peaceman–Rachford.

## 5 The Peaceman–Rachford method for solving the problem (6)–(9)

Let the grid be square $x = ih$, $y = jh$, $i, j = \pm 1, \pm 2, \ldots$ Then the left side of the difference equation inside the domain on a five-point template with an error of $O(h^2)$ will have the form [12]:

$$h^2 L_k(\sigma_m, p)_{i,j} = \sigma_{k,i+\frac{1}{2},j}\left(p_{i+1,j} - p_{i,j}\right) + \sigma_{k,i-\frac{1}{2},j}\left(p_{i-1,j} - p_{i,j}\right)$$

$$+\sigma_{k,i,j+\frac{1}{2}} + (p_{i,j+1} - p_{i,j}) + \sigma_{k,i,j-\frac{1}{2}}(p_{i,j-1} - p_{i,j}), \tag{29}$$

here $k = 1, 2$ layer numbers.

Let's write down the boundary conditions on the well

$$h^2 L_1(\sigma_1, p)_{i_0,j_0} + h^2 L_2(\sigma_2, p)_{i_0,j_0} = \frac{2}{\pi} Q_0 \ln \frac{h}{r_c},$$

$$p_{2,i_0,j_0} = p_{1,i_0,j_0} + \rho h H. \tag{30}$$

On the contour $r = R$ of two-layered interlayers

$$p_k\left(r\right) = \varphi_0, \quad k = 1, 2. \tag{31}$$

Let us write equation (29) with constant hydraulic conductivity coefficient $\sigma = \text{const}$. Let us introduce the difference operators:

$$p_{\pm 1} p \equiv p_{\pm 2} p \equiv p_{i,j\pm 1}, \quad Ep \equiv p_{i,j}.$$

Let us represent the operator $L_k$ as a sum, e.g. for $k = 1$.

$$L_1 = A_1 + A_2, \quad A_1 = \frac{\sigma_1}{h^2}(p_{+1} - 2E + p_{-1}), \quad A_2 = \frac{\sigma_1}{h^2}(p_{+2} - 2E + p_{-2}).$$

If $p^k = \left\{p_{i,j}^k\right\}$, is known, it is done in two steps through finding the intermediate value $p^{k+1/2} = \left\{p_{i,j}^{k+1/2}\right\}$:

$$\frac{p_{i,j}^{k+1/2} - p_{i,j}^k}{\omega} = h^2 A_1 p^{k+1/2} + h^2 A_2 p^k, \quad 1 \le i \le N-1,\ 1 \le j \le N-1, \tag{32}$$

the corresponding entry is the same at $k = 2$.

Condition (33) at the intermediate stage

$$A_{10} p^{k+1/2} + A_{20} p^k = L_2(\sigma, p)_{i_0,j_0} + \frac{2}{\pi} Q_0 \ln \frac{h}{r_c} / h^2. \tag{33}$$

On the contour of the layers, $p_{i,j}^{k+1/2} = \varphi_0$.

Similarly, the Peaceman–Rachford method is constructed for the second layer in the case of the operator $L_2(\sigma_2, p)$, [15]. The computational algorithm of the problem (6)–(9)

consists of two stages – internal and external iteration.In the case of internal iteration, the pressure function in individual formations is determined. And the outer iteration ends when the nonlocal boundary condition (30) is satisfied.

During the computational process, for example, for the first layer at a well point, from expression (23) we determine $L_1(\sigma_1, p)_{i_0,j_0}$ with fixed $L_2(\sigma_2, p)_{i_0,j_0}$ and we solve $L_1(\sigma_1, p)_{i,j}$ i.e. we perform an internal iteration for the first layer. Then we carry out the internal iteration for the second layer with fixed $L_1(\sigma_1, p)_{i_0,j_0}$ and solve $L_2(\sigma_2, p)_{i,j}$. This procedure continues until condition (30) is satisfied, at which point the outer iteration ends.

Let us briefly dwell on the issue of convergence of the Peaceman–Rachford method [13]. Matrices $A_1$ and $A_2$ are symmetric and negative – definite and have a common complete orthonormal system of eigenvectors

$$2\sin(i\pi_k h_1)\sin(i\pi l h); \quad k = \overline{1, N_1 - 1}; \ l = \overline{1, N_2 - 1}.$$

The eigenvalues of the operator $A_1$ and $A_2$ are $(\lambda_i)_{A_1} = -\frac{4}{h_1^2}\sin^2\left(\frac{i\pi h_1}{2}\right)$, $(\lambda_j)_{A_2} = -\frac{4}{h_2^2}\sin^2\left(\frac{j\pi h_2}{2}\right)$, $1 \le i \le N_1 - 1$, $1 \le j \le N_2 - 1$.

In the case of a tridiagonal matrix, excluding $p^{k+1/2}$, we can write

$$(E - \omega A_1)(E - \omega A_2)p^{k+1} = (E + \omega A_1)(E + \omega A_2)p^k,$$

Hence $p^{k+1} = (E - \omega A_2)^{-1}(E - \omega A_1)^{-1}(E + \omega A_1)(E + \omega A_2)p^k$. It is known that the eigenvalues of the transition operator are taken

$$B = (E - \omega A_1)(E - \omega A_2)(E + \omega A_1)(E + \omega A_2)$$

equal

$$(\lambda_{i,j})_B = \frac{1 + \omega(\lambda_i)_{A_1}}{1 - \omega(\lambda)_{A_1}} \cdot \frac{1 + \omega(\lambda_j)_{A_2}}{1 - \omega(\lambda_j)_{A_1}}. \tag{34}$$

Since $(\lambda_i)_{A_1} < 0$, $(\lambda_j)_{A_2} < 0$, then $(\lambda_{i,j})_B < 1$ and $\omega > 0$ for any $\omega > 0$. Therefore, the Peaceman–Rachford method converges.

If the internal iteration associated with the Peaceman-Reckford method converges, then obviously the external iteration also converges.

In this method, the question of the optimal choice of $\omega$ is a complex issue that is not resolved in all cases. You can proceed as follows: for the first N-1 iterations put [14, 15]

$$\omega_{k+1} = -\frac{1}{(\lambda_i)_{A_1}}, \quad k = \overline{0, N_1 - 2}.$$

Then $(\lambda_{i,j})_B$, $1 \le i \le N_1 - 1$ from (34) will vanish. If at the same time the inequality

$$\max_{\substack{1 \le i \le N_1 - 1 \\ 1 \le j \le N_2 - 1}} \left|p_{i,j}^{k+1} - p_{i,j}^k\right| < \varepsilon,$$

then $\omega$ is then chosen to be equal to

$$\omega_k = -\frac{1}{(\lambda_{k-N_1+2})_{A_2}}, \quad k = N_1 - 1, N_1, \dots, N_1 + N_2 - 3.$$

$R^k$ will be a value of the order of $e^{-\gamma_N k}$. The value $\gamma_N$ characterizes the quality of the iterative method. As is known, for the simplest iteration process $P^{k+1} = P^k + \alpha(Au^k - h^2 f)$ under certain restrictions on $\alpha$ we have $\gamma_N = 1/N^2$.

## Conclusion

The solution of the problem (6)–(9) was carried out with the following parameters. On the contour of the layers, the same pressure was maintained at 15 MPa, the fluid flow rate from the two layers was 80 t/day, the thickness of the layers was 10 m, 15 m, and the thickness of the impermeable interlayer bridge was 0.5 m, the fluid viscosity was 4 Sp, the permeability's were different 0.3 D, 0.5 D. The results obtained by finite difference method were compared with the exact solution. In the filtration area of the radius of influence of the well, the error averaged 0.1 per cent. However, this error did not exceed 0.05 per cent when approaching the reservoir contour, i.e. it decreased. For $\frac{1}{4}$ area they obtained solutions by formulas (18)–(19) are shown in Figures 1 and 2. The same pressure of 14 MPa was maintained at the contour of the two layers, and the well radius was 12 cm. Here, the isobaric surfaces of the pressure field $P(x, y)$ differ by 0.7 MPa. In Figure 4, the isobaric surfaces are obtained with a contour pressure of 13 MPa. Naturally, when the contour pressure decreases, the concentric surfaces occupy less area.

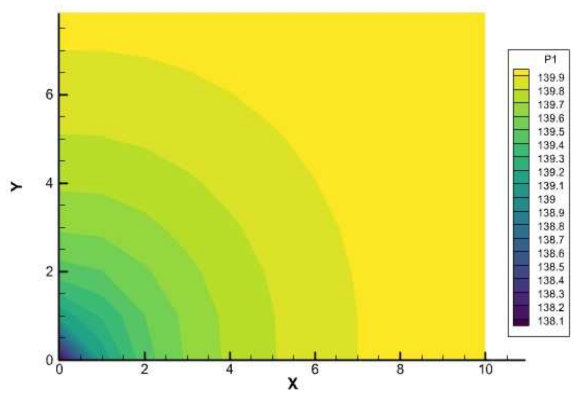


**Figure 3.** Pressure field for the first.

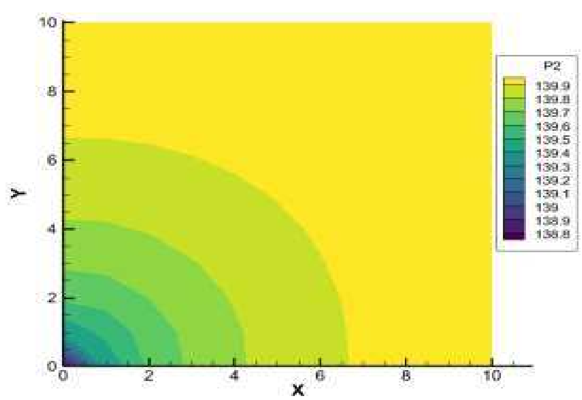


**Figure 4.** Pressure field for the second layer with permeability of 0.5 D.
Layer with permeability of 0.3 D. The contour pressure is 14 MPa.
The contour pressure is 14 MPa.

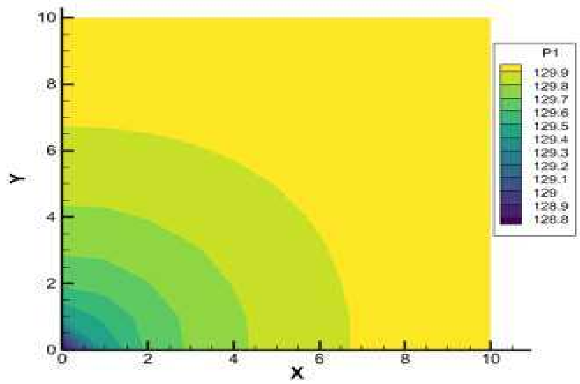


**Figure 5.** Pressure field for the first layer with a permeability of 0.5 D.
Pressure on the contour is 13 MPa.

Joint development of several formations with one well can be cost-effective, especially for low-productivity formations that cannot be exploited separately because they are not economically feasible. The results obtained with the analytical method show the correctness and high accuracy of the numerical finite difference method.

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