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A REGULARIZED TRACE OF A TWO-FOLD DIFFERENTIATION OPERATOR WITH NON-LOCAL MATCHING CONDITIONS ON A STAR GRAPH WITH ARCS OF THE SAME LENGTH

In this paper, we study the regularized trace of a two-fold differentiation operator with non-local matching conditions on a star graph consisting of arcs of the same length. We consider both the integrable case, when the potentials belong to the space L_1, and the singular case, in which the potentials admit more general features, including distributions. The main attention is paid to the derivation of the asymptotic decomposition of the characteristic function corresponding to the boundary value problem on a graph and the calculation of regularized traces using spectral theory methods. The main goal is to calculate the first regularized trace of an operator, which is defined as the limit of the sum of the differences of the eigenvalues of the operator and its modification. It is shown that in the integrable case, the regularized trace is a linear functional of the potential coefficients, whereas in the singular case (when the potentials are represented as generalized functions), it acquires a nonlinear dependence. Explicit formulas for the regularized trace using characteristic determinants and integral representation methods are derived. The results of this work generalize the well-known formulas of regularized traces applied to operators on a segment to the case of more complex structures such as graphs. The work is of interest to specialists in the field of spectral theory of operators and differential equations on graphs.
Key words: Regularized trace, star graph, differential operator, Sturm-Liouville operator.

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Ұзындығы бірдей доғалары бар граф жұлдызындағы локальды емес сәйкестік шарттары бар екі еселенген дифференциалдау операторының реттелген ізі

Бұл жұмыста бірдей ұзындықтағы доғалардан тұратын граф-жұлдыздағы локальды емес сәйкестендіру шарттары бар екі реттік дифференциалдау операторының реттелген ізі зерттеледі. Потенциалдар L_1 кеңістігіне жататын интегралданатын жағдай да, потенциалдар үлестіруді қоса алғанда, жалпы ерекшеліктерге мүмкіндік беретін сингулярлық жағдай да қарастырылады. Графтағы шеткі есептерге сәйкес келетін сипаттамалық функцияның асимптотикалық ыдырауын анықтауға және спектрлік теория әдістерін қолдана отырып, реттелген іздерді есептеуге баса назар аударылады. Негізгі мақсат-оператордың меншікті мәндерінің айырмашылықтарының қосындысының шегі және оның модификациясы ретінде анықталатын оператордың бірінші реттелген ізін есептеу. Интегралданған жағдайда реттелген із потенциал коэффициенттерінің сызықтық функционалы болып табылады, ал сингулярлық жағдайда (потенциалдар жалпыланған функциялар түрінде ұсынылған кезде) ол сызықтық емес тәуелділікке ие болады. Сипаттамалық анықтауыштар мен интегралды бейнелеу әдістерін қолдана отырып, реттелген ізге арналған нақты формулалар шығарылды. Бұл жұмыстың нәтижелері графтар сияқты күрделі құрылымдар жағдайында кесіндідегі операторларға қолданылатын реттелген іздердің белгілі формулаларын жинақтайды. Жұмыс операторлардың спектрлік теориясы және графтардағы дифференциалдық теңдеулер саласындағы мамандарды қызықтырады.
Түйін сөздер: Реттелген із, граф жұлдызы, дифференциалды оператор, Штурм-Лиувилль операторы.

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Регуляризованный след оператора двух кратного дифференцирования с нелокальными условиями согласования на граф-звезде с дугами одинаковой длины

В данной работе исследуется регуляризованный след оператора двукратного дифференцирования с нелокальными условиями согласования на графе-звезде, состоящем из дуг одинаковой длины. Рассматриваются как интегрируемый случай, когда потенциалы принадлежат пространству L_1, так и сингулярный случай, при котором потенциалы допускают более общие особенности, включая распределения. Основное внимание уделяется выводу асимптотического разложения характеристической функции, соответствующей краевой задаче на графе, и вычислению регуляризованных следов с использованием методов спектральной теории. Основной целью является вычисление первого регуляризованного следа оператора, который определяется как предел суммы разностей собственных значений оператора и его модификации. Показано, что в интегрируемом случае регуляризованный след является линейным функционалом от коэффициентов потенциала, тогда как в сингулярном случае (когда потенциалы представлены в виде обобщённых функций) он приобретает нелинейную зависимость. Выведены явные формулы для регуляризованного следа, использующие характеристические определители и методы интегрального представления. Результаты данной работы обобщают известные формулы регуляризованных следов, применяемые к операторам на отрезке, на случай более сложных структур, таких как графы. Работа представляет интерес для специалистов в области спектральной теории операторов и дифференциальных уравнений на графах.

Ключевые слова: регуляризованный след, граф-звезда, дифференциальный оператор, оператор Штурма-Лиувилля.

1 Formulation of the Problem

In [1], the first regularized trace of the Sturm-Liouville operator B was calculated, generated by the differential expression

$$l(y) = -y^{(2)}(x) + \left(h\delta\left(x - \frac{\pi}{2}\right) - \frac{h}{\pi}\right) y(x)$$

on the segment $[0, \pi]$ with Dirichlet boundary conditions. The eigenvalues of operator B are denoted by λ_n for $n > 0$. Then the formula is valid

$$\sum_{n=1}^{\infty}(\lambda_n - n^2 - \frac{1}{\pi} + (-1)^n \frac{1}{\pi}) = -\frac{h^2}{8}. \tag{1}$$

Further generalizations of A.M.Savchuk's formula can be found in [2,3]. In this paper, the formula (1) is generalized to the case of a differential operator on a star graph.

N.P.Bondarenko [4] considers a star graph with more than two arcs. The lengths of the arcs are considered equal to π. In the article [4], the eigenvalue problem B for a twofold differentiation operator on a graph is investigated

$$-y_j^{(2)}(x) = \lambda y_j(x), x \in (0, \pi), \qquad j = \overline{1, m},$$

with Robin conditions in the boundary vertices

$$y_j^{(1)}(0) - h_j y_j(0) = 0, \qquad j = \overline{1,m},$$

with continuity conditions in the inner vertex

$$y_j(\pi) = y_1(\pi), \qquad j = \overline{2,m},$$

and the matching conditions in the inner vertex

$$\sum_{j=1}^{m} \left(y_j^{(1)}(\pi) + \int_0^{\pi} p_j(x)\, y_j(x)\, dx \right) = 0.$$

Here λ is a spectral parameter, and nonzero numbers h_j are complex numbers. In the first part of the article, the functions $p_j(x)$ belong to the space $L_2(0,\pi)$. The eigenvalues of operator B are denoted by $[\lambda_n, n \geq 1]$. Along with operator B, we also consider operator B_0, which is obtained from operator B when $p_j(x) \equiv 0$, $h_j = 0$, $j = \overline{1,m}$. The eigenvalues of the operator B_0 are denoted by $\{\lambda_n^0, n \geq 1\}$.

The purpose of this article is to calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} \left(\lambda_k - \lambda_k^0 - a_k \right),$$

where are the sequences $\{a_k\}$ and $\{m_n\}$ they are selected in a special way. Thus, the calculated sum is called the first regularized trace of the operator B defined on the star graph. Formulas of regularized traces for different classes of differential operators can be found in the works of V.A. Sadovnichy and his students [5].

The work consists of two parts. First, we study the integrable case when the functions $p_j(x)$ belong to the space $L_2(0,\pi)$. In the second part, we study the singular case when the functions $p_j(x)$ represent distributions. In this case, it is assumed that the generalized primitive $q_j(x) = \int p_j(x)\, dx$ are functions of limited variation. It is proved that in the integrable case, the regularized trace is a linear functional of the functions $p_j(x)$. At the same time, the regularized trace in the singular case is a nonlinear functional of the functions $p_j(x)$.

2 The main result in the integrable case

It is convenient to introduce notation to formulate the results. Let

$$P(x) = \sum_{j=1}^{m} p_j(x), \qquad P_1(x) = \sum_{j=1}^{m} h_j p_j(x), \qquad P_2(x) = \sum_{j=1}^{m} \frac{p_j(x)}{h_j},$$

$$H_1 = \sum_{j=1}^{m} h_j, \qquad H_2 = \sum_{j=1}^{m} \frac{1}{h_j}, \qquad \lambda = z^2.$$

Theorem 1 (integrable case) . *Let the nonzero numbers h_j be complex, and the functionsp$_j(x)$ belong to the space $L_2(0,\pi)$. Also assume that the function $P(x)=\sum_{j=1}^{m} p_j(x)$ satisfies the Dini condition at the point $x=\pi$. The, for $m>2$, the formula for the regularized trace is valid*

$$\lim_{n\to\infty}\sum_{k=1}^{m_n}\left(\lambda_k-\lambda_k^0\right)=P(\pi).$$

Here m_n is the number of eigenvalues of the original problem in the circle λ plane of radius $\left(n+\frac{1}{4}\right)^2$ centered at zero.

It was shown in [4] that the characteristic determinants of the operators B_0 and B are given by the equalities

$$\varDelta_0\left(z^2\right)=-mz\sin(z\pi)\left(\cos(z\pi)\right)^{m-1},$$

$$\varDelta(\lambda)=\varDelta_0(\lambda)\left(1+F(\lambda)\right), \tag{2}$$

where

$$F\left(z^2\right)=-\int_0^\pi P(x)\frac{\cos(zx)}{z\sin(z\pi)}dx-\int_0^\pi P_1(x)\frac{\sin(zx)}{z^2\sin(z\pi)}dx-H_2\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}$$

$$-H_1\left(\frac{tg(z\pi)}{z}\right)^{m-1}\int_0^\pi P_2(x)\frac{\cos(zx)}{z\sin(z\pi)}dx-H_1\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}\int_0^\pi P(x)\frac{\sin(zx)}{z^2\sin(z\pi)}dx.$$

In fact, the characteristic determinants $\Delta_0(\lambda)$ and $\Delta(\lambda)$ are integer functions of λ. The zeros of the integer functions $\Delta_0(\lambda)$ and$\Delta(\lambda)$ represent the eigenvalues of operators B_0 and B. Thus, the sequence $\{\lambda_n^0, n\geq 1\}$ represents a sequence of zeros taking into account their multiplicities of the whole function $\Delta_0(\lambda)$. Similarly, the sequence $\{\lambda_n, n\geq 1\}$ it is associated with the zeros of the whole function $\Delta(\lambda)$. Their asymptotic behavior was clarified in [4]. Since the zeros of the whole function $\Delta_0(\lambda)$ break up into series, the zeros of the whole function $\Delta(\lambda)$ also have an asymptotically serial structure (see Theorem 1.2 from [4]). Their asymptotic behavior is clarified in the work [4].

Let us use γ_n to denote a circle in the z-plane of radius $n+\frac{1}{4}$ centered at zero. It is easy to understand that the function $\operatorname{tg}(z\pi)$ on the circles γ_n is bounded by a constant independent of n. If x is fixed between zero and π, then the functions $\frac{\cos(zx)}{\sin(z\pi)}$ and $\frac{\sin(zx)}{\sin(z\pi)}$ on the circles γ_n are bounded by a constant independent of n. For sufficiently large n, the function $\ln(1+F(\lambda))$ is holomorphic on the circle γ_n. Now let's try to calculate the integral $\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\varDelta\left(z^2\right)$. According to the principle of the argument, we have

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\varDelta\left(z^2\right)=2\sum_{k=1}^{m_n}\lambda_k. \tag{3}$$

Here $m_n = \frac{1}{2\pi i}\oint_{\gamma_n} d\ln\Delta\left(z^2\right) = \frac{1}{2\pi i}\oint_{\gamma_n} d\ln\Delta_0\left(z^2\right)$ for sufficiently large n. On the other hand, the ratio (2) implies

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\Delta\left(z^2\right) = \frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\Delta_0\left(z^2\right) + \frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right).$$

It is clear that

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\Delta_0\left(z^2\right) = 2\sum_{k=1}^{m_n}\lambda_k^0. \tag{4}$$

Applying the piecemeal integration formula allows us to write the ratio

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right) = -2\frac{1}{2\pi i}\oint_{\gamma_n} z\ln\left(1+F\left(z^2\right)\right)dz.$$

Let $m>3$. For sufficiently large n, we rewrite the last equality as

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right) = -2\frac{1}{2\pi i}\oint_{\gamma_n} zF\left(z^2\right)dz + o(1). \tag{5}$$

It is taken into account here that for $n\to\infty$ on the circles γ_n, the function $F\left(z^2\right) = o\left(\frac{1}{n}\right)$, since the functions $p_j(x)$ belong to the space $L_2(0,\pi)$. It remains to calculate the integral $\frac{1}{2\pi i}\oint_{\gamma_n} zF\left(z^2\right)dz$ using the deduction theorem. When $n\to\infty$ we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\gamma_n} zF\left(z^2\right)dz &= \frac{1}{2\pi i}\oint_{\gamma_n} z\left(-\int_0^\pi P(x)\frac{\cos(zx)}{z\sin(z\pi)}dx\right)dz + o(1)\\ &= -\int_0^\pi P(x)\left(\frac{1}{2\pi i}\oint_{\gamma_n}\frac{\cos(zx)}{z\sin(z\pi)}dz\right)dx + o(1)\\ &= -\frac{1}{\pi}\int_0^\pi P(x)\left(1+2\sum_{k=1}^{n}(-1)^k\cos(kx)\right)dx + o(1).\end{aligned} \tag{6}$$

Thus, it follows from the relations (3)–(6) that for $m>3$ the formula of the regularized trace has the form

$$\lim_{n\to\infty}\sum_{k=1}^{m_n}\left(\lambda_k-\lambda_k^0\right) = P(\pi). \tag{7}$$

Now consider the case of $m=3$. For sufficiently large n, equality (5) can be written as

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right)\\ &= -2\frac{1}{2\pi i}\oint_{\gamma_n} z\left(-\int_0^\pi P(x)\frac{\cos(zx)}{z\sin(z\pi)}dx\right)dz + 2H_2\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}dz + o(1).\end{aligned}$$

We need to calculate the integral $\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^2 dz$ using the deduction theorem:

$$\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^2 dz = 0$$

Thus, the contribution from the integral $\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^2 dz$ in the sum (7) is missing. In the case of $m = 3$, the formula (7) is preserved.

3 The singular case

In this paragraph, the generalized primitive $q_j(x) = \int p_j(x)\,dx$ are functions of limited variation. In particular, the case when the generalized primordial $q_j(x)$ represent the jump functions is studied in detail. In this case, the Stieltjes integral is calculated using the formula

$$\int_0^\pi y(x)\,dQ(x) = \sum_{s=1}^{r} t_s y(x_s),$$

where $dQ(x) = P(x)\,dx$.

Then

$$F\left(z^2\right) = -\sum_{s=1}^{r} t_s\frac{\cos(zx_s)}{z\sin(z\pi)} - \sum_{s=1}^{r_1} t_{s1}\frac{\sin(zx_{s1})}{z^2\sin(z\pi)} - H_2\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}$$

$$-H_1\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}\sum_{s=1}^{r_2} t_{s2}\frac{\cos(zx_{s2})}{z\sin(z\pi)} - H_1\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}\sum_{s=1}^{r} t_s\frac{\sin(zx_s)}{z^2\sin(z\pi)}.$$

In this case, the ratio (5) will be written as

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right) &= -2\frac{1}{2\pi i}\oint_{\gamma_n} z\ln\left(1+F\left(z^2\right)\right)dz\\
&= 2\frac{1}{2\pi i}\oint_{\gamma_n} z\sum_{s=1}^{r} t_s\frac{\cos(zx_s)}{z\sin(z\pi)}dz + 2\frac{1}{2\pi i}\oint_{\gamma_n} z\sum_{s=1}^{r_1} t_{s1}\frac{\sin(zx_{s1})}{z^2\sin(z\pi)}dz\\
&+ 2H_2\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1} dz-\\
&- \frac{1}{2\pi i}\oint_{\gamma_n} z\left(\sum_{s=1}^{r} t_s\frac{\cos(zx_s)}{z\sin(z\pi)}\right)^2 dz + o(1), \qquad n\to\infty.\end{aligned}$$

It remains to calculate the required integrals using the deduction theorem:

$$2\frac{1}{2\pi i}\oint_{\gamma_n}\sum_{s=1}^{r} t_s\frac{\cos(zx_s)}{\sin(z\pi)}dz = 2\sum_{s=1}^{r}\left(2\sum_{k=1}^{n}\frac{\cos(kx_s)}{\pi(-1)^k} + \frac{1}{\pi}\right)t_s,$$

$$2\frac{1}{2\pi i}\oint_{\gamma_n} z\sum_{s=1}^{r_1} t_{s1}\frac{\sin(zx_{s1})}{z^2\sin(z\pi)}dz = 2\sum_{s=1}^{r_1} t_{s1}\left(\frac{x_{s1}}{\pi}+2\sum_{k=1}^{n}(-1)^k\frac{\sin(kx_{s1})}{k\pi}\right),$$

$$\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\sum_{s=1}^{r} t_s\frac{\cos(zx_s)}{z\sin(z\pi)}\right)^2 dz = 2\sum_{k=1}^{r}\frac{1}{k}\left(\sum_{s=1}^{r} t_s\cos(kx_s)\right)^2.$$

Thus, in the singular case, the regularized trace formula has the form

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{m_n}\left(\lambda_k-\lambda_k^0\right)-\sum_{s=1}^{r}\left(2\sum_{k=1}^{n}\frac{\cos(kx_s)}{\pi(-1)^k}+\frac{1}{\pi}\right)t_s\right\}$$
$$=-\sum_{w=1}^{r}\sum_{s=1}^{r} t_w t_s\left(C(x_w+x_s)+C(x_w-x_s)\right)+\sum_{s=1}^{r_1} t_{s1}A_s, \tag{8}$$

where $A_s=\frac{x_{s1}}{\pi}+2\sum_{k=1}^{\infty}(-1)^k\frac{\sin(kx_{s1})}{k\pi}$, $C(x_s)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos(kx_s)}{k}$. Since $A_s=0$. Then (8) will take the form

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{m_n}\left(\lambda_k-\lambda_k^0\right)-\sum_{s=1}^{r}\left(2\sum_{k=1}^{n}\frac{\cos(kx_s)}{\pi(-1)^k}+\frac{1}{\pi}\right)t_s\right\}$$
$$=-\sum_{w=1}^{r}\sum_{s=1}^{r} t_w t_s\left(C(x_w+x_s)+C(x_w-x_s)\right) \tag{9}$$

Let requirement 1 be fulfilled: for an arbitrary continuous function $y(x)$, the integrals satisfy the equalities

$$\int_0^\pi y(x)\,dQ(x)=\sum_{s=1}^{r} t_s y(x_s),$$

$$\int_0^\pi y(x)\,dQ_1(x)=\sum_{s=1}^{r_1} t_{s1} y(x_{s1}),$$

where $dQ(x)=P(x)\,dx$ and $dQ_1(x)=P_1(x)\,dx$.

Theorem 2 (the singular case) *Let the nonzero numbers h_j be complex, the set of functions $\{p_j(x)\}$ is subject to requirement 1. Then, for $m>2$, the formula (9) is valid for a regularized trace.*

Thus, from formula (9) we see that the regularized trace in the singular case is a nonlinear functional from jumps $\{t_s\}$. At the same time, from the theorem 1 we see that the regularized trace in the integrable case is a linear functional of the functions $\{p_j(x)\}$. A similar effect in the case of differential operators on a segment was noted in [1, 2, 3]. In this paper, it is shown that the A. M. Savchuk effect is also preserved for second-order differential operators on a star graph.

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References

[1] Savchuk A.M., ”A regularized first–order trace of the Sturm-Liouville operator with a δ-potential“, *UMN*, 55:6, (2000): 155–156.

[2] Savchuk A.M., Shkalikov A. A., ”Trace formula for Sturm–Liouville operators with singular potentials“, *Math. Notes*, 69:3, (2001): 427-442.

[3] Galkovsky E. D., Nazarov A. I., ”On the trace formula for high-order ordinary differential operators“, *Mathematical Collection*, 212:5, (2021): 80-101.

[4] Bondarenko N.P., ”Inverse problem for a differential operator on a star-shaped graph with nonlocal matching condition“, *Boletin de la Sociedad Matematica Mexican*, 29(1):2, (2023).

[5] Sadovnichy V.A., Podolsky V.E., ”Traces of operators“, *UMN*, 61:5(371), (2006): 89-156.

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