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INVERSE SOURCE RECOVERY IN A CLASS OF SINGULAR DIFFUSION EQUATIONS VIA OPTIMAL CONTROL

This paper addresses the inverse problem of identifying a space-dependent source term in a singular parabolic equation involving an inverse-square potential, knowing final time measurement data. The problem is reformulated within an optimal control framework, minimizing a Tikhonov-regularized functional to ensure stability. Theoretical contributions include existence and uniqueness of weak solutions for the direct problem, along with a stability estimate for the inverse problem under a first-order optimality condition. A Landweber-type iterative algorithm is designed for numerical reconstruction, validated through synthetic examples with both exact and noisy data.

Key words: Inverse problem, singular parabolic equation, stability; regularization, Landweber method.

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Сингулярлық диффузиялық теңдеулер класы үшін көзді оңтайлы басқару әдісімен қалпына келтіру

Бұл жұмыста кері квадратты потенциалы бар сингулярлық параболалық теңдеудегі кеңістіктік тәуелді көзді анықтаудың кері есебі қарастырылады, ол ақырлы уақыт мезетіндегі өлшеу деректерін пайдаланады. Есеп тиімді басқарудағы орнықтылықты қамтамасыз ету үшін Тихоновтың регуляризацияланған функционалын минимизациялауға негізделіп тұжырымдалған. Теориялық нәтижелері ретінде тура есеп үшін әлсіз шешімнің бар және жалғыздығы дәлелденуін, сонымен қатар, бірінші реттік оптималдық шарты орындалған жағдайда кері есептің орнықтылығының бағалауын айтуға болады. Сандық нәтижелері ретінде дәл және шулы деректермен синтетикалық мысалдарда тексерілген Ландвебер типіндегі итерациялық алгоритм әзірленді.

Түйінді сөздер: Кері есеп, сингулярлы параболалық теңдеу, тұрақтылық, регуляризация, Ландвебер әдісі.

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Восстановление источника в классе сингулярных уравнений диффузии с использованием метода оптимального управления

В данной работе рассматривается обратная задача идентификации пространственно-зависимого источника в сингулярном параболическом уравнении с обратно-квадратичным потенциалом на основе данных измерений в конечный момент времени. Задача переформулируется в рамках оптимального управления путём минимизации регуляризованного функционала Тихонова, что обеспечивает устойчивость решения. Теоретические результаты включают доказательство существования и единственности слабого решения для прямой задачи, а также оценку устойчивости для обратной задачи, основанную на условии оптимальности первого порядка. Для численной реконструкции разработан итерационный алгоритм типа Ландвебера, эффективность которого подтверждена на синтетических примерах с точными и зашумлёнными данными.

Түйінді сөздер: Обратная задача, сингулярное параболическое уравнение, стабильность, регуляризация, Метод Ландвебера.

1 introduction

Inverse problems are concerned with the identification of unknown inputs or sources from partial or indirect observations of the system's response, in contrast to forward problems, where the output is computed from given inputs. It is well known that inverse problems are often ill-posed in the sense of Hadamard; that is, the solution may not exist, may not be unique, or may not depend continuously on the data. Consequently, small perturbations in the measurements—such as those due to noise—can lead to significant errors in the solution

In the present work, we investigate the inverse problem of identifying a spatially dependent source term in a singular parabolic equation from measurements of the solution at a fixed final time. More precisely, we consider the following initial-boundary value problem

$$\begin{cases} \partial_t \theta(x,t) - \theta_{xx}(x,t) - \dfrac{\mu}{|x|^2}\theta(x,t) = f(x), \quad (x,t) \in Q_T := \Omega \times (0,T), \\ \theta(0,t) = \theta(1,t) = 0, \quad t \in (0,T), \\ \theta(x,0) = \theta_0(x), x \in \Omega, \end{cases} \tag{1}$$

where $\Omega := (0,1)$, $0 < T < \infty$ is an arbitrary final fixed time, θ_0 is a given smooth function describe the initial state, $f(x)$ represents the unknown source term which is assumed to be kept independent of time variable t.

We are particularly concerned with the inverse problem of recovering the spatially dependent source term $f(x)$ appearing in the governing parabolic equation. To this end, we assume that the solution $u(x,t)$ is observed at the final time $t = T$ over the spatial domain Ω, that is

$$u(x,T) = \omega(x), \quad x \in \Omega, \tag{2}$$

where $\omega \in L^2(\Omega)$ denotes the final-time observation. When the source term $f(x)$ is known, the associated initial-boundary value problem (1) defines the so-called direct (or forward) problem. In the present study, however, $f(x)$ is unknown and must be identified from the final observation (2). Accordingly, we formulate the inverse problem as the determination of $f(x)$ from a prescribed admissible class such that the corresponding solution to (1) satisfies the final-time constraint (2).

Singular inverse-square potentials have attracted considerable attention in recent years due to their relevance in modeling various physical phenomena across multiple disciplines, including quantum cosmology [5], combustion theory [6], electron capture processes [8], and quantum mechanics [7]. Moreover, such potentials naturally arise in the linearization of certain reaction–diffusion systems governed by the heat equation involving supercritical source terms [1].

In the context of inverse problems for parabolic equations, a substantial body of literature has addressed issues related to stability and well-posedness for various classes of equations using a range of analytical and numerical techniques [12, 17–21].

Concerning inverse problems for singular parabolic equations, we mention, among other works, the study in [15], where the inverse source problem for the model (1) was investigated in a multidimensional setting. In [11], the author addressed the inverse problem of identifying a source term in degenerate singular parabolic equations, with degeneracy and singularity occurring in the interior of the spatial domain. More recently, in [14], the inverse source problem for a heat equation involving multipolar inverse-square potentials was considered.

From a numerical perspective, it is worth noting that only a limited number of works have been devoted to the identification of source terms or coefficients in parabolic equations with inverse-square potentials, despite the fact that such models arise naturally in both theoretical studies and applied contexts.

In contrast to the aforementioned studies, which commonly rely on techniques based on Carleman estimates [17], our approach is framed within the context of optimal control theory—a widely used methodology for addressing inverse source problems in a broad class of evolution equations [1, 10, 13, 22]. Specifically, we recast the inverse problem as an optimal control problem, where the unknown source term is treated as a control variable. The objective is then to minimize a suitably defined cost functional, which yields a quasi-solution to the original inverse problem.

By deriving and analyzing the first-order necessary optimality conditions, we establish both the local stability and uniqueness of the quasi-solution. More precisely, our main stability result can be stated as follows: let (U, f) and $(\tilde{U}, \tilde{f})$ be two solutions to the inverse problem (1)–(2) corresponding to final-time observations ω and $\tilde{\omega}$, respectively. Then, there exists a constant $C > 0$, independent of the final time T, such that

$$\|f - \tilde{f}\|^2_{L^2(\Omega)} \leq C\|\omega - \tilde{\omega}\|^2_{L^2(\Omega)}.$$

The second main contribution of this work concerns the numerical reconstruction of the unknown source term in the problem (1), based on the final-time observation (2). To this end, we develop a numerical scheme built upon the well-known Landweber iterative method. This approach has proven to be both reliable and efficient, as demonstrated through a series of numerical experiments.

The remainder of the paper is organized as follows. In Section 2, we establish the well-posedness of the direct problem (1). Section 3 is devoted to the analysis of the inverse problem within an optimal control framework; in particular, we prove the existence of a minimizer for the cost functional and derive the associated first-order necessary optimality condition. In Section 4, using the optimality condition, we establish a stability result for the inverse problem. Section 5 is concerned with the numerical reconstruction of the unknown source term. To this end, we implement a Landweber-type iterative method to compute an approximate solution to the inverse problem based on the final-time data.

2 Analysis of the direct problem

2.1 Functional framework

As is well known in the analysis of parabolic equations involving singular inverse-square potentials, the constant μ plays a crucial role in determining the well-posedness of the associated problem. Specifically, there exists a critical threshold $\mu^* > 0$ beyond which the problem becomes ill-posed. This upper bound is given by the optimal constant in the Hardy inequality, which ensures that for any function $z \in H_0^1(\Omega)$, the weighted function $\frac{z}{x} \in L^2(\Omega)$, and the following inequality holds:

$$\mu^* \int_\Omega \frac{z^2(x)}{x^2}\, dx \leq \int_\Omega |z_x(x)|^2\, dx. \tag{3}$$

In the one-dimensional setting $\Omega = (0,1)$, it is known that the critical constant is $\mu^* = \frac{1}{4}$. For fixed $\mu \in (0, \mu^*]$, we define the following functional space:

$$H^1_{\mu,0}(\Omega) := \left\{ z \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega) \,:\, z(0) = z(1) = 0, \quad \int_\Omega \left(z_x^2(x) - \mu \frac{z^2(x)}{x^2} \right) dx < +\infty \right\}.$$

This space is a Hilbert space when equipped with the inner product

$$(z_1, z_2)_\mu := \int_\Omega \left(z_{1,x}(x) z_{2,x}(x) - \mu \frac{z_1(x) z_2(x)}{x^2} \right) dx,$$

and the corresponding norm

$$\|z\|_\mu := \left(\int_\Omega \left(z_x^2(x) - \mu \frac{z^2(x)}{x^2} \right) dx \right)^{1/2}.$$

By standard arguments, one can show that there exist positive constants $C_1, C_2 > 0$, depending on μ, such that

$$(1 - 4\mu) \int_\Omega z_x^2\, dx + C_1 \int_\Omega z^2\, dx \leq \|z\|_\mu^2 \leq (1 + 4\mu) \int_\Omega z_x^2\, dx + C_2 \int_\Omega z^2\, dx.$$

This implies that for the subcritical case $\mu < \mu^*$, the spaces $H^1_{\mu,0}(\Omega)$ and $H^1_0(\Omega)$ are topologically equivalent with respect to their norms. However, in the critical case $\mu = \mu^*$, the space $H^1_{\mu,0}(\Omega)$ strictly contains $H^1_0(\Omega)$, that is,

$$H^1_0(\Omega) \subsetneq H^1_{\mu,0}(\Omega).$$

In this work, we restrict our attention to the subcritical case $0 < \mu < \mu^*$. Now, le us define the space $H^1_\mu(\Omega)$ as the completion of $H^1(\Omega)$ with respect to the norm

$$\|z\|_{H^1_\mu(\Omega)} := \left(\|z\|^2_{L^2(\Omega)} + \|z\|^2_\mu\right)^{1/2}.$$

Accordingly, we may write

$$H^1_{\mu,0}(\Omega) = \left\{z \in H^1_\mu(\Omega) \,:\, z(0) = z(1) = 0\right\}.$$

Under the assumption $\mu < \frac{1}{4}$, it is known that $H^1_\mu(\Omega)$ embeds continuously into the Sobolev space $W^{1,q}_0(\Omega)$ for all $1 \leq q < 2$, and also into the fractional Sobolev spaces $H^s_0(\Omega)$ for all $0 \leq s < 1$. Moreover, due to the compact embedding $W^{1,q}_0(\Omega) \hookrightarrow H^s_0(\Omega)$ for suitable $q = q(s)$ sufficiently close to 2, and the compactness of $H^s_0(\Omega) \hookrightarrow L^2(\Omega)$, we conclude that

$$H^1_\mu(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega),$$

where the embedding is compact. For more details on the properties of $H^1_\mu(\Omega)$, we refer the reader to [2] and [15].

2.2 Well-posedness of the Direct Problem

In order to analyze the inverse problem associated with the differential equation under consideration, a thorough understanding of the corresponding direct problem is essential. Therefore, we begin by establishing the well-posedness of the direct problem, with a detailed analysis of the existence, uniqueness, and regularity of its solutions.

To define a weak solution, we multiply equation (1) by a test function $\phi \in H^1_{\mu,0}(\Omega)$, integrate over Ω, and use integration by parts. This leads to the following variational formulation.

Definition 1. *Let $\theta_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$. A function θ is said to be a* weak solution *to problem* (1) *if*

$$\theta \in L^2(0,T; H^1_{\mu,0}(\Omega)), \quad \theta_t \in L^2(0,T; H^{-1}_\mu(\Omega)),$$

and for all test functions $\phi \in L^2(0,T; H^1_{\mu,0}(\Omega))$, the following variational identity holds:

$$\iint_{Q_T} \theta_t \phi \, dx \, dt + \iint_{Q_T} \theta_x \phi_x \, dx \, dt - \mu \iint_{Q_T} \frac{\theta\phi}{x^2} \, dx \, dt = \iint_{Q_T} f\phi \, dx \, dt, \tag{4}$$

with the initial condition $\theta(0) = \theta_0$ satisfied in $L^2(\Omega)$.

Remark 1. *The use of the weighted Sobolev space $H^1_{\mu,0}(\Omega)$ is crucial due to the singularity of the potential term $\mu x^{-2}\theta$, which renders the classical space $H^1_0(\Omega)$ inadequate when $\mu > 0$. For $\mu < \mu^*$, the Hardy inequality ensures that the bilinear form associated with the operator is coercive on $H^1_{\mu,0}(\Omega)$.*

Before formulating the inverse problem, it is necessary to establish that the associated direct problem is well posed.
This ensures that for any admissible source term, the governing singular parabolic equation admits a unique weak solution that depends continuously on the data.
Such a result guarantees that the forward operator is mathematically well defined, which is a fundamental prerequisite for the subsequent optimal control framework.

Theorem 1. *Let $\theta_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$. Then, problem* (1) *admits a unique weak solution θ in the sense of Definition 1, satisfying*

$$\theta \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_{\mu,0}(\Omega)), \quad \theta_t \in L^2(0,T; H^{-1}_\mu(\Omega)).$$

Moreover, the following a priori energy estimate *holds:*

$$\sup_{t\in[0,T]} \|\theta(t)\|^2_{L^2(\Omega)} + \int_0^T \|\theta(t)\|^2_\mu \, dt + \int_0^T \|\theta_t(t)\|^2_{H^{-1}_\mu(\Omega)} \, dt \le C \left(\|\theta_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q_T)} \right), \quad (5)$$

where the constant $C > 0$ depends only on μ, Ω, and T.

3 Optimal control

The inverse problem under consideration is ill-posed in the sense of Hadamard, meaning that uniqueness and stability of solutions cannot be guaranteed without introducing additional constraints.
A widely used strategy in such cases is to recast the inverse problem as an optimal control problem, where the unknown source term is treated as a control variable.
This approach allows us to incorporate a regularization mechanism that stabilizes the inversion procedure.

More precisely, the inverse problem is reformulated as the minimization of a Tikhonov-type cost functional, consisting of two terms: a data misfit term that enforces consistency with the final-time observation, and a penalty term that ensures stability by controlling the norm of the source.
The admissible set of controls is restricted to bounded functions in $L^2(\Omega)$, which reflects a priori physical knowledge about the source.

This optimal control formulation serves as the foundation for the subsequent analysis. In particular, it allows us to establish the existence of minimizers (**Th**2), to derive necessary optimality conditions (**Th**3), and to prove stability estimates for the reconstructed source (**Th**4). Hence, Section 3 plays a crucial role in bridging the direct analysis of the forward problem with the theoretical and numerical treatment of the inverse problem.

4 Formulation of the Inverse Problem

The inverse problem addressed in this work can be stated as follows: given an initial condition $\theta_0(x) \in L^2(\Omega)$ and a final-time observation $\omega(x) \in L^2(\Omega)$, determine the spatially dependent source term $f(x)$ such that the corresponding solution θ to the initial-boundary value problem (1) satisfies the over-specified final condition

$$\theta(x,T) = \omega(x), \quad \text{for all } x \in \Omega. \tag{6}$$

To tackle this ill-posed problem, we adopt an optimal control framework. The inverse problem is reformulated as the following constrained optimization problem: find $f^* \in \mathcal{A}$ such that

$$\min_{f\in\mathcal{A}} \mathcal{J}(f) = \mathcal{J}(f^*), \quad \text{subject to } \theta[f] \text{ solving (1)}, \tag{7}$$

where the cost functional $\mathcal{J} : L^2(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{J}(f) := \frac{1}{2}\,\|\theta[f](\cdot,T) - \omega\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|f\|^2_{L^2(\Omega)}, \tag{8}$$

and $\gamma > 0$ is a regularization parameter. The admissible set $\mathcal{A} \subset L^2(\Omega)$ is given by

$$\mathcal{A} := \left\{ f \in L^2(\Omega) \,:\, c_0 \le f(x) \le c_1 \text{ a.e. in } \Omega \right\}, \tag{9}$$

for some constants $0 < c_0 < c_1$. The regularization term in (8) ensures the stability of the minimization problem and reflects a priori bounds on the unknown source.

Next, we establish the existence of an optimal solution to the minimization problem (7) by means of the following result.

Theorem 2. *Let $\theta_0 \in L^2(\Omega)$, $\omega \in L^2(\Omega)$, and assume that the direct problem (1) admits a unique weak solution $\theta[f]$ for every $f \in \mathcal{A}$, as guaranteed by Theorem 1. Then, the optimal control problem (7) admits at least one solution; that is, there exists $f^* \in \mathcal{A}$ such that*

$$\mathcal{J}(f^*) = \min_{f\in\mathcal{A}} \mathcal{J}(f).$$

Proof 1. *Since $\mathcal{J}(f) \ge 0$ for all $f \in \mathcal{A}$, the cost functional $\mathcal{J}$ admits an infimum over the admissible set $\mathcal{A}$, denoted by*

$$d := \inf_{f\in\mathcal{A}} \mathcal{J}(f).$$

Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence such that

$$d < \mathcal{J}(f_n) \le d + \frac{1}{n}, \quad \text{for all } n \in \mathbb{N}^*. \tag{10}$$

Since $\mathcal{A} \subset L^2(\Omega)$ is closed, convex, and bounded, there exists a subsequence (still denoted f_n) and a limit $f^* \in \mathcal{A}$ such that

$$f_n \rightharpoonup f^* \quad \text{weakly in } L^2(\Omega). \tag{11}$$

Let $\theta_n := \theta[f_n]$ denote the unique weak solution to problem (1) with source term f_n. By Theorem 1, the sequence (θ_n) is uniformly bounded in the spaces

$$L^2(0,T;H^1_{\mu,0}(\Omega)), \quad L^\infty(0,T;L^2(\Omega)), \quad \text{and} \quad L^2(0,T;H^{-1}_\mu(\Omega)).$$

Hence, up to a subsequence, there exists $\theta^* \in L^2(0,T;H^1_{\mu,0}(\Omega))$ *such that*

$$\begin{aligned} &\theta_n \rightharpoonup \theta^* \quad \textit{weakly in } L^2(0,T;H^1_{\mu,0}(\Omega)), \\ &\theta_n \overset{*}{\rightharpoonup} \theta^* \quad \textit{weakly-* in } L^\infty(0,T;L^2(\Omega)), \\ &\partial_t\theta_n \rightharpoonup \partial_t\theta^* \quad \textit{weakly in } L^2(0,T;H^{-1}_\mu(\Omega)). \end{aligned} \tag{12}$$

Furthermore, by the Aubin–Lions lemma and the compact embedding $H^1_{\mu,0}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$*, we also obtain the strong convergence*

$$\theta_n \to \theta^* \quad \textit{strongly in } L^2(Q_T). \tag{13}$$

Now, subtracting the weak formulations satisfied by $\theta^* = \theta[f^*]$ *and* $\theta_n = \theta[f_n]$*, and testing the resulting equation by* $\phi = \theta^* - \theta_n$*, we obtain the energy inequality:*

$$\frac{1}{2}\frac{d}{dt}\|\theta^*(t) - \theta_n(t)\|^2_{L^2(\Omega)} \le h(t)\int_\Omega (f^*(x) - f_n(x))(\theta^*(x,t) - \theta_n(x,t))\,dx. \tag{14}$$

Integrating both sides over $(0,T)$*, we get*

$$\|\theta^*(T) - \theta_n(T)\|^2_{L^2(\Omega)} \le \iint_{Q_T} h(t)(f^*(x) - f_n(x))(\theta^*(x,t) - \theta_n(x,t))\,dx\,dt.$$

Using the weak convergence $f_n \rightharpoonup f^*$ *in* $L^2(\Omega)$ *and strong convergence* $\theta_n \to \theta^*$ *in* $L^2(Q_T)$*, we deduce that the right-hand side vanishes as* $n \to \infty$*, hence:*

$$\|\theta^*(T) - \theta_n(T)\|_{L^2(\Omega)} \to 0 \quad \textit{as } n \to \infty. \tag{15}$$

To conclude, we analyze the convergence of the misfit term. Define:

$$\begin{aligned} I_n &:= \left| \|\theta^*(T) - \omega\|^2_{L^2(\Omega)} - \|\theta_n(T) - \omega\|^2_{L^2(\Omega)} \right| \\ &\le \|\theta^*(T) - \theta_n(T)\|_{L^2(\Omega)} \cdot \|\theta^*(T) + \theta_n(T) - 2\omega\|_{L^2(\Omega)}. \end{aligned}$$

Due to (15)*, we conclude:*

$$\lim_{n\to\infty} \|\theta_n(T) - \omega\|^2_{L^2(\Omega)} = \|\theta^*(T) - \omega\|^2_{L^2(\Omega)}. \tag{16}$$

Finally, applying weak lower semi-continuity of the L^2*-norm to* (11)*, and using* (16)*, we obtain:*

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{J}(f_n) &= \liminf_{n\to\infty} \left(\frac{1}{2}\|\theta_n(T) - \omega\|^2 + \frac{\gamma}{2}\|f_n\|^2 \right) \\ &\ge \frac{1}{2}\|\theta^*(T) - \omega\|^2 + \frac{\gamma}{2}\|f^*\|^2 = \mathcal{J}(f^*). \end{aligned}$$

Combining this with the minimality of the sequence (10)*, we conclude that* f^* *is indeed a minimizer of the functional* $\mathcal{J}$*, i.e.,* $\mathcal{J}(f^*) = d$*. This completes the proof.*

Theorem 3 provides the first-order necessary condition characterizing the optimal control. This condition links the unknown source term with the adjoint state and plays a central role both in the theoretical analysis of stability and in the numerical implementation of the Landweber-type method.

Theorem 3. *Let $f^* \in \mathcal{A}$ be an optimal solution to the control problem (7), and let $\theta^* := \theta[f^*]$ denote the corresponding solution to the state equation (1). Then, the following variational inequality holds:*

$$\int_\Omega [\theta^*(x,T) - \omega(x)]\, \xi(x,T)\, dx + \gamma \int_\Omega f^*(x)\,(h(x) - f^*(x))\, dx \geq 0, \quad \forall h \in \mathcal{A}, \tag{17}$$

where $\xi \in L^2(0,T; H^1_{\mu,0}(\Omega)) \cap C([0,T]; L^2(\Omega))$ is the unique weak solution to the following adjoint problem:

$$\begin{cases} \partial_t \xi(x,t) - \xi_{xx}(x,t) - \dfrac{\mu}{x^2}\xi(x,t) = h(x) - f^*(x), & \text{in } Q_T := \Omega \times (0,T), \\ \xi(0,t) = \xi(1,t) = 0, & \text{for } t \in (0,T), \\ \xi(x,0) = 0, & \text{for } x \in \Omega := (0,1). \end{cases} \tag{18}$$

Proof 2. *Let $h \in \mathcal{A}$ and $\delta \in [0,1]$, and define a convex perturbation of the optimal control f^* by*

$$f_\delta := f^* + \delta(h - f^*).$$

Since $\mathcal{A}$ is convex, it follows that $f_\delta \in \mathcal{A}$ for all $\delta \in [0,1]$. Let $\theta_\delta := \theta[f_\delta]$ denote the unique weak solution to problem (1) associated with the control f_δ.

We define the perturbed cost functional

$$\mathcal{J}_\delta := \mathcal{J}(f_\delta) = \frac{1}{2}\int_\Omega |\theta_\delta(x,T) - \omega(x)|^2 dx + \frac{\gamma}{2}\int_\Omega |f_\delta(x)|^2 dx. \tag{19}$$

Since f^* is an optimal control, the function $\delta \mapsto \mathcal{J}(f_\delta)$ attains its minimum at $\delta = 0$. Therefore, the derivative of $\mathcal{J}_\delta$ with respect to δ satisfies

$$\frac{d}{d\delta}\mathcal{J}(f_\delta)\bigg|_{\delta=0} \geq 0. \tag{20}$$

We now compute this derivative. By differentiating under the integral sign and using the chain rule, we obtain:

$$\frac{d}{d\delta}\mathcal{J}(f_\delta) = \int_\Omega [\theta_\delta(x,T) - \omega(x)]\, \frac{\partial \theta_\delta}{\partial \delta}(x,T)\, dx + \gamma \int_\Omega f_\delta(x)(h(x) - f^*(x))\, dx. \tag{21}$$

Evaluating (21) at $\delta = 0$, we define $\xi := \frac{\partial \theta_\delta}{\partial \delta}\big|_{\delta=0}$. Then inequality (20) becomes:

$$\int_\Omega [\theta^*(x,T) - \omega(x)]\, \xi(x,T)\, dx + \gamma \int_\Omega f^*(x)(h(x) - f^*(x))\, dx \geq 0, \tag{22}$$

which is precisely the desired variational inequality (17).

It remains to characterize ξ. Differentiating the state equation with respect to δ, we find that ξ satisfies the following linearized problem:

$$\begin{cases} \partial_t \xi(x,t) - \xi_{xx}(x,t) - \dfrac{\mu}{x^2}\xi(x,t) = h(x) - f^*(x), & \text{in } Q_T, \\ \xi(0,t) = \xi(1,t) = 0, & t \in (0,T), \\ \xi(x,0) = 0, & x \in \Omega, \end{cases}$$

which coincides with problem (18)*. This concludes the proof.*

5 Stability Results

In this section, we investigate the stability of the inverse problem with respect to perturbations in the final-time observation data. Stability plays a central role in inverse problems, especially due to their inherent ill-posedness in the sense of Hadamard. In our context, the goal is to assess how the optimal solution f^* depends continuously on the measured data $\omega \in L^2(\Omega)$.

We consider two final-time observations $\omega, \tilde{\omega} \in L^2(\Omega)$, and analyze the corresponding solutions $f^*, \tilde{f}^* \in \mathcal{A}$ obtained by minimizing the cost functional (7). Under appropriate assumptions, we prove that small perturbations in the data lead to small changes in the recovered source, thereby establishing a Lipschitz-type stability estimate for the inverse problem.

Inverse problems are typically unstable with respect to perturbations in the data.

Theorem 4 demonstrates that, under the proposed optimal control formulation, the recovered source satisfies a Lipschitz-type stability estimate.
This result ensures robustness of the reconstruction and provides a rigorous theoretical justification for the numerical performance observed in Section 6.

Theorem 4. *Let $f, \tilde{f} \in \mathcal{A}$ be two optimal controls corresponding to the final observations $\omega, \tilde{\omega} \in L^2(\Omega)$, respectively, and let $\theta := \theta[f]$, $\tilde{\theta} := \theta[\tilde{f}]$ be the associated solutions to the state equation* (1)*. Then, the following Lipschitz-type stability estimate holds:*

$$\|f - \tilde{f}\|^2_{L^2(\Omega)} \leq \frac{1}{2\gamma}\|\omega - \tilde{\omega}\|^2_{L^2(\Omega)}. \tag{23}$$

Proof 3. *Let $f, \tilde{f} \in \mathcal{A}$ be two optimal controls corresponding to the final-time data $\omega, \tilde{\omega} \in L^2(\Omega)$, and let $\theta := \theta[f]$, $\tilde{\theta} := \theta[\tilde{f}]$ be the associated solutions to the state problem* (1)*.*

We apply the first-order optimality condition (17) *with $f^* = f$ and $h = \tilde{f}$, yielding:*

$$\int_\Omega [\theta(x,T) - \omega(x)]\,\xi(x,T)\,dx + \gamma \int_\Omega f(x)(\tilde{f}(x) - f(x))\,dx \geq 0, \tag{24}$$

where ξ solves the adjoint problem:

$$\begin{cases} \partial_t \xi - \xi_{xx} - \dfrac{\mu}{x^2}\xi = \tilde{f} - f, & \text{in } Q_T, \\ \xi(0,t) = \xi(1,t) = 0, & t \in (0,T), \\ \xi(x,0) = 0, & x \in \Omega. \end{cases} \tag{25}$$

Similarly, applying (17) *with* $f^* = \tilde{f}$ *and* $h = f$*, we obtain:*

$$\int_\Omega \left[\tilde{\theta}(x,T) - \tilde{\omega}(x)\right] \tilde{\xi}(x,T)\, dx + \gamma \int_\Omega \tilde{f}(x)(f(x) - \tilde{f}(x))\, dx \geq 0, \tag{26}$$

where $\tilde{\xi}$ *solves:*

$$\begin{cases} \partial_t \tilde{\xi} - \tilde{\xi}_{xx} - \dfrac{\mu}{x^2}\tilde{\xi} = f - \tilde{f}, & \text{in } Q_T, \\ \tilde{\xi}(0,t) = \tilde{\xi}(1,t) = 0, & t \in (0,T), \\ \tilde{\xi}(x,0) = 0, & x \in \Omega. \end{cases} \tag{27}$$

Adding inequalities (24) *and* (26) *yields:*

$$\gamma \|f - \tilde{f}\|^2_{L^2(\Omega)} \leq \int_\Omega \left[\theta(T) - \omega\right] \xi(T)\, dx + \int_\Omega \left[\tilde{\theta}(T) - \tilde{\omega}\right] \tilde{\xi}(T)\, dx. \tag{28}$$

Now, define the error functions $E := \theta - \tilde{\theta}$*, and* $X := \xi - \tilde{\xi}$*. Then,* E *solves:*

$$\begin{cases} \partial_t E - E_{xx} - \dfrac{\mu}{x^2} E = f - \tilde{f}, & \text{in } Q_T, \\ E(0,t) = E(1,t) = 0, & t \in (0,T), \\ E(x,0) = 0, & x \in \Omega, \end{cases} \tag{29}$$

and X *solves the homogeneous problem:*

$$\begin{cases} \partial_t X - X_{xx} - \dfrac{\mu}{x^2} X = 0, & \text{in } Q_T, \\ X(0,t) = X(1,t) = 0, & t \in (0,T), \\ X(x,0) = 0, & x \in \Omega. \end{cases} \tag{30}$$

Hence, by uniqueness of weak solutions, we conclude $X = 0$*, i.e.,* $\xi = \tilde{\xi}$*, and similarly* $E = -\xi$ *from comparing* (29) *and* (25).

Substituting into (28) *and using* $E = -\xi$*, we obtain:*

$$\begin{aligned} \gamma \|f - \tilde{f}\|^2 &\leq \int_\Omega E(x,T)\xi(x,T)\, dx + \int_\Omega (\omega(x) - \tilde{\omega}(x))\xi(x,T)\, dx \\ &= -\|\xi(\cdot,T)\|^2_{L^2(\Omega)} + \int_\Omega (\omega - \tilde{\omega})\xi(\cdot,T)\, dx. \end{aligned}$$

Applying the Cauchy–Schwarz and Young inequalities, we get:

$$\begin{aligned} \gamma \|f - \tilde{f}\|^2 &\leq -\|\xi(T)\|^2 + \|\omega - \tilde{\omega}\| \cdot \|\xi(T)\| \\ &\leq -\|\xi(T)\|^2 + \frac{1}{2}\|\omega - \tilde{\omega}\|^2 + \frac{1}{2}\|\xi(T)\|^2 \\ &= -\frac{1}{2}\|\xi(T)\|^2 + \frac{1}{2}\|\omega - \tilde{\omega}\|^2 \\ &\leq \frac{1}{2}\|\omega - \tilde{\omega}\|^2. \end{aligned}$$

Dividing both sides by $\gamma > 0$, we conclude:

$$\|f - \tilde{f}\|^2_{L^2(\Omega)} \leq \frac{1}{2\gamma}\|\omega - \tilde{\omega}\|^2_{L^2(\Omega)},$$

which completes the proof.

Corollary 1. *Assume that assumptions of Theorem (4) hold. Furthermore, suppose that ω matches $\tilde{\omega}$ over Ω then $f = \tilde{f}$*

6 Numerical identification

In this section, we present a numerical strategy for identifying the unknown source term $f(x)$ in the singular parabolic problem (1), based on the final-time observation $\omega(x)$. Due to the ill-posedness of the inverse problem, direct inversion is highly unstable, and regularization techniques are essential to obtain stable and meaningful numerical approximations.

To this end, we implement an iterative regularization scheme based on the classical Landweber method, which is widely used in inverse problems due to its simplicity and robustness. The approach consists of iteratively updating the source term by moving along the negative gradient direction of the cost functional (7), evaluated via the solution of the associated forward and adjoint problems.

6.1 Landweber iteration method

Let us define the input-output operator $\mathcal{T}$ associated with the parabolic problem (1), which maps a source term to the final-time state of the corresponding solution. For simplicity of computation, we assume the initial condition is homogeneous, i.e., $\theta_0 = 0$. Then, the operator $\mathcal{T}$ is given by:

$$\begin{aligned} \mathcal{T}\colon\ & L^2(\Omega) \longrightarrow H^1_{\mu,0}(\Omega), \\ & f \mapsto \mathcal{T}f := \theta[f](\cdot, T), \end{aligned}$$

where $\theta[f]$ denotes the weak solution to problem (1) with source term $f \in L^2(\Omega)$, and $\theta_0 = 0$ as initial data. In this framework, $\mathcal{T}f$ represents the output measurement at the final time $t = T$.

In view of the above considerations, our inverse problem can be equivalently reformulated as the operator equation

$$\text{Find } f^\dagger \in \mathcal{A} \text{ such that } \mathcal{T}f^\dagger = \omega,$$

where $\mathcal{T}\colon L^2(\Omega) \to H^1_{\mu,0}(\Omega)$ is the input-output operator defined in the previous subsection, and $\omega \in L^2(\Omega)$ denotes the measured final-time data. Formally, the exact solution $f^\dagger$ satisfies the associated normal equation

$$\mathcal{T}^*\mathcal{T}f^\dagger = \mathcal{T}^*\omega,$$

where $\mathcal{T}^*$ denotes the adjoint of the operator $\mathcal{T}$. This normal equation can be interpreted as a fixed-point problem of the form

$$f^\dagger = f^\dagger - \beta\,\mathcal{T}^*\left(\mathcal{T}f^\dagger - \omega\right),$$

where $\beta > 0$ is a relaxation parameter. Based on this formulation, we construct an iterative Landweber-type method to approximate $f^\dagger$. Starting from an initial guess $f_0 \in L^2(\Omega)$, the iteration proceeds as:

$$\begin{aligned} f_{m+1} &= f_m - \beta\,\mathcal{T}^*\left(\mathcal{T}(f_m) - \omega\right) \\ &= f_m - \beta\,\mathcal{T}^*\left(\theta_m(\cdot, T) - \omega\right), \end{aligned} \tag{31}$$

where $\theta_m := \theta[f_m]$ is the solution of the forward problem (1) associated with the current iterate f_m.

It is well known (see, e.g., [9]) that the Landweber iteration (31) converges strongly to the minimum-norm solution $f^\dagger$, provided that $0 < \beta < 1/\|\mathcal{T}\|^2$ and the initial guess $f_0 \in \mathcal{D}(\mathcal{T})$. In practice, the iteration is terminated according to a suitable discrepancy principle or tolerance-based stopping rule.

For the numerical implementation of the Landweber algorithm, it is essential to compute the adjoint of the input–output operator.

Lemma 1 provides an explicit characterization of this adjoint in terms of the solution of an auxiliary boundary value problem.
This result enables the efficient numerical realization of the iterative reconstruction scheme.

Lemma 1. *Let $\psi \in L^2(\Omega)$, and let $\eta \in L^2(0,T; H^1_{\mu,0}(\Omega))$ be the unique weak solution of the following initial-boundary value problem:*

$$\begin{cases} \partial_t \eta(x,t) - \partial_{xx}\eta(x,t) + \dfrac{\mu}{x^2}\eta(x,t) = \psi(x), & \text{in } Q_T := \Omega \times (0,T), \\ \eta(x,0) = 0, & x \in \Omega, \\ \eta(0,t) = \eta(1,t) = 0, & t \in (0,T). \end{cases} \tag{32}$$

Then, the adjoint operator $\mathcal{T}^* \colon L^2(\Omega) \to L^2(\Omega)$, corresponding to the input-output operator $\mathcal{T}f = \theta[f](\cdot, T)$, is given by

$$\mathcal{T}^*\psi = \eta(\cdot, T),$$

Proof 4. *Let $f \in L^2(\Omega)$, and denote by $\theta = \theta[f] \in L^2(0,T; H^1_{\mu,0}(\Omega))$ the unique weak solution of the forward problem:*

$$\begin{cases} \partial_t \theta(x,t) - \partial_{xx}\theta(x,t) + \dfrac{\mu}{x^2}\theta(x,t) = f(x), & \text{in } Q_T, \\ \theta(x,0) = 0, & x \in \Omega, \\ \theta(0,t) = \theta(1,t) = 0, & t \in (0,T). \end{cases} \tag{33}$$

Then the input-output operator $\mathcal{T} \colon L^2(\Omega) \to L^2(\Omega)$ is defined by

$$\mathcal{T}f = \theta(\cdot, T).$$

Let $\psi \in L^2(\Omega)$, and let $\eta \in L^2(0,T;H^1_{\mu,0}(\Omega))$ be the solution of the following adjoint problem:

$$\begin{cases} \partial_t \eta(x,t) - \partial_{xx}\eta(x,t) + \dfrac{\mu}{x^2}\eta(x,t) = \psi(x), & \text{in } Q_T, \\ \eta(x,0) = 0, & x \in \Omega, \\ \eta(0,t) = \eta(1,t) = 0, & t \in (0,T). \end{cases} \tag{34}$$

We want to compute $\mathcal{T}^*\psi$ using the definition of the adjoint. By definition, $\mathcal{T}^*$ is the operator such that

$$\langle \mathcal{T} f, \psi \rangle_{L^2(\Omega)} = \langle f, \mathcal{T}^*\psi \rangle_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega).$$

Now, compute the left-hand side:

$$\langle \mathcal{T} f, \psi \rangle_{L^2(\Omega)} = \int_\Omega \theta(x,T)\psi(x)\,dx.$$

We aim to express this quantity in terms of f and η, and thereby identify $\mathcal{T}^*\psi$. To this end, we define the auxiliary function $v(x,t) := \eta(x,T-t)$. It is easy to verify (by direct substitution) that v satisfies the backward parabolic problem:

$$\begin{cases} -\partial_t v(x,t) - \partial_{xx} v(x,t) + \dfrac{\mu}{x^2} v(x,t) = \psi(x), & \text{in } Q_T, \\ v(x,T) = 0, & x \in \Omega, \\ v(0,t) = v(1,t) = 0, & t \in (0,T). \end{cases} \tag{35}$$

We now multiply the equation for θ by v, integrate over Q_T, and use integration by parts in time and space. We obtain:

$$\begin{aligned} \iint_{Q_T} f(x)v(x,t)\,dxdt &= \iint_{Q_T} \left(\partial_t\theta \cdot v + \partial_x\theta \cdot \partial_x v + \frac{\mu}{x^2}\theta v\right) dxdt \\ &= \iint_{Q_T} \left(-\partial_t v \cdot \theta + \partial_x\theta \cdot \partial_x v + \frac{\mu}{x^2}\theta v\right) dxdt, \end{aligned}$$

where we have used the fact that $\theta(x,0) = v(x,T) = 0$.

Since v satisfies (35), the right-hand side becomes:

$$\iint_{Q_T} \psi(x)\theta(x,t)\,dxdt.$$

Thus, we have established the identity:

$$\iint_{Q_T} f(x)v(x,t)\,dxdt = \iint_{Q_T} \psi(x)\theta(x,t)\,dxdt.$$

Now, reversing the change of variables $v(x,t) = \eta(x,T-t)$, we have:

$$\int_0^T v(x,t)\,dt = \int_0^T \eta(x,s)\,ds.$$

Similarly,

$$\iint_{Q_T} f(x)v(x,t)\,dxdt = \int_\Omega f(x) \int_0^T \eta(x,s)\,ds\,dx,$$

and

$$\iint_{Q_T} \psi(x)\theta(x,t)\,dxdt = \int_\Omega \psi(x) \int_0^T \theta(x,t)\,dt\,dx.$$

Assuming that this identity holds for all $T > 0$, we formally differentiate both sides with respect to T, obtaining:

$$\int_\Omega \psi(x)\theta(x,T)\,dx = \int_\Omega f(x)\eta(x,T)\,dx.$$

Therefore, we have:

$$(\mathcal{T} f, \psi)_{L^2(\Omega)} = (\theta(\cdot,T), \psi)_{L^2(\Omega)} = (f, \eta(\cdot,T))_{L^2(\Omega)},$$

and since this holds for all $f \in L^2(\Omega)$, we conclude:

$$\mathcal{T}^*\psi = \eta(\cdot,T).$$

To summarize, we now outline the main steps of the iterative procedure used to numerically reconstruct the unknown source term f in problem (1), based on the Landweber method.

Algorithm 1 Iterative Landweber Method for Source Identification

Require: Relaxation parameter $\beta > 0$, tolerance $\varepsilon > 0$, final-time data $\omega \in L^2(\Omega)$
Ensure: Approximate solution $f^\dagger$ and corresponding state $\theta^\dagger$ to the inverse problem

```
Initialization: Choose an initial guess f_0 ∈ A, and set k = 0
Solve Forward Problem: Compute θ_0 := θ[f_0] by solving (1)
Solve Adjoint Problem: Compute η_0 by solving (32) with source ψ = θ_0(·,T) − ω
Update Control: Set

        f_1 := f_0 − β η_0(·,T)

for k = 1, 2, ... until convergence do
    Solve θ_k := θ[f_k] from (1)
    if ‖θ_k(·,T) − ω‖_{L²(Ω)} < ε then
        Set f† := f_k, θ† := θ_k, and stop
    else
        Solve η_k from (32) with ψ = θ_k(·,T) − ω
        Update f_{k+1} := f_k − β η_k(·,T)
    end if
end for
```

6.2 Numerical results and discussions

In this subsection, we present numerical experiments that illustrate the performance of the proposed Landweber algorithm for reconstructing the space-dependent source term. The experiments are designed to validate both the accuracy and stability of the method under noise-free and noisy final-time data.

We begin with Example 1, where the exact solution of the forward problem is available in closed form. This allows for a direct comparison between the reconstructed and exact source profiles. In Example 2, the forward solution is generated numerically, thereby testing the algorithm in a more realistic setting. In both cases, the reconstructions confirm the theoretical predictions: the Landweber method converges towards the true source when noise-free data are used, while in the presence of noisy data, the algorithm still yields stable and accurate approximations, as shown in Figures 6.1–6.3.

The relative error $E_2(k)$ is also monitored as a function of the iteration index k. The error curves demonstrate a rapid initial decrease followed by saturation, which is consistent with the discrepancy principle and the finite accuracy of the numerical discretization. Overall, these results validate the effectiveness and robustness of the proposed method.

6.3 Numerical Implementation and Discretization

This subsection is devoted to numerical examples that illustrate the performance of the proposed Landweber algorithm for reconstructing the space-dependent source term $f(x)$ in the inverse problem (1). The solutions to both the direct and adjoint problems are approximated using finite-difference methods.

We fix the final time $T = 1$, so that the spatio-temporal domain is $Q_T = (0,1) \times (0,1)$. Let $M, N \in \mathbb{N}^*$ denote the number of spatial and temporal subdivisions, respectively. Define the mesh sizes

$$\Delta x = \frac{1}{M}, \quad \Delta t = \frac{1}{N}.$$

The spatial and temporal grid points are given by:

$$x_i = i\Delta x, \quad \text{for } i = 0, 1, \ldots, M, \qquad t_j = j\Delta t, \quad \text{for } j = 0, 1, \ldots, N.$$

The functions $\theta(x,t)$ (solution of the forward problem) and $\eta(x,t)$ (solution of the adjoint problem) are evaluated at these grid points. The numerical schemes employed for the discretization are based on finite-difference approximations of second-order spatial derivatives and backward or Crank–Nicolson schemes in time, ensuring stability in the presence of the singular potential μ/x^2. Boundary conditions are imposed explicitly at $x = 0$ and $x = 1$.

In the numerical tests, we measure the accuracy of the reconstructed source using the relative error at iteration k, defined by

$$E_2(k) := \|f^k - f\|^2_{L^2(\Omega)} = \frac{1}{M+1} \sum_{i=0}^{M} \left(f(x_i) - f^k(x_i) \right)^2,$$

where f is the exact source function and f^k is the reconstructed approximation at the k-th iteration, evaluated on the discrete grid $\{x_i\}_{i=0}^{M}$.

To test robustness against measurement errors, we also consider noisy data. The perturbed observation $\omega_\varepsilon(x)$ is generated from the exact final state $\omega(x) = \theta(x,T)$ by injecting a multiplicative random noise:

$$\omega_\varepsilon(x) = \omega(x) + \varepsilon \cdot \omega(x) \cdot \mathrm{rand}(x), \quad x \in \Omega, \tag{36}$$

where $\varepsilon \in (0,1)$ denotes the noise level and $\mathrm{rand}(x) \in (0,1)$ is a uniformly distributed random function over the spatial domain. This simulates realistic data perturbations encountered in practice.

Example 1. *In this first test case, we consider the inverse problem* (1)–(2) *with singularity parameter* $\mu = \frac{1}{5}$*, and a source term given by*

$$f(x,t) = -5\sin(\pi t)\left((x^2 - \pi^2 x^2 + \mu)\sin(\pi x)\right), \quad (x,t) \in Q_T.$$

It is easy to verify that the corresponding exact solution of the forward problem (1) *is*

$$\theta(x,t) = x^2 \sin(\pi x)(1 - e^{-t}), \quad x \in \Omega, \quad t \in [0,T].$$

Consequently, the final-time observation used in the inverse problem is computed as $\omega(x) = u(x,T)$*. This example allows for direct comparison between the reconstructed and exact source terms.*

Example 2. *In this second test, we consider a synthetic example in which the exact source term is prescribed as*

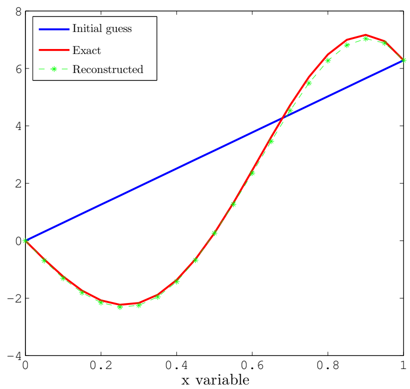
$$f(x) = \sin(\pi x), \quad x \in \Omega.$$

We set the singularity parameter to $\mu = \frac{1}{6}$*. The final-time data* $\omega(x) = u(x,T)$ *is generated by solving the direct problem* (1) *using this exact source. This test serves to validate the reconstruction algorithm when the forward solution is numerically simulated, without using an explicit expression for* $u(x,t)$*.*

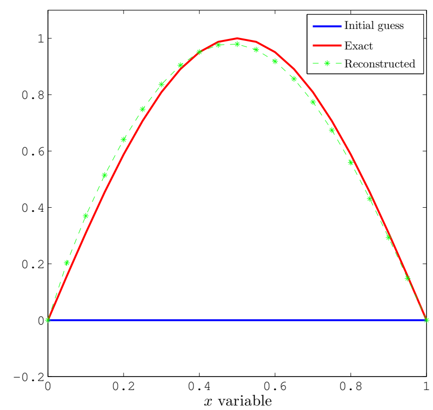
Discussion on Example 1

For the inversion process, we employ moderate discretization parameters, setting $\Delta t = 10^{-3}$ and $\Delta x = 5 \times 10^{-2}$. The Landweber iteration is initialized with the admissible guess $f_0(x) = x^2$. Figure 6.1 (a) shows a comparison between the exact source $f^\dagger$ and the reconstructed profile f_k in Example 1 after $k = 8000$ iterations. The agreement is notably close, confirming the convergence of the algorithm in the noise-free setting.

To assess the robustness of the method under measurement perturbations, we conduct additional experiments using noisy final-time data ω, generated according to the perturbation model (36). The reconstruction is evaluated after $k = 400$ iterations. As shown in Figure 6.2-(a), the reconstructions remain satisfactory under moderate noise levels, and the computed state $\theta_k(\cdot,T)$ matches the perturbed observations ω with high accuracy. However, for higher noise levels, the reconstruction quality deteriorates significantly. The evolution of $E_2(k)$ is shown in Figure 6.3-(a). We observe a monotonic decay of the error up to around $k = 400$, after which the reduction halts due to accumulated discretization errors in the numerical solution of the direct and adjoint problems.

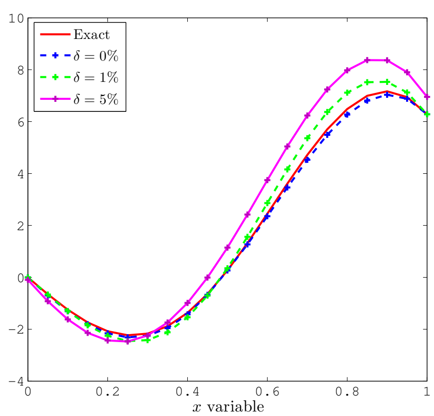


(a) Example 1 with $k = 800$

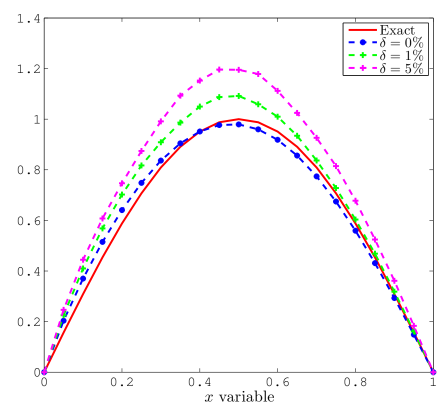


(b) Example 2 with $k = 400$

Figure 6.1: Numerical reconstruction.

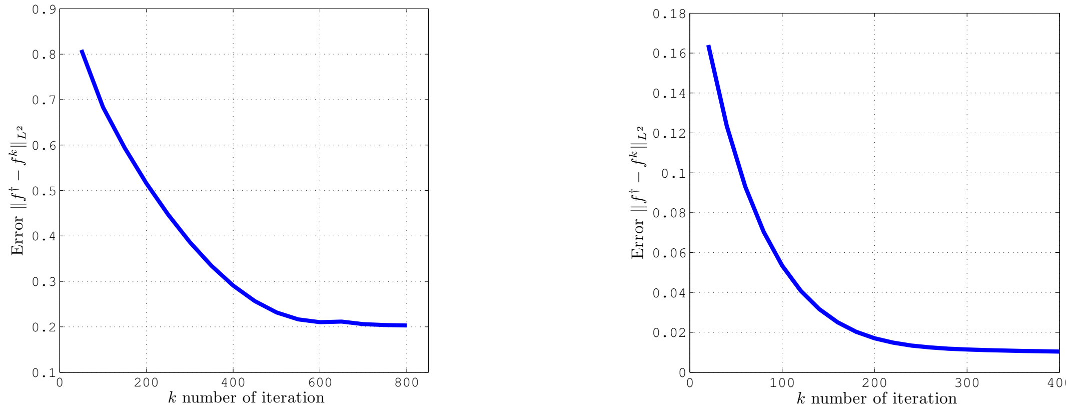


(a) Example 1



(b) Example 2

Figure 6.2: The numerical results with noise



(a) Example 1 (b) Example 2

Figure 6.3: Behaviour of reconstruction error $E_2(k)$ as a function of k.

Discussion on Example 2

For the second example, we consider a synthetic source term $f^\dagger(x) = \sin(\pi x)$ with singularity parameter $\mu = \frac{1}{6}$. The final-time observation $\omega(x)$ is generated by numerically solving the direct problem (1). The Landweber iteration is initialized with the same admissible guess $f_0(x) = 0$, and discretization parameters are set to $\Delta t = 10^{-3}$ and $\Delta x = 5 \times 10^{-2}$, as in the previous example. Figure 6.1 (b) displays the comparison between the exact source $f^\dagger$ and the reconstructed solution f_k after $k = 400$ iterations. The reconstruction achieves high accuracy with significantly fewer iterations than in Example 1, which is attributed to the simpler spectral content of the source.

To evaluate stability with respect to data perturbations, we introduce noisy observations based on the same noise model (36). The reconstruction after $k = 400$ iterations is reported in Figure 6.2 (b). The results indicate that the reconstructed state $\theta_k(\cdot, T)$ approximates the noisy data ω well for low to moderate noise levels. However, as the noise amplitude increases, the reconstruction degrades, consistent with the sensitivity of the inverse problem to measurement errors.

The convergence history of the relative error $E_2(k)$ is depicted in Figure 6.3 (b). Similar to the first example, we observe a rapid decay of the error up to $k \approx 300$, followed by stagnation. The early saturation is again due to the discretization effects and the finite resolution of the spatial grid, which limit further improvements in accuracy despite continued iteration.

Conclusion

In this work, we have addressed an inverse problem concerned with the identification of a space-dependent source term in a diffusion equation governed by a singular inverse-square potential. The proposed approach is based on an optimal control framework.

We began by establishing the existence and uniqueness of weak solutions to the direct problem. The inverse problem was then reformulated as a constrained optimization problem, for which we proved the existence of a minimizer and derived a first-order necessary optimality condition. This condition was further employed to demonstrate a Lipschitz-type stability result with respect to perturbations in the final-time data.

On the numerical side, we developed an iterative Landweber-type algorithm to reconstruct the unknown source term from noisy final measurements. A series of numerical experiments were carried out, confirming the effectiveness, stability, and robustness of the proposed reconstruction method, even in the presence of data perturbations.

As directions for future work, we plan to extend the current methodology to more complex models, including systems of coupled singular parabolic equations and fractional-order singular diffusion problems.

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