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ASYMPTOTIC SOLUTIONS TO INITIAL VALUE PROBLEMS FOR SINGULARLY PERTURBED QUASI-LINEAR IMPULSIVE SYSTEMS

This paper investigates a singularly perturbed quasi-linear impulsive differential system with singularities present both in the differential equations and in the impulse functions. The boundary function method is employed to derive the main results. A uniform asymptotic approximation with higher accuracy is constructed and a complete asymptotic expansion is obtained. Theoretical findings are supported by illustrative examples and numerical simulations. The analysis reveals the presence of boundary and interior layers caused by the singular perturbation and impulsive effects. Sufficient conditions for the existence and uniqueness of the solution are established. The results contribute to the theoretical understanding of impulsive systems with complex singular structures and may be applicable to various problems in applied mathematics.
Key words:singularly perturbed systems, impulsive differential equations with singularities, small parameter, the boundary function method.

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Сингулярлы ауытқыған квазисызықты импульсті жүйелер үшін бастапқы есептің асимптотикалық шешімі

Бұл мақала дифференциалдық теңдеуімен қатар импульстік функциясында кіші параметрі бар сингулярлы ауытқыған квази-сызықты импульстік дифференциалдық жүйені қарастырады. Негізгі нәтижелерді алу үшін шекаралық функциялар әдісі қолданылады. Шешімнің кез-келген дәлдіктегі асимптотикалық жуықтауы алынды және толық асимптотикалық жіктелуі құрылады. Теориялық тұжырымдар иллюстрациялық мысалдармен және сандық модельдеу нәтижелерімен расталады. Зерттеу нәтижесі сингулярлық ауытқу мен импульстік әсерлерден туындайтын шекаралық және ішкі қабаттардың болуын анықтайды. Шешімнің бар және жалғыз болуының жеткілікті шарттары анықталады. Алынған нәтижелер күрделі сингулярлық құрылымы бар импульстік жүйелер туралы теориялық түсініктің дамуына ықпал етеді және қолданбалы математика мәселелерінде қолдануға болады.
Түйін сөздер: сингулярлы ауытқыған жүйелер, сингулярлы ауытқыған импульсті дифференциалдық теңдеулер, кіші параметр, шекаралық функциялар әдісі.

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Асимптотические решения начальных задач для сингулярно возмущённых квазилинейных импульсных систем

В данной работе исследуется сингулярно возмущённая квазилинейная импульсная дифференциальная система, в которой сингулярности присутствуют как в дифференциальных уравнениях, так и в импульсных функциях. Для получения основных результатов применяется метод граничных функций. Построено равномерное асимптотическое приближение повышенной точности и получено полное асимптотическое разложение. Теоретические выводы подтверждаются иллюстративными примерами и результатами численного моделирования. Анализ выявляет наличие как граничных, так и внутренних слоёв, возникающих в результате сингулярных возмущений и импульсных эффектов.

Установлены достаточные условия существования и единственности решения. Полученные результаты способствуют развитию теоретического понимания импульсных систем со сложной сингулярной структурой и могут быть применимы в задачах прикладной математики.

Ключевые слова: сингулярно возмущённые системы, импульсные дифференциальные уравнения с сингулярностями, малый параметр, метод граничных функций.

1 Introduction

Perturbation methods deal with problems that contain a small parameter, usually denoted by ε, which perturbs or slightly modifies a simpler, well-understood problem. These problems arise frequently in applied mathematics [1, 2], physics and engineering [3]. There are two main types of perturbation problems: Regular perturbation problems – the solution varies smoothly with ε.

Singular perturbation problems – the small parameter multiplies the highest derivative [4], causing drastic changes in the nature of the solution as $\varepsilon \to 0$.

Singularly perturbed differential equations represent a challenging and fascinating class of problems where small parameters significantly impact the solution behavior. These equations require specialized methods like matched asymptotic expansions to accurately capture the full dynamics of the solution across different scales.

This work is associated with one of the effective asymptotic methods in the theory of singular perturbations about the method of boundary functions, the mathematical foundations of which were laid in the works [5, 6]. The boundary layer method is a powerful analytical technique used to study differential equations with rapid changes in a small region of the domain — typically near a boundary. This method is especially useful in fluid dynamics, applied mathematics, and singular perturbation theory. In many physical systems (especially fluid flow), variables like velocity or temperature change very sharply near boundaries (e.g., surfaces), but slowly elsewhere. The thin region of rapid change is called the boundary layer. Outside this layer, the solution varies smoothly this is the outer region.

Impulse effects describe the response or reaction of a system to a sudden, short-duration force or signal. These effects are critical in understanding how systems behave under rapid or transient conditions. Impulse differential equations (or impulsive differential equations) are used to model systems that experience sudden changes (impulses) at specific moments in time [7]. These equations combine continuous dynamics (ordinary differential equations) with discrete jumps or instantaneous changes.

Singularly perturbed impulsive systems present significant difficulties. An exact solution of impulsive differential equations with singular perturbations is elusive, which explains the relatively small number of studies in this area. Major works in this field were performed before 2000 (see [8–13]), including the research of Kulev (1992) and Bainov et al. (1996) on uniform asymptotic stability, as well as the work of Zhu et al. (2007) on the exponential stability of singularly perturbed equations with impulsive delay.

In [14–17], singularly perturbed Tikhonov-type systems with impulsive effects are studied. These systems are distinguished by the presence of both slow and fast dynamics, as well as by discrete state discontinuities occurring at fixed time instants. The combination of multi-scale behavior and impulsive phenomena provides a rigorous mathematical framework for the analysis and modeling of complex processes exhibiting rapid transitions and time-scale separation induced by a small perturbation parameter.

Akhmet and Çağ [18–20] extended the Tikhonov theorem to a class of singularly perturbed impulsive systems of the form

$$\begin{aligned} \mu\dot{z} &= f(z,y,t), \qquad \dot{y} = g(z,y,t), \\ \mu\Delta z|_{t=\theta_i} &= I(z,y,\mu), \qquad \Delta y|_{t=\eta_j} = J(z,y), \end{aligned} \tag{1}$$

with initial condition

$$z(0,\mu) = z^0, \quad y(0,\mu) = y^0, \tag{2}$$

where z, f and I are m-dimensional vector valued functions, y, g and J are n-dimensional vector valued functions, $\theta_{i}{}_{i=1}^{p}, 0 < \theta_1 < \theta_2 < ... < \theta_p < T$, and $\eta_{j}{}_{j=1}^{k}$, are distinct discontinuity moments in $(0,T)$.

Unlike the study referenced in [10], the authors considered systems in which not only the differential part but also the impulsive parts are singularly perturbed. In this framework, the impulsive function depends explicitly on the small parameter μ, and the moments of discontinuity for the functions z and y are not coincident. The extension of Tikhonov's theorem to such systems necessitates the treatment of additional complexities arising from the perturbation of impulses.

In [18], two types of singular behavior are analyzed: single-layer and multi-layers structures, both arising due to the nature of the impulse functions. The singularities in the impulsive part are addressed using techniques from singular perturbation theory. Stability of the reduced system in the fast (rescaled) time is established through Lyapunov's second method.

Papers [21–23] are devoted to the study of impulsive systems with singularities. Using the boundary layer method, the authors constructed a uniform asymptotic approximation of the solution for $0 < t < T$, and obtained higher-order approximations as well as complete asymptotic expansions for systems with singularly perturbed impulses.

2 Formalities of approximation

Let us consider on the segment $[0,T]$ the following system

$$\begin{aligned} \mu z' &= F(y,t)z + G(y,t), \quad \mu\Delta z|_{t=\theta_i} = I_1(y,\mu)z + I_2(y,\mu), \\ y' &= f(y,t)z + g(y,t), \qquad \Delta y|_{t=\theta_i} = J_1(y,\mu)z + J_2(y,\mu) \end{aligned} \tag{3}$$

with initial condition

$$z(0,\mu) = z^0, \quad y(0,\mu) = y^0, \tag{4}$$

where μ is a small positive real number, z^0 and y^0 are assumed to be independent of μ, $\theta_{i}{}_{i=1}^{p}, 0 < \theta_1 < \theta_2 < ... < \theta_p < T$, are distinct discontinuity moments in $(0,T)$. We define $\Delta x|_{t=\theta_i} = x(\theta_i+) - x(\theta_i)$, assuming that the right-hand limit $x(\theta_i+) = \lim\limits_{t\to\theta_i+} x(t)$ exists and that the left-hand limit satisfies $x(\theta_i-) = x(\theta_i)$.

Assume that $\mu = 0$ in equation (3). In this case, system (3) reduces to the following system

$$\begin{aligned} &0 = F(\overline{y}, t)\overline{z} + G(\overline{y}, t), \qquad 0 = I_1(\overline{y}, 0)\overline{z} + I_2(\overline{y}, 0), \\ &\overline{y}' = f(\overline{y}, t)\overline{z} + g(\overline{y}, t), \qquad \Delta \overline{y}|_{t=\theta_i} = J_1(\overline{y}, 0)\overline{z} + J_2(\overline{y}, 0), \end{aligned} \tag{5}$$

which is called to as a degenerate system, since its order is lower than that of system (3). Therefore, for system (5) the number of initial conditions to be less than the number of initial conditions for (3). For system (5) we should retain only the initial condition for $\overline{y}$ since no initial condition for $\overline{z}$ is needed:

$$\overline{y}(0) = y^0. \tag{6}$$

In order to solve system (5), one needs to find $\overline{z}$ from the equations $0 = F(\overline{y}, t)\overline{z} + G(\overline{y}, t)$ and $0 = I_1(\overline{y}, 0)\overline{z} + I_2(\overline{y}, 0)$. Then, one selects a root of the system in the form $\overline{z} = \varphi(\overline{y}(t), t) = -\frac{G(\overline{y}, t)}{F(\overline{y}, t)}$, which satisfies the equations $0 = F(\overline{y}, t)\varphi(\overline{y}(t), t) + G(\overline{y}, t)$ and $0 = I_1(\overline{y}, 0)\varphi(\overline{y}(t), t) + I_2(\overline{y}, 0)$. Substituting this expression into equation (5) together with the initial condition (6) yield system

$$\begin{aligned} &\overline{y}' = f(\overline{y}, t)\varphi(\overline{y}(t), t) + g(\overline{y}, t), \qquad \Delta \overline{y}|_{t=\theta_i} = J_1(\overline{y}, 0)\varphi(\overline{y}(t), t) + J_2(\overline{y}, 0), \\ &\overline{y}(0) = y^0. \end{aligned} \tag{7}$$

The following conditions are assumed to hold.

(C1) The functions $F(y, t), G(y, t), f(y, t), g(y, t)$ and $I_i(y, \varepsilon), J_i(y, \varepsilon), i = 1, 2$ are infinitely differentiable on the interval $0 \leq t \leq T$.

(C2) $F(y, t) < 0, 0 \leq t \leq T$.

(C3) The system (7) has a unique solution $\overline{y}(t)$ on $0 \leq t \leq T$.

(C4) $1 + \frac{\partial}{\partial \overline{y}}(J_1(\overline{y}, 0)\varphi(\overline{y}(t), t) + J_2(\overline{y}, 0)) \neq 0$.

(C5) $\lim\limits_{(z,y,)\to(\varphi,\overline{y},0)} \dfrac{I_1(y, \mu)z + I_2(y, \mu)}{\mu} = 0$, where $\overline{y} = \overline{y}(\theta_i)$ are the values for each impulse moment at the points $t = \theta_i, i = 1, 2, ..., p$.

An asymptotic approximation to the solution $z(t, \mu), y(t, \mu)$ of problem (3)–(4) will be sought in the form

$$\begin{aligned} &z(t, \mu) = \overline{z}(t, \mu) + \omega^{(i)}(\tau_i, \mu), \quad \tau_i = \frac{t - \theta_i}{\mu}, \quad i = 0, 1, 2, ..., p, \\ &y(t, \mu) = \overline{y}(t, \mu) + \mu\nu^{(i)}(\tau_i, \mu), \quad \theta_i < t \leq \theta_{i+1}, \quad \theta_0 \equiv 0, \theta_{p+1} \equiv T. \end{aligned} \tag{8}$$

where

$$\begin{aligned} &\overline{z}(t, \mu) = \sum_{k=0}^{\infty} \mu^k \overline{z}_k(t), \qquad \overline{y}(t, \mu) = \sum_{k=0}^{\infty} \mu^k \overline{y}_k(t), \\ &\omega^{(i)}(\tau_i, \mu) = \sum_{k=0}^{\infty} \mu^k \omega_k^{(i)}(\tau_i), \qquad \nu^{(i)}(\tau_i, \mu) = \sum_{k=0}^{\infty} \mu^k \nu_k^{(i)}(\tau_i). \end{aligned} \tag{9}$$

The coefficients $\omega_k^{(i)}(\tau_i)$ and $\nu_k^{(i)}(\tau_i)$ in (9) are called boundary functions, for which the following additional condition is imposed,

$$\omega_k^{(i)}(\infty) = 0,\ \nu_k^{(i)}(\infty) = 0, i = 0, 1, 2, ..., p. \tag{10}$$

By substituting the expansions (8) into system (3), we get at the following equalities

$$\begin{aligned}
&\mu\overline{z}'(t,\mu) + \dot{\omega}^{(i)}(\tau_i,\mu) = F(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t)(\overline{z}(t,\mu) + \omega^{(i)}(\tau_i,\mu)) - F(\overline{y}(t,\mu), t)\overline{z}(t,\mu) + \\
&+ F(\overline{y}(t,\mu), t)\overline{z}(t,\mu) + [G(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t) - G(\overline{y}(t,\mu), t)] + G(\tilde{y}(t,\mu), t),\\
&\overline{y}'(t,\mu) + \dot{\nu}^{(i)}(\tau_i,\mu) = f(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t)(\overline{z}(t,\mu) + \omega^{(i)}(\tau_i,\mu)) - f(\overline{y}(t,\mu), t)\overline{z}(t,\mu) + \\
&+ f(\overline{y}(t,\mu), t)\overline{z}(t,\mu) + [g(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t) - g(\overline{y}(t,\mu), t)] + g(\overline{y}(t,\mu), t).
\end{aligned}$$

Separating the expressions with respect to the variables t and τ_i, we derive two systems

$$\begin{aligned}
\mu\overline{z}'(t,\mu) &= F(\overline{y}(t,\mu), t)\overline{z}(t,\mu) + G(\overline{y}(t,\mu), t),\\
\overline{y}'(t,\mu) &= f(\overline{y}(t,\mu), t)\overline{z}(t,\mu) + g(\overline{y}(t,\mu), t),
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\dot{\omega}^{(i)}(\tau_i,\mu) &= F(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t)\omega^{(i)}(\tau_i,\mu) + [F(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t) - F(\overline{y}(t,\mu), t)]\overline{z}(t,\mu) + \\
&+ G(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t) - G(\overline{y}(t,\mu), t),\\
\dot{\nu}^{(i)}(\tau_i,\mu) &= f(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t)\omega^{(i)}(\tau_i,\mu) + [f(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t) - f(\overline{y}(t,\mu), t)]\overline{z}(t,\mu) + \\
&+ g(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t) - g(\overline{y}(t,\mu), t).
\end{aligned} \tag{12}$$

Let us express F, f, I_1 and I_2 in the form of power series in μ as follows:

$$\begin{aligned}
&F(\overline{y}(t,\mu), t)\overline{z}(t,\mu) + G(\overline{y}(t,\mu), t) =\\
&= F(\overline{y}_0(t) + \mu\overline{y}_1(t) + ..., t)\overline{z}(t,\mu) + G(\overline{y}_0(t) + \mu\overline{y}_1(t) + ..., t) =\\
&= \left(F(\overline{y}_0(t), t) + \mu F_y(\overline{y}_0(t), t)\overline{y}_1(t) + ... + \mu^k F_y(\overline{y}_0(t), t)\overline{y}_k(t) + ...\right)(\overline{z}_0(t) + \mu\overline{z}_1(t) + ...) + \\
&+ \left(G(\overline{y}_0(t), t) + \mu G_y(\overline{y}_0(t), t)\overline{y}_1(t) + ... + \mu^k G_y(\overline{y}_0(t), t)\overline{y}_k(t) + ...\right) =\\
&= F(\overline{y}_0(t), t)\overline{z}_0(t) + G(\overline{y}_0(t), t) + \mu[F(\overline{y}_0(t), t)\overline{z}_1(t) + (F_y(t)\overline{z}_0(t) + G_y(t))\overline{y}_1(t)] + \\
&+ \mu^k[F(\overline{y}_0(t), t)\overline{z}_k(t) + (F_y(t)\overline{z}_0(t) + G_y(t))\overline{y}_k(t) + H_k(t)] + ... =\\
&= F(\overline{y}_0(t), t)\overline{z}_0(t) + G(\overline{y}_0(t), t) + \mu\overline{H}_1(t) + \ldots \mu^k\overline{H}_k(t) + ...,
\end{aligned}$$

where functions $F_y(t)$ and $G_y(t)$ are calculated at the point $(\overline{y}_0(t), t)$ and $H_k(t)$ are defined recursively in terms of $\overline{z}_j(t)$ and $\overline{y}_j(t)$ for $j < k$,

$$\begin{aligned}
&F(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), t) - F(\overline{y}(t,\mu), t) =\\
&= F(\overline{y}_0(\theta_i + \mu\tau_i) + \mu\overline{y}_1(\theta_i + \mu\tau_i) + ... + \mu\nu_0^{(i)}(\tau_i) + \mu^2\nu_1^{(i)}(\tau_i) + ..., \theta_i + \mu\tau_i) -\\
&- F(\overline{y}_0(\theta_i + \mu\tau_i) + \mu\overline{y}_1(\theta_i + \mu\tau_i) + ..., \theta_i + \mu\tau_i) =\\
&= \mu F_y(\overline{y}_0(\theta_i), \theta_i)\nu_0^{(i)}(\tau_i) + \mu^2[F_y(\overline{y}_0(\theta_i), \theta_i)\nu_1^{(i)}(\tau_i) + F_2(\theta_i)] + ... + \\
&+ \mu^k[F_y(\overline{y}_0(\theta_i), \theta_i)\nu_{k-1}^{(i)}(\tau_i) + F_k(\theta_i)] + ... = \mu\Pi_1 F(\tau_i) + \ldots + \mu^k\Pi_k F(\tau_i) + ...,
\end{aligned}$$

$$F(\overline{y}(t,\mu)+\mu\nu^{(i)}(\tau_i,\mu),t)\omega^{(i)}(\tau_i,\mu)+[F(\overline{y}(t,\mu)+\mu\nu^{(i)}(\tau_i,\mu),t)-F(\overline{y}(t,\mu),t)]\overline{z}(t,\varepsilon)+$$
$$+G(\overline{y}(t,\mu)+\mu\nu^{(i)}(\tau_i,\mu),t)-G(\overline{y}(t,\mu),t)=$$
$$=F(\overline{y}(\theta_i+\mu\tau_i,\mu)+\mu\nu^{(i)}(\tau_i,\mu),\theta_i+\mu\tau_i)\omega^{(i)}(\tau_i,\mu)+[\mu\Pi_1F(\tau_i)+\ldots+\mu^k\Pi_kF(\tau_i)+...]\overline{z}(\theta_i+\mu\tau_i,\mu)+$$
$$+\mu\Pi_1G(\tau_i)+\ldots+\mu^k\Pi_kG(\tau_i)+...=F(\overline{y}_0(\theta_i),\theta_i)\omega_0^{(i)}(\tau_i)+\mu F(\overline{y}_0(\theta_i),\theta_i)\omega_1^{(i)}(\tau_i)+...+$$
$$+[\mu\Pi_1F(\tau_i)+\ldots+\mu^k\Pi_kF(\tau_i)+...](\overline{z}_0(\theta_i)+\mu\overline{z}_1(\theta_i)+...)+\mu\Pi_1G(\tau_i)+\ldots+\mu^k\Pi_kG(\tau_i)+...=$$
$$=F(\overline{y}_0(\theta_i),\theta_i)\omega_0^{(i)}(\tau_i)+\mu[F(\overline{y}_0(\theta_i),\theta_i)\omega_1^{(i)}(\tau_i)+\Pi_1F(\tau_i)\overline{z}_0(\theta_i)+\Pi_1G(\tau_i)]+...+$$
$$+\mu^k[F(\overline{y}_0(\theta_i),\theta_i)\omega_k^{(i)}(\tau_i)+\Pi_kF(\tau_i)\overline{z}_0(\theta_i)+\Pi_kG(\tau_i)]+...=$$
$$=\Pi_0H(\tau_i)+\mu\Pi_1H(\tau_i)+\ldots+\mu^k\Pi_kH(\tau_i)+...,$$

$$\overline{z}(\theta_i+\mu\tau_i,\mu)=\overline{z}_0(\theta_i+\mu\tau_i)+\mu\overline{z}_1(\theta_i+\mu\tau_i)+...=\overline{z}_0(\theta_i)+\mu\tau_i\overline{z}_0'(\theta_i)+...+$$
$$+\mu(\overline{z}_1(\theta_i)+\mu\tau_i\overline{z}_1'(\theta_i)+...)+...=\overline{z}_0(\theta_i)+\mu[\overline{z}_1(\theta_i)+\overline{z}_0'(\theta_i)\tau_i]+$$
$$+\mu^2[\overline{z}_2(\theta_i)+\overline{z}_1'(\theta_i)\tau_i+\overline{z}_0''(\theta_i)\frac{\tau_i}{2}]+...=\overline{\omega}_0(\tau_i)+\mu\overline{\omega}_1(\tau_i)+\mu^2\overline{\omega}_2(\tau_i)+...$$

where the functions $F_k(\theta_i)$ are calculated at the point $(\overline{y}_0(\theta_i),\theta_i), i=1,2,...,p,$ and $\Pi_kF(\tau_i),\Pi_kG(\tau_i), i=1,2,...,p,$ are defined recursively in terms of $\omega_j^{(i)}(\tau_i)$ and $\nu_j^{(i)}(\tau_i)$ for $j<k$. Analogously, one can get that

$$I_1(y(\theta_i,\mu),\mu)z(\theta_i,\mu)+I_2(y(\theta_i,\mu),\mu)=I_1(y(\theta_i-,\mu),\mu)z(\theta-,\mu)+I_2(y(\theta_i-,\mu),\mu)=$$
$$=I_1\left(\overline{y}(\theta_i,\mu)+\mu\nu^{(i-1)}\left(\frac{\theta_i-\theta_{i-1}}{\mu},\mu\right),\mu\right)\left(\overline{z}(\theta_i,\mu)+\omega^{(i-1)}\left(\frac{\theta_i-\theta_{i-1}}{\mu},\mu\right)\right)+$$
$$+I_2\left(\overline{y}(\theta_i,\mu)+\mu\nu^{(i-1)}\left(\frac{\theta_i-\theta_{i-1}}{\mu},\mu\right),\mu\right)=$$
$$=I_1(\overline{y}(\theta_i,\mu),\mu)\overline{z}(\theta_i,\mu)+I_2(\overline{y}(\theta_i,\mu),\mu)=I_1(\overline{y}_0(\theta_i),0)\overline{z}_0(\theta_i)+I_2(\overline{y}_0(\theta_i),0)+ \tag{13}$$
$$+\mu[I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i)+I_{1y}(\theta_i)\overline{z}_0(\theta_i)\overline{y}_1(\theta_i)+I_{1\varepsilon}(\theta_i)]+\varepsilon[I_{2y}(\theta_i)\overline{y}_1(\theta_i)+I_{2\varepsilon}(\theta_i)]+...+$$
$$+\mu^k[I_1(\overline{y}_0(\theta_i),0)\overline{z}_k(\theta_i)+I_{1y}(\theta_i)\overline{z}_0(\theta_i)\overline{y}_k(\theta_i)+I_{1k}(\theta_i)]+\mu^k[I_{2y}(\theta_i)\overline{y}_k(\theta_i)+I_{2k}(\theta_i)]+...=$$
$$=I_1(\overline{y}_0(\theta_i),0)\overline{z}_0(\theta_i)+I_2(\overline{y}_0(\theta_i),0)+$$
$$+\mu[I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i)+(I_{1y}(\theta_i)\overline{z}_0(\theta_i)+I_{2y}(\theta_i))\overline{y}_1(\theta_i))+\overline{I}_\mu(\theta_i)]+...+$$
$$+\mu^k[I_1(\overline{y}_0(\theta_i),0)\overline{z}_k(\theta_i)+(I_{1y}(\theta_i)\tilde{z}_0(\theta_i)+I_{2y}(\theta_i))\tilde{y}_k(\theta_i))+\overline{I}_k(\theta_i)]+...=$$
$$=\overline{T}_0(\theta_i)+\mu\overline{T}_1(\theta_i)+\ldots+\mu^k\overline{T}_k(\theta_i)+\ldots,$$

where the terms $I_{1y}(\theta_i)$, $I_{2y}(\theta_i), I_{1k}(\theta_i)$ and $I_{2k}(\theta_i)$ are calculated at the point $(\overline{y}_0(\theta_i),0), i=1,2,...,p,$ and $I_{1k}(\theta_i), I_{2k}(\theta_i)$ are defined recursively in terms of $\overline{z}_j(\theta_i)$ and $\overline{y}_j(\theta_i)$ for $j<k$. Analogous expansions hold for the expression $J_1(y,\mu)z+J_2(y,\mu)$.

The problems (3), (4) with (11) and (12) can be rewritten in the following form

$$\begin{gathered}\mu(\overline{z}_0'(t)+\mu\overline{z}_1'(t)+\ldots+\mu^k\tilde{z}_k'(t)+\ldots)=\overline{H}_0(t)+\mu\overline{H}_1(t)+\ldots\mu^k\overline{H}_k(t)+\ldots,\\ \overline{y}_0'(t)+\mu\overline{y}_1'(t)+\ldots+\mu^k\overline{y}_k'(t)+\ldots=\overline{h}_0(t)+\mu\overline{h}_1(t)+\ldots\mu^k\overline{h}_k(t)+\ldots,\\ \dot{\omega}_0^{(i)}(\tau_i)+\mu\dot{\omega}_1^{(i)}(\tau_i)+\ldots+\mu^k\dot{\omega}_k^{(i)}(\tau_i)+\ldots=\Pi_0H(\tau_i)+\mu\Pi_1H(\tau_i)+\ldots+\mu^k\Pi_kH(\tau_i)+\ldots,\\ \dot{\nu}_0^{(i)}(\tau_i)+\varepsilon\dot{\nu}_1^{(i)}(\tau_i)+\ldots+\varepsilon^k\dot{\nu}_k^{(i)}(\tau_i)+\ldots=\Pi_0h(\tau_i)+\varepsilon\Pi_1h(\tau_i)+\ldots+\mu^k\Pi_kh(\tau_i)+\ldots,\\ \mu(\sum_{k=0}^{\infty}\mu^k\Delta\overline{z}_k|_{t=\theta_i}+\sum_{k=0}^{\infty}\mu^k\omega_k^{(i)}(0))=\overline{T}_0(\theta_i)+\mu\overline{T}_1(\theta_i)+\ldots+\mu^k\tilde{T}_k(\theta_i)+\ldots,\\ \sum_{k=0}^{\infty}\mu^k\Delta\overline{y}_k|_{t=\theta_i}+\mu\sum_{k=0}^{\infty}\mu^k\nu_k^{(i)}(0)=\overline{S}_0(\theta_i)+\mu\overline{S}_1(\theta_i)+\ldots+\mu^k\overline{S}_k(\theta_i)+\ldots.\end{gathered}$$

By inserting the expansion (9) into conditions (4), we get

$$z(0,\mu)=\sum_{k=0}^{\infty}\mu^k\overline{z}_k(0)+\sum_{k=0}^{\infty}\mu^k\omega_k^{(0)}(0)=z^0,$$

and

$$y(0,\mu)=\sum_{k=0}^{\infty}\mu^k\overline{y}_k(0)+\mu\sum_{k=0}^{\infty}\mu^k\nu_k^{(0)}(0)=y^0.$$

The expansions are performed up to order n and the coefficients are equated by powers of μ. For the zero-order approximation $\overline{z}_0(t),\overline{y}_0(t),\omega_0^{(i)}(\tau_i)$ and $\nu_0^{(i)}(\tau_i), i=1,2,...,p$, the following systems are obtained:

$$\begin{aligned}0&=F(\tilde{y}_0(t),t)\overline{z}_0(t)+G(\overline{y}_0(t),t),\\ \overline{y}_0'(t)&=f(\overline{y}_0(t),t)\overline{z}_0(t)+g(\overline{y}_0(t),t),\end{aligned}\tag{14}$$

$$\begin{aligned}\dot{\omega}_0^{(i)}(\tau_i)&=F(\overline{y}_0(\theta_i),\theta_i)\omega_0^{(i)}(\tau_i)=\Pi_0H(\tau_i),\\ \dot{\nu}_0^{(i)}(\tau_i)&=f(\overline{y}_0(\theta_i),\theta_i)\omega_0^{(i)}(\tau_i)=\Pi_0h(\tau_i),\end{aligned}\tag{15}$$

$$0=\frac{I_1(\overline{y}_0(\theta_i),0)\overline{z}_0(\theta_i)+I_2(\overline{y}_0(\theta_i),0)}{\mu},\tag{16}$$

$$\begin{aligned}&\Delta\overline{z}_0|_{t=\theta_i}+\omega_0^{(i)}(0)=I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i)+(I_{1y}(\theta_i)\overline{z}_0(\theta_i)+I_{2y}(\theta_i))\overline{y}_1(\theta_i)+\overline{I}_\mu(\theta_i)=\overline{T}_1(\theta_i),\\ &\Delta\overline{y}_0|_{t=\theta_i}=J_1(\overline{y}_0(\theta_i),0)\overline{z}_0(\theta_i)+J_2(\overline{y}_0(\theta_i),0)=\overline{S}_0(\theta_i),\\ &\overline{z}_0(0)+\omega_0^{(0)}(0)=z^0,\ \ \overline{y}_0(0)=y^0.\end{aligned}\tag{17}$$

To find the coefficients of $\mu^k(k\geq 1)$, the following equations are used

$$\begin{aligned}\overline{z}_{k-1}'(t)&=F(\overline{y}_0(t),t)\overline{z}_k(t)+(F_y(t)\overline{z}_0(t)+G_y(t))\overline{y}_k(t)+H_k(t),\\ \overline{y}_k'(t)&=f(\overline{y}_0(t),t)\overline{z}_k(t)+(f_y(t)\overline{z}_0(t)+g_y(t))\overline{y}_k(t)+h_k(t),\end{aligned}\tag{18}$$

$$\dot{\omega}_k^{(i)}(\tau_i) = F(\overline{y}_0(\theta_i),\theta_i)\omega_k^{(i)}(\tau_i) + \Pi_k F(\tau_i)\overline{z}_0(\theta_i) + \Pi_k G(\tau_i) = \Pi_k H(\tau_i),$$
$$\dot{\nu}_k^{(i)}(\tau_i) = f(\overline{y}_0(\theta_i),\theta_i)\omega_k^{(i)}(\tau_i) + \Pi_k f(\tau_i)\overline{z}_0(\theta_i) + \Pi_k g(\tau_i) = \Pi_k h(\tau_i),$$

$$\begin{aligned}
&\Delta\overline{z}_k|_{t=\theta_i} + \omega_k^{(i)}(0) = I_1(\overline{y}_0(\theta_i),0)\overline{z}_{k+1}(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_{k+1}(\theta_i)) + \overline{I}_{k+1}(\theta_i),\\
&\Delta\overline{y}_k|_{t=\theta_i} + \nu_{k-1}^{(i)}(0) = J_1(\overline{y}_0(\theta_i),0)\overline{z}_k(\theta_i) + (J_{1y}(\theta_i)\overline{z}_0(\theta_i) + J_{2y}(\theta_i))\overline{y}_k(\theta_i)) + \overline{J}_k(\theta_i),\\
&\overline{z}_k(0) + \omega_k^{(0)}(0) = 0, \;\; \overline{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0.
\end{aligned} \tag{19}$$

Now we consider the interval $t \in [0,\theta_1]$. To obtain the leading-order approximations $\overline{z}_0(t) = \overline{z}(t)$ and $\overline{y}_0(t) = \overline{y}(t)$, we solve system

$$0 = F(\overline{y}_0(t),t)\overline{z}_0(t) + G(\overline{y}_0(t),t),$$
$$\overline{y}_0'(t) = f(\overline{y}_0(t),t)\overline{z}_0(t) + g(\overline{y}_0(t),\; \overline{y}_0(0) = y^0.$$

By virtue of the first equation in (14), equation (15) can be rewritten in the form

$$\dot{\omega}_0^{(0)}(\tau_0) = F(\overline{y}_0(0),0)\omega_0^{(0)}(\tau_0).$$

From the last equation, together with the initial condition

$$\omega_0^{(0)}(0) = z^0 - \overline{z}_0(0)$$

the function $\omega_0^{(0)}(\tau_0)$ can be determined. According to condition $(C5)$, $\omega_0^{(0)}(\tau_0)$ admits the exponential estimate

$$|\omega_0^{(0)}(\tau_0)| \le c\exp(-\kappa\tau_0), \tag{20}$$

where $c > 0$ and $\kappa > 0$.

The final step is to solve equation

$$\dot{\nu}_0^{(0)}(\tau_0) = F(\overline{y}_0(0),0)\omega_0^{(0)}(\tau_0) \equiv \Pi_0 h(\tau_0).$$

In view of condition (10), the initial condition is given by

$$\nu_0^{(0)}(0) = -\int_0^\infty \Pi_0 h(s)ds,$$

from which it follows that

$$\nu_0^{(0)}(\tau_0) = -\int_{\tau_0}^\infty \Pi_0 h(s)ds.$$

Since $\Pi_0 f(\tau_0)$ decays exponentially, i.e., $|\Pi_0 f(\tau_0)| \le c\exp(-\kappa\tau_0)$ the same holds for $\nu_0^{(0)}(\tau_0):$

$$|\nu_0^{(0)}(\tau_0)| \le c\exp(-\kappa\tau_0).$$

The coefficients of μ^k in the approximations $\overline{z}_k(t)$ and $\overline{y}_k(t)$ are obtained by applying system

$$\overline{z}_{k-1}'(t) = F(\overline{y}_0(t),t)\overline{z}_k(t) + (F_y(t)\overline{z}_0(t) + G_y(t))\overline{y}_k(t) + H_k(t),$$
$$\overline{y}_k'(t) = f(\overline{y}_0(t),t)\overline{z}_k(t) + (f_y(t)\overline{z}_0(t) + g_y(t))\overline{y}_k(t) + h_k(t),$$
$$\overline{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0.$$

To get $\omega_k^{(0)}(\tau_0)$, the following system must be solved

$$\dot{\omega}_k^{(0)}(\tau_0) = F(\overline{y}_0(0), 0)\omega_k^{(0)}(\tau_0) + \Pi_k F(\tau_0)\overline{z}_0(0) + \Pi_k G(\tau_0) = \Pi_k H(\tau_0),$$
$$\omega_k^{(0)}(0) = -\overline{z}_k(0).$$

The remaining task is to solve the equation

$$\dot{\nu}_k^{(0)}(\tau_0) = f(\overline{y}_0(0), 0)\omega_k^{(0)}(\tau_0) + \Pi_k f(\tau_0)\overline{z}_0(0) + \Pi_k g(\tau_0) = \Pi_k h(\tau_0)$$

Taking into account condition (10), the initial condition is given by

$$\nu_k^{(0)}(0) = -\int_0^{\infty} \Pi_k h(s) ds,$$

from which it follows that

$$\nu_k^{(0)}(\tau_0) = -\int_0^{\tau_\infty} \Pi_k h(s) ds.$$

Both $\Pi_k H(\tau_0)$ and $\Pi_k h(\tau_0)$ satisfy exponential estimates of the type given in (20). As a consequence, the following inequalities are satisfied,

$$|\omega_k^{(0)}(\tau_0)| \leq c \exp(-\kappa\tau_0),$$
$$|\nu_k^{(0)}(\tau_0)| \leq c \exp(-\kappa\tau_0).$$

Let us now consider the interval $t \in (\theta_i, \theta_{i+1}], i = 1, 2, ..., p$. To obtain the leading-order terms $\overline{z}_0(t) = \overline{z}(t)$ and $\overline{y}_0(t) = \overline{y}(t)$, corresponding to the power ε^0, we make use of system

$$0 = F(\overline{y}_0(t), t)\overline{z}_0(t) + G(\overline{y}_0(t), t), \qquad 0 = I_1(\overline{y}_0(\theta_i), 0)\overline{z}_0(\theta_i) + I_2(\overline{y}_0(\theta_i), 0),$$
$$\overline{y}_0'(t) = f(\overline{y}_0(t), t)\overline{z}_0(t) + g(\overline{y}_0(t), t), \qquad \Delta\overline{y}_0|_{t=\theta_i} = J_1(\overline{y}_0(\theta_i), 0)\overline{z}_0(\theta_i) + J_2(\overline{y}_0(\theta_i), 0).$$

In view of the first equation in (14), equation (15) takes the form

$$\dot{\omega}_0^{(i)}(\tau_i) = F(\overline{y}_0(\theta_i), \theta_i)\omega_0^{(i)}(\tau_i), i = 1, 2, ..., p.$$

Based on the last equation and the initial condition

$$\omega_0^{(i)}(0) = I_1(\overline{y}_0(\theta_i), 0)\overline{z}_1(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_1(\theta_i) + \overline{I}_\mu(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p,$$

the function $\omega_0^{(i)}(\tau_i)$ is to be determined, where $\omega_0^{(i)}(0)$ represented in the modified form below. Differentiating both sides of the first equations in (14) and (16) yields the following

$$\begin{aligned} F_y(\overline{y}_0(\theta_i), \theta_i)\overline{z}_0(\theta_i) + G_y(\overline{y}_0(\theta_i), \theta_i) &= -F(\overline{y}_0(\theta_i), \theta_i)\frac{dz}{dy}, \\ I_{1y}(\overline{y}_0(\theta_i), 0)\overline{z}_0(\theta_i) + I_{2y}(\overline{y}_0(\theta_i), 0) &= -I_1(\overline{y}_0(\theta_i), 0)\frac{dz}{dy}. \end{aligned} \tag{21}$$

Inserting the first equation of (21) into (18) results in

$$\overline{z}_0'(\theta_i) = F(\overline{y}_0(\theta_i), \theta_i)\overline{z}_1(\theta_i) + (F_y(\theta_i)\overline{z}_0(\theta_i) + G_y(\theta_i))\overline{y}_1(\theta_i) + H_1(\theta_i).$$

Hence, it follows that

$$\overline{z}_1(\theta_i) - \overline{y}_1(\theta_i)\frac{dz}{dy} = \frac{\overline{z}_0'(\theta_i) - H_1(\theta_i)}{F(\overline{y}_0(\theta_i), \theta_i)}. \tag{22}$$

Inserting the second equation of (21) into (19) gives

$$\omega_0^{(i)}(0) = I_1(\overline{y}_0(\theta_i), 0)[\overline{z}_1(\theta_i) - \overline{y}_1(\theta_i)\frac{dz}{dy}] + \overline{I}_\mu(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

Substituting equation (22) in place of the square bracket yields

$$\omega_0^{(i)}(0) = \frac{I_1(\overline{y}_0(\theta_i), 0)}{F(\overline{y}_0(\theta_i), \theta_i)}(\overline{z}_0'(\theta_i) - H_1(\theta_i)) + \overline{I}_\mu(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

According to condition $(C5)$, the function $\omega_0^{(i)}(\tau_i)$ satisfies an exponential estimate of the form

$$|\omega_0^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p, \tag{23}$$

where c and κ denote positive constants, which values may differ across various inequalities.

The remaining task is to solve the following equation

$$\dot{\nu}_0^{(i)}(\tau_i) = f(\overline{y}_0(\theta_i), \theta_i)\omega_0^{(i)}(\tau_i) = \Pi_0 h(\tau_i), i = 1, 2, ..., p.$$

Using condition (10), we determine the initial condition as follows

$$\nu_0^{(i)}(0) = -\int_0^\infty \Pi_0 h(s)ds.$$

Consequently, the following result is derived

$$\nu_0^{(i)}(\tau_i) = -\int_{\tau_i}^\infty \Pi_0 h(s)ds.$$

Since $|\Pi_0 h(\tau_i)| \le c\exp(-\kappa\tau_i)$, it holds that

$$|\nu_0^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p.$$

The coefficients of ε^k in the approximations $\overline{z}_k(t)$ and $\overline{y}_k(t)$ are determined from the following system

$$\begin{gathered}\overline{z}_{k-1}'(t) = F(\overline{y}_0(t), t)\overline{z}_k(t) + (F_y(t)\overline{z}_0(t) + G_y(t))\overline{y}_k(t) + H_k(t),\\ \overline{y}_k'(t) = f(\overline{y}_0(t), t)\overline{z}_k(t) + (f_y(t)\overline{z}_0(t) + g_y(t))\overline{y}_k(t) + h_k(t),\\ \Delta\overline{y}_k|_{t=\theta_i} + \nu_{k-1}^{(i)}(0) = J_1(\overline{y}_0(\theta_i), 0)\overline{z}_k(\theta_i) + (J_{1y}(\theta_i)\overline{z}_0(\theta_i) + J_{2y}(\theta_i))\overline{y}_k(\theta_i)) + \overline{J}_k(\theta_i).\end{gathered}$$

The functions $\omega_k^{(i)}(\tau_i)$ are determined as the solutions of the following system

$$\begin{aligned}&\dot{\omega}_k^{(i)}(\tau_i) = F(\overline{y}_0(\theta_i), \theta_i)\omega_k^{(i)}(\tau_i) + \Pi_k F(\tau_i)\overline{z}_0(\theta_i) + \Pi_k G(\tau_i) = \Pi_k H(\tau_i),\\ &\omega_k^{(i)}(0) = I_1(\overline{y}_0(\theta_i), 0)\overline{z}_{k+1}(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_{k+1}(\theta_i) + \overline{I}_{k+1}(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i},\end{aligned}$$

where the initial value $\omega_k^{(i)}(0)$ can be represented in an equivalent form below

$$\omega_k^{(i)}(0) = \frac{I_1(\overline{y}_0(\theta_i),0)}{F(\overline{y}_0(\theta_i),\theta_i)}(\overline{z}_k'(\theta_i) - H_{k+1}(\theta_i)) + \overline{I}_{k+1}(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i},$$

Finally, it is necessary to solve the equation

$$\dot{\nu}_k^{(i)}(\tau_i) = f(\overline{y}_0(\theta_i),\theta_i)\omega_k^{(i)}(\tau_i) + \Pi_k f(\tau_i)\overline{z}_0(\theta_i) + \Pi_k g(\tau_i) = \Pi_k h(\tau_i), i = 1,2,...,p.$$

By applying condition (10), we obtain

$$\nu_k^{(i)}(0) = -\int_0^\infty \Pi_k h(s)ds,$$

and

$$\nu_k^{(i)}(\tau_i) = -\int_{\tau_i}^\infty \Pi_k h(s)ds.$$

The functions $\Pi_k H(\tau_i)$ and $\Pi_k h(\tau_i)$ admit exponential estimates of the form (23). Accordingly, one can prove that the following inequalities are satisfied,

$$\begin{aligned} &|\omega_k^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), i = 1,2,...,p,\\ &|\nu_k^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), i = 1,2,...,p. \end{aligned} \tag{24}$$

Hence, the expansions in (9) are constructed at least up to the terms of order $k = n$.

3 Main Results

In this section, we prove Theorems 1 and Theorem 2, which address two different behaviors: a single layer singularity and a multi-layers singularity. The first behavior corresponds to a layer concentrated near $t = 0$, while the second deals with the presence of multiple layers near $t = 0$ and at the points $t = \theta_i$, $i = 1, 2, \ldots, p$. It is demonstrated that the partial sums of the series (8) form a sequence of uniform approximations to the solution of the problem (3)–(4).

3.1 Asymptotic expansion of singularity with a single layer

We consider the case in which the convergence of the solution is non-uniform in a neighborhood of $t = 0$, as a result of the initial condition $z(0,\mu) = z^0$ satisfying $z^0 \neq \varphi$ for all $\mu > 0$. The interval where this non-uniformity occurs is referred to as the *initial layer*.

In accordance with condition ($C5$) of (13), the following identity holds,

$$I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_1(\theta_i) + \overline{I}_\mu(\theta_i) = 0, i = 1,2,...,p.$$

As a result, the first equation of (17) becomes

$$\omega_0^{(i)}(0) = -\Delta\overline{z}_0|_{t=\theta_i}, i = 1,2,...,p.$$

Substituting the above expression into (8), we obtain

$$z(\theta_i+,\mu) = \overline{z}_0(\theta_i+) + \omega_0^{(i)}(0) + O(\mu) = \overline{z}_0(\theta_i) + O(\mu), i = 1,2,...,p.$$

It can be concluded that the region of non-uniform convergence has a thickness of order $O(\mu)$, since for $t > 0$ the estimate $|z(t,\mu) - \varphi| = O(\mu)$ holds and can be made arbitrarily small by choosing sufficiently small μ. This indicates that, for sufficiently small values of μ, the solution $z(t,\mu)$ to the problem (3), (4) does not exhibit boundary layer behavior in the vicinity of the points $t = \theta_i$, $i = 1,2,\ldots,p$.

Theorem 1 *Let conditions $(C1) - (C4)$ and $(C5)$ be satisfied. Then there exist positive constants μ_0 and c such that, for all $\mu \in (0,\mu_0]$, the problem (3), (4) admits a unique solution $z(t,\mu), y(t,\mu)$ that satisfies the inequality*

$$\begin{aligned} &|z(t,\mu) - Z_n(t,\mu)| \le c\mu^{n+1}, \quad 0 \le t \le T, \\ &|y(t,\mu) - Y_n(t,\mu)| \le c\mu^{n+1}, \quad 0 \le t \le T, \end{aligned} \tag{25}$$

where

$$\begin{aligned} &Z_n(t,\mu) = Z_n^{(i)}(t,\mu), Y_n(t,\mu) = Y_n^{(i)}(t,\mu), \theta_i < t \le \theta_{i+1}, \\ &Z_n^{(i)}(t,\mu) = \sum_{k=0}^{n} \mu^k \overline{z}_k(t) + \sum_{k=0}^{n} \mu^k \omega_k^{(i)}(\tau_i), \tau_i = \frac{t-\theta_i}{\mu}, \\ &Y_n^{(i)}(t,\mu) = \sum_{k=0}^{n} \mu^k \overline{y}_k(t) + \mu \sum_{k=0}^{n} \mu^k \nu_k^{(i)}(\tau_i), i = 1,2,...,p. \end{aligned}$$

Proof 1 *Substituting the expressions $z(t,\mu) = u(t,\mu) + Z_n(t,\mu)$ and $y(t,\mu) = v(t,\mu) + Y_n(t,\mu)$ into equations (3) and (4), we derive the following system*

$$\begin{aligned} \mu\frac{du}{dt} &= F(Y_0,t)u + [F_y(Y_0,t)Z_0 + G_y(Y_0,t)]v + T_1(u,v,t,\mu), \\ \frac{dv}{dt} &= f(Y_0,t)u + [f_y(Y_0,t)Z_0 + g_y(Y_0,t)]v + T_2(u,v,t,\mu), \\ \mu\Delta u|_{t=\theta_i} &= I_1(Y_0,0)u + [I_{1y}(Y_0,0)Z_0 + I_{2y}(Y_0,0)]v + S_1(u,v,\theta_i,\mu), \\ \Delta v|_{t=\theta_i} &= J_1(Y_0,0)u + [J_{1y}(Y_0,0)Z_0 + J_{2y}(Y_0,0)]v + S_2(u,v,\theta_i,\mu), \end{aligned} \tag{26}$$

with initial condition

$$u(0,\mu) = 0, \quad v(0,\mu) = 0, \tag{27}$$

where the components of the functions F_z, F_y, f_z and f_y are calculated at the points $(\overline{z}_0(t) +$

$\omega_0^{(i)}(\tau_i), \overline{y}_0(t), 0), i = 1, 2, ..., p,$

$$\begin{aligned}
T_1(u,v,t,\mu) =& F(v+Y_n,t)(u+Z_n) + G(v+Y_n,t) - F(Y_0,t)u- \\
& - [F_y(Y_0,t)Z_0 + G_y(Y_0,t)]v - \mu\frac{dZ_n}{dt}, \\
T_2(u,v,t,\mu) =& f(v+Y_n,t)(u+Z_n) + g(v+Y_n,t) - f(Y_0,t)u- \\
& - [f_y(Y_0,t)Z_0 + g_y(Y_0,t)]v - \frac{dY_n}{dt}, \\
S_1(u,v,\theta_i,\mu) =& I_1(v+Y_n^{(i-1)},\mu)(u+Z_n^{(i-1)}) + I_2(v+Y_n^{(i-1)},\mu) - I_1(Y_0,0)u- \\
& - [I_{1y}(Y_0,0)Z_0 + I_{2y}(Y_0,0)]v + \mu Z_n^{(i-1)} - \mu Z_n^{(i)}, \\
S_2(u,v,\theta_i,\mu) =& J_1(v+Y_n^{(i-1)},\mu)(u+Z_n^{(i-1)}) + J_2(v+Y_n^{(i-1)},\mu) - J_1(Y_0,0)u- \\
& - [J_{1y}(Y_0,0)Z_0 + J_{2y}(Y_0,0)]v + Y_n^{(i-1)} - Y_n^{(i)}.
\end{aligned}$$

The functions $T(u,v,t,\mu)$ possess the following two properties,

1) $T_1(0,0,t,\mu) = O(\mu^{n+1}), T_2(0,0,t,\mu) = O(\mu^{n+1})$.

2) *For any $\mu > 0$, there exist constants $c_2 > 0$ and $\mu_0 > 0$ such that, for all $\mu \in (0,\mu_0)$ and for $u_i, v_i, i = 1,2$, the following inequalities are satisfied,*

$|T_i(u_1,v_1,t,\mu) - T_i(u_2,v_2,t,\mu)| \le c_2\mu(|u_2-u_1| + |v_2-v_1|),\ i = 1,2.$

We now proceed to prove property 1). For $t \in (\theta_i, \theta_{i+1}]$, it follows that

$$\begin{aligned}
&T_1(0,0,t,\mu) = F(v+Y_n,t)Z_n + G(v+Y_n,t) - \mu\frac{dZ_n}{dt} = G(\sum_{k=0}^{n}\mu^k(\overline{y}_k(t) + \mu\nu_k^{(i)}(\tau_i)),t) + \\
&+ F(\sum_{k=0}^{n}\mu^k(\overline{y}_k(t) + \mu\nu_k^{(i)}(\tau_i)),t)(\sum_{k=0}^{n}\mu^k(\overline{z}_k(t) + \omega_k^{(i)}(\tau_i))) - \sum_{k=0}^{n}\mu^k(\overline{z}'_k(t) + \dot{\omega}_k^{(i)}(\tau_i)) = \\
&= F(\sum_{k=0}^{n}\mu^k(\overline{y}_k(t),t)\sum_{k=0}^{n}\mu^k\overline{z}_k(t) + G(\sum_{k=0}^{n}\mu^k(\overline{y}_k(t),t) - \sum_{k=0}^{n}\mu^k\overline{z}'_k(t) + \\
&F(\overline{y}(\theta_i+\mu\tau_i,\mu) + \mu\nu^{(i)}(\tau_i,\mu), \theta_i+\mu\tau_i)\sum_{k=0}^{n}\mu^k\omega_k^{(i)}(\tau_i) + \sum_{k=0}^{n}\mu^k(\Pi_kF(\tau_i)\overline{z}_0(\theta_i) + \Pi_kG(\tau_i)) - \\
&- \sum_{k=0}^{n}\mu^k\dot{\omega}_k^{(i)}(\tau_i) = [\sum_{k=0}^{n}\mu^k\overline{H}_k(t) + O(\mu^{n+1}) - \sum_{k=0}^{n}\mu^k\overline{z}'_k(t)] + \\
&+ [\sum_{k=0}^{n}\mu^k\Pi_kH(\tau_i) + O(\mu^{n+1}) - \sum_{k=0}^{n}\mu^k\dot{\omega}_k^{(i)}(\tau_i)] = O(\mu^{n+1}),
\end{aligned}$$

similarly to that for the functions $\overline{y}_k(t), \nu_k^{(i)}(\tau_i), i = 1,2,...,p$. The validity of the second property of the functions $T_j, j = 1,2$, can be derived by applying the mean value theorem. In fact,

$$T_i(u_1,v_1,t,\mu) - T_i(u_2,v_2,t,\mu) = \sup_{[0;T]}|\partial_u^* T|\cdot(u_1-u_2) + \sup_{[0;T]}|\partial_v^* T|\cdot(v_1-v_2),$$

where $\partial_u^* T = \partial_u^* T(u^*(s), v^*(s), t, \mu)$, $\partial_v^* T = \partial_v^* T(u^*(s), v^*(s), t, \mu)$, $u^*(s) = u_2 + s(u_1 - u_2)$, $v^*(s) = u^*(s) = v_2 + s(v_1 - v_2)$, $0 < s < 1$. *But*

$$\partial_u T_i(u^*(s), v^*(s), t, \mu) = F(v + Y_n, t) - F(Y_0, t),$$
$$\partial_v T_i(u^*(s), v^*(s), t, \mu) = F_y(v + Y_n, t)(u + Z_n) - F_y(Y_0, t)Z_0 + G_y(v + Y_n, t) - G_y(Y_0, t),$$

and

$$|u^*(s) + Z_n(t, \mu) - Z_0(t)| \leq |u^*(s)| + C\mu,$$
$$|v^*(s) + Y_n(t, \mu) - Y_0(t)| \leq |v^*(s)| + C\mu.$$

The continuity of the first-order partial derivatives of the functions $F(y,t), G(y,t), f(y,t)$ *and* $g(y,t)$ *ensures the validity of property* 2)*. The functions* $S_i(u, v, \theta_i, \mu)$, $i = 1, 2$, *possess the following two properties,*

1^*) *For* $0 < \mu < \mu_0$

$$S_1(0, 0, \theta_i, \mu) = O(\mu^{n+1}), S_2(0, 0, \theta_i, \mu) = O(\mu^{n+1}).$$

2^*) *For any* $\mu > 0$, *there exist constants* $c_2 > 0$ *and* $\mu_0 > 0$ *such that, for all* $\mu \in (0, \mu_0)$ *and for* $u_i, v_i, i = 1, 2$, *the following inequalities are satisfied,*

$$|S_i(u_1, v_1, t, \mu) - S_i(u_2, v_2, t, \mu)| \leq c_2\mu(|u_2 - u_1| + |v_2 - v_1|), \; i = 1, 2.$$

The proofs of properties 1^*) *and* 2^*) *follow analogously to those of properties* 1) *and* 2)*, respectively.*

We now reformulate the impulsive system (26)–(27) as an equivalent system of integral equations

$$u(t, \mu) = \frac{1}{\mu}\int_0^t \Phi(t, s, \mu)[(F_y(Y_0, s)Z_0 + G_y(Y_0, s))v(s, \mu) + T_1(u, v, s, \mu)]ds + \\ + \sum_{0<\theta_i<t} \Phi(t, \theta_i, \mu)(1 + \frac{I_1(Y_0, 0)}{\mu})^{-1}([I_{1y}(Y_0, 0)Z_0 + I_{2y}(Y_0, 0)]v(\theta_i, \mu) + S_1(u, v, \theta_i, \mu)), \tag{28}$$

$$v(t, \mu) = \int_0^t \Psi(t, s, \mu)[f(Y_0, s)u(s, \mu) + T_2(u, v, t, \mu)]ds + \\ + \sum_{0<\theta_i<t} \Psi(t, \theta_i, \mu)(1 + J_{1y}(Y_0, 0)Z_0 + J_{2y}(Y_0, 0))^{-1}(J_1(Y_0, 0)u(\theta_i, \mu) + S_2(u, v, \theta_i, \mu)), \tag{29}$$

where $\Phi(t, s, \mu)$ *and* $\Psi(t, s, \mu)$ *denote the fundamental matrices of the corresponding system*

$$\mu\frac{d\Phi}{dt} = F(Y_0, t)\Phi, \; t \neq \theta_i, \; \mu\Delta\Phi|_{t=\theta_i} = I_1(Y_0, 0)\Phi, \; \Phi(s, s, \mu) = 1,$$
$$\frac{d\Psi}{dt} = (f_y(Y_0, t)Z_0 + g_y(Y_0, t))\Psi, \; t \neq \theta_i, \; \Delta\Psi|_{t=\theta_i} = (J_{1y}(Y_0, 0)Z_0 + J_{2y}(Y_0, 0))\Psi, \; \Psi(s, s, \mu) = 1.$$

The following holds for the fundamental matrix $\Phi(t, s, \mu)$

$$|\Phi(t, s, \mu)| \leq c\exp(-\frac{\kappa}{\mu}(t - s)), \;\; 0 \leq s \leq t \leq T.$$

By inserting the representation of $v(t,\mu)$ from equation (29) into the first equation, we derive

$$u(t,\mu) = \int_0^t H(t,s,\mu)u(s,\mu)ds + N_1(u,v,t,\mu),$$

where H denotes a bounded kernel, and the function N_1 satisfies the same two properties as the function $T(u,v,t,\mu)$. The last equation may be replaced by an equivalent one of the form

$$u(t,\mu) = \int_0^t R(t,s,\mu)N_1(u,v,s,\mu)ds + N_1(u,v,t,\mu) = M_1(u,v,t,\mu), \tag{30}$$

where R is the resolvent corresponding to the kernel H. Substituting the representation (30) for $u(t,\mu)$ into equation (29) yields

$$\begin{aligned} v(t,\mu) = &\int_0^t \Psi(t,s,\mu)[f(Y_0,s)M_1(u,v,s,\mu) + T_2(u,v,s,\mu)]ds + \\ &+ \sum_{0<\theta_i<t} \Psi(t,\theta_i,\mu)(1 + J_{1y}(Y_0,0)Z_0 + J_{2y}(Y_0,0))^{-1}(J_1(Y_0,0)M_1(u,v,\theta_i,\mu) + \\ &+ S_2(u,v,\theta_i,\mu)) = M_2(u,v,t,\mu). \end{aligned} \tag{31}$$

The functions M_1 and M_2 possess the same two properties as the function $T(u,v,t,\mu)$. The method of successive approximations applied to systems (30) and (31) yields a unique solution that fulfills the corresponding estimates

$$\begin{aligned} |u(t,\mu)| &= |z(t,\mu) - Z_n(t,\mu)| \le c\mu^{n+1}, \quad 0 \le t \le T, \\ |v(t,\mu)| &= |y(t,\mu) - Y_n(t,\mu)| \le c\mu^{n+1}, \quad 0 \le t \le T. \end{aligned}$$

The theorem is proven.

3.2 Asymptotic expansion of singularity with multi-layers

In the previous subsection, it was shown that there exists a single initial layer. Using an impulse function, the convergence can be nonuniform near several points, that is to say, that *multi-layers* emerge. These layers occur on the neighborhoods of $t = 0$ and $t = \theta_{i i=1}^{p}$. In the preceding subsection, the existence of a single initial layer was demonstrated. The introduction of an impulse function leads to nonuniform convergence in the vicinity of multiple points, resulting in the formation of multi-layer structures. These layers are localized near $t = 0$ and $t = \theta_i, i = 1, 2, \ldots, p$.

In order to generate a singularity exhibiting a multi-layer structure, we examine system (3) subject to conditions(C1)–(C4) along with the additional requirement condition

$$\text{(C6)} \quad \lim_{(z,y,\mu)\to(\varphi,\overline{y},0)} \frac{I_1(y,\mu)z + I_2(y,\mu)}{\mu} = l_i \neq 0,$$

where l_i is a constant, $\varphi(\overline{y}(\theta_i),\theta_i)+l_i, i = 1, 2, ..., p$, are the values for each impulse moment at the points $t = \theta_i, i = 1, 2, ..., p$. By virtue of condition (C6) from equation (13), the following equality holds

$$I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_1(\theta_i) + \overline{I}_\mu(\theta_i) = l_i \neq 0, i = 1, 2, ..., p.$$

Accordingly, the first equation of system (17) can be rewritten in the following form

$$\omega_0^{(i)}(0) = l_i - \Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

By substituting the previously derived expression into (8), we arrive at

$$z(\theta_i+, \mu) = \overline{z}_0(\theta_i+) + \omega_0^{(i)}(0) + O(\mu) = \overline{z}_0(\theta_i) + l_i + O(\mu), i = 1, 2, ..., p.$$

According to condition $(C6)$, after each impulse moment θ_i, the difference $|z(\theta_i+, \mu) - \varphi| = l_i + O(\mu)$ does not vanish as $\mu \to 0$. Consequently, the convergence is nonuniform. Therefore, it can be concluded that the solution $z(t, \mu)$ of system (3) with the initial condition (4) exhibits a multi-layer structure, with layers forming in the neighborhoods of $t = 0$ and $t = \theta_i$ for $i = 1, 2, \ldots, p$.

The proof of the next theorem follows by analogy with the proof of Theorem 1.

Theorem 2 *Let conditions $(C1) - (C4)$ and $(C6)$ be satisfied. Then there exist positive constants μ_0 and c such that, for all $\mu \in (0, \mu_0]$, the problem (3), (4) admits a unique solution $z(t, \mu), y(t, \mu)$ that satisfies the inequality*

$$|z(t, \mu) - Z_n(t, \mu)| \leq c\mu^{n+1}, \quad 0 \leq t \leq T,$$
$$|y(t, \mu) - Y_n(t, \mu)| \leq c\mu^{n+1}, \quad 0 \leq t \leq T,$$

where

$$Z_n(t, \mu) = Z_n^{(i)}(t, \mu), Y_n(t, \mu) = Y_n^{(i)}(t, \mu), \theta_i < t \leq \theta_{i+1},$$
$$Z_n^{(i)}(t, \mu) = \sum_{k=0}^{n} \mu^k \overline{z}_k(t) + \sum_{k=0}^{n} \mu^k \omega_k^{(i)}(\tau_i), \tau_i = \frac{t - \theta_i}{\mu},$$
$$Y_n^{(i)}(t, \mu) = \sum_{k=0}^{n} \mu^k \overline{y}_k(t) + \mu \sum_{k=0}^{n} \mu^k \nu_k^{(i)}(\tau_i), i = 1, 2, ..., p.$$

4 Numerical examples

4.1 Example 1

Consider the impulsive system with singularities

$$\begin{aligned} \mu z' &= -y^2 z + y^2 - 5\mu^2 y, \qquad \mu\Delta z|_{t=\theta_i} = zy - y - 2\mu^2 y^3, \\ y' &= 2zy - 8y, \qquad \Delta y|_{t=\theta_i} = 2yz - 8y, \end{aligned} \tag{32}$$

initial conditions

$$z(0, \mu) = 2, \quad y(0, \mu) = 3, \tag{33}$$

where $\theta_i = i/5, i = 1, 2, ...7$. Assume that $\mu = 0$ in the considered problem. In this case, the first equation of system (32) reduces to the form $-\overline{y}^2\overline{z} + \overline{y}^2 = 0, \overline{zy} - \overline{y} = 0.$ which yields the solution $\overline{z} = \varphi = 1$. Nevertheless, according to condition $(C2)$, the root $\overline{z} = 1$ is uniformly asymptotically stable. Inserting the value $\overline{z} = 1$ into the second equation of (32) yields the following result

$$\begin{aligned} &\overline{y}' = -6\overline{y}, \quad \Delta\overline{y}|_{t=\theta_i} = \overline{y} + 1, \\ &\overline{y}(0) = 3. \end{aligned} \tag{34}$$

This system possesses a unique solution $\overline{y}(t)$. Next, we examine the validity of condition $(C5)$

$$\lim_{(z,y,\mu)\to(\varphi,\overline{y},0)} \frac{zy - y - 2\mu^2 y^3}{\mu} = 0.$$

The solution $z(t,\mu)$ of system (32) with the initial condition (33) exhibits a single initial layer at $t = 0$. The simulation results presented in Figure 1 confirm the presence of this single-layer behavior. As $\mu \to 0$, Figure 2 shows that the solution to problem (32), (33) converges to the solution of the corresponding degenerate system (34).

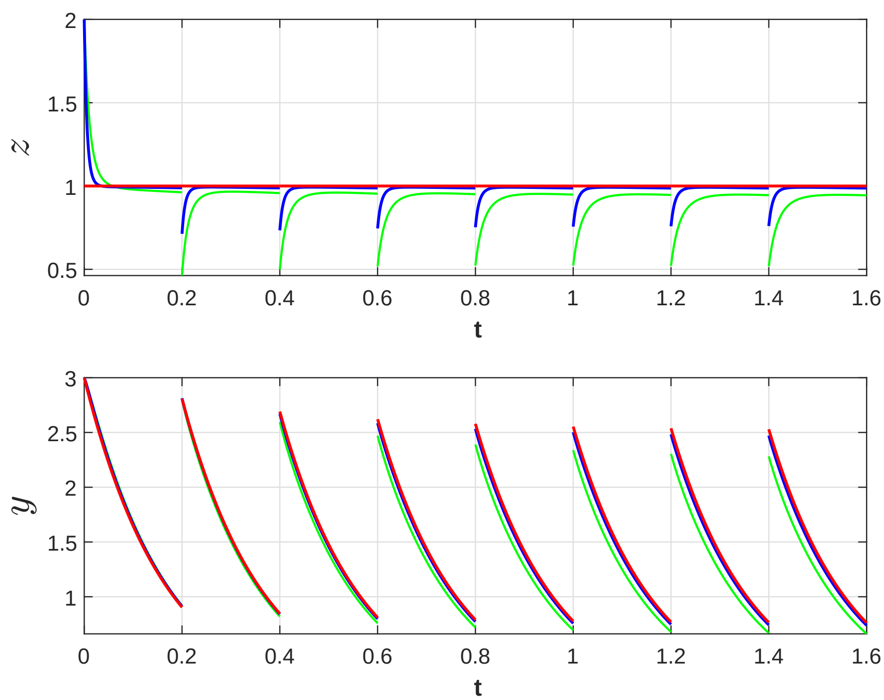

Figure 1: The blue and green curves illustrate the solutions of system (32) with initial conditions (33), corresponding to the values $\mu = 0.1$ and $\mu = 0.05$, respectively. The red line represents the solution to problem (34).

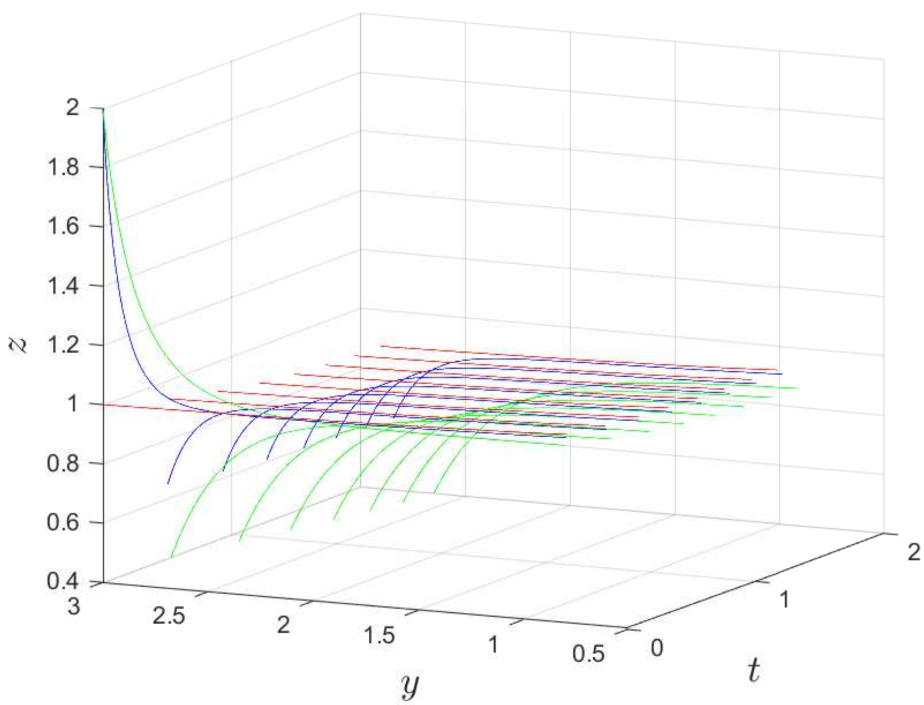

Figure 2: The blue and green curves illustrate the solutions of system (32) with initial conditions (33), corresponding to the values $\mu = 0.1$ and $\mu = 0.05$, respectively. The red line represents the solution to problem (34).

4.2 Example 2

Now, we now consider the following system

$$\begin{aligned} \mu z' &= -y^2 z - 3y^2 - 4\mu^2 yz, \qquad \mu\Delta z|_{t=\theta_i} = zy + 3y - 6\mu^2 yz - 4sin(2\mu), \\ y' &= 2zy - 8y, \qquad \Delta y|_{t=\theta_i} = 2y - z, \end{aligned} \tag{35}$$

initial conditions

$$z(0,\mu) = -1, \quad y(0,\mu) = 3, \tag{36}$$

where $\theta_i = i/5, i = 1, 2, ...7$. Setting $\mu = 0$ in (35) transforms the first equation into $-\overline{y}^2\overline{z} - 3\overline{y}^2 = 0$, which simplifies to $\overline{zy} + 3\overline{y} = 0$. This yields the root $\overline{z} = -3$. The corresponding root $\varphi = -3$ is uniformly asymptotically stable, as it satisfies condition $(C2)$. Inserting $\overline{z} = -3$ into the second equation of system (35) yields

$$\begin{aligned} &\overline{y}' = -14\overline{y}, \ \ \Delta\overline{y}|_{t=\theta_i} = 2\overline{y} + 3, \\ &\overline{y}(0) = 3. \end{aligned} \tag{37}$$

One can confirm that condition $(C6)$ is satisfied

$$\lim_{(z,y,\mu)\to(\varphi,\overline{y},0)} \frac{zy + 3y - 6\mu^2 yz - 4sin(2\mu)}{\mu} = -8 \neq 0.$$

The solution $z(t,\mu)$ of system (35) with initial condition (36) exhibits multi-layers near $t=0$ and at each point $t=\theta_i^+, i=1,2,...,7$. Figure 3 reveals the presence of multi-layer behavior in the solution, while Figure reffig4 shows that, as $\mu \to 0$, the solution of the original problem (35), (36) approaches the solution of the degenerate system (37).

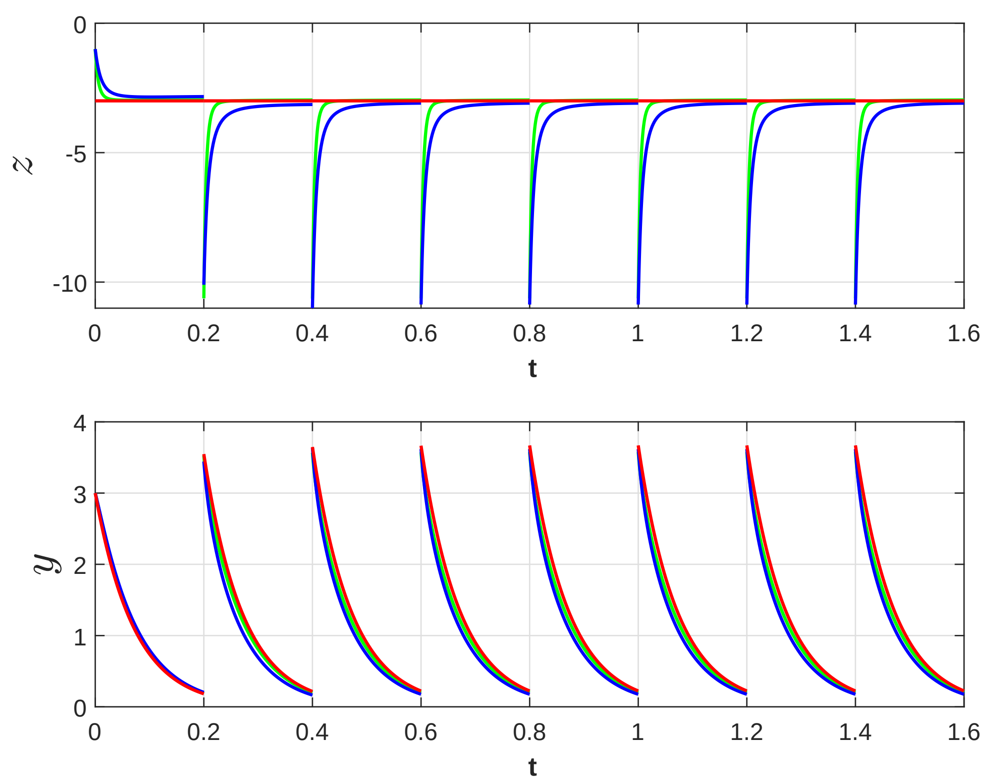

Figure 3: The green and blue curves illustrate the solutions of system (35) with initial conditions (36), corresponding to the values $\mu = 0.1$ and $\mu = 0.05$, respectively. The red line represents the solution to problem (37).

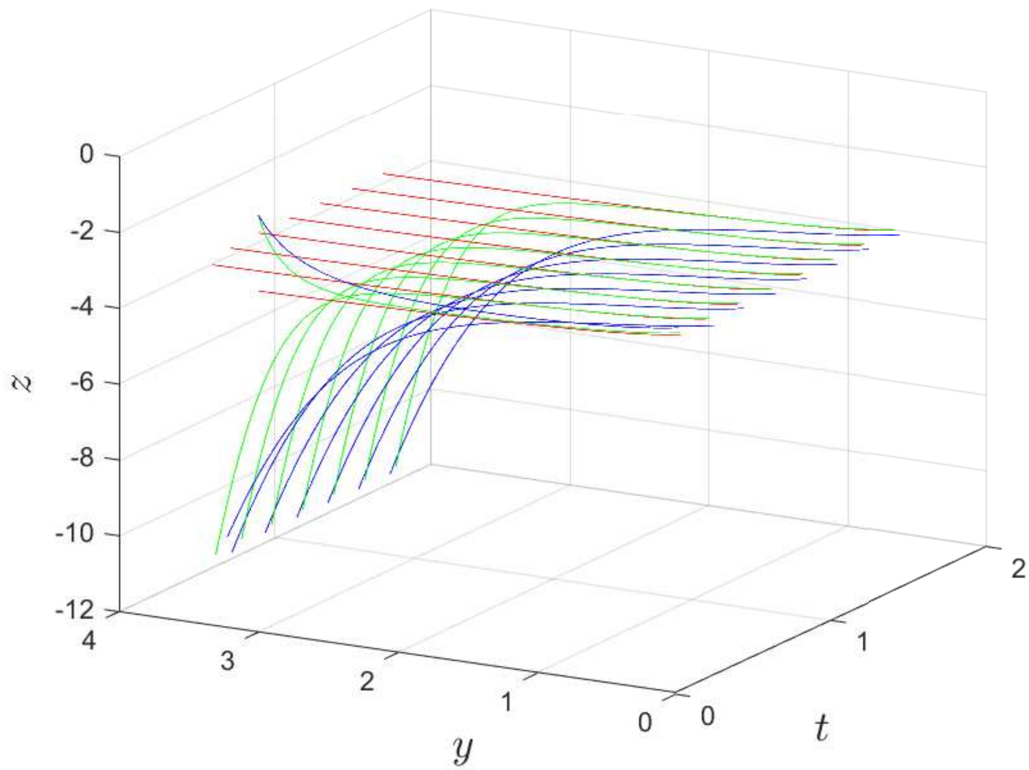

Figure 4: The green and blue curves illustrate the solutions of system (35) with initial conditions (36), corresponding to the values $\mu = 0.1$ and $\mu = 0.05$, respectively. The red line represents the solution to problem (37).

5 Conclusion

In this paper, the singularly perturbed quasi-linear impulsive differential equation is considered. The boundary function method is employed to construct asymptotic solutions with arbitrary accuracy. Both single-layer and multi-layers phenomena are analyzed within the framework of asymptotic expansions. This approach allows a detailed description of the solution behavior in regions characterized by rapid transitions and boundary layers. The theoretical results are supported by illustrative examples and numerical simulations.

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