МРНТИ 27.39.21 DOI: https://doi.org/10.26577/JMMCS202512738

N.S. Tokmagambetov
Karaganda Buketov University, Karaganda, Kazakhstan
e-mail: nariman.tokmagambetov@gmail.com

ON q-DEFORMATED HÖRMANDER MULTIPLIER THEOREM

Abstract. The main purposes of this work, we introduce the q-deformed Fourier multiplier A_g defined on the space $L^2_q(\mathbb{R}_q)$ through the framework of the q^2-Fourier transform, while also extending the functional setting of $L^p_q(\mathbb{R}_q)$ with $1 \leq p < \infty$. Our approach provides a natural extension of classical Fourier multiplier theory into the q-deformed setting, which is relevant in the context of quantum groups and noncommutative analysis. Furthermore, we establish several key q-analogues of classical harmonic analysis inequalities for the q^2-Fourier transform, including the Paley inequality, Hausdorff-Young inequality, Hausdorff-Young-Paley inequality, and Hardy-Littlewood inequality. These results not only generalize their classical counterparts but also open new avenues for analysis on q-deformed spaces. As a significant application, we prove a q-deformed version of the Hörmander multiplier theorem, which provides sufficient conditions for the boundedness of multipliers in the q-deformed setting. This work sets the stage for further developments in the field of q-deformed harmonic analysis.

Key words: q-Jackson integral, q-caclulus, Fourier multiplier, inequality, multiplier, Hausdorff-Young inequality.

Н.С. Токмагамбетов
Қарағанды Бөкетов университеті, Қарағанды, Қазақстан
e-mail: nariman.tokmagambetov@gmail.com

q-деформацияланған Хёрмандердің мультипликаторлар теоремасы туралы

Аннотация. Бұл жұмыстың негізгі мақсаттары: біз $L^2_q(\mathbb{R}_q)$ кеңістігінде анықталған q^2-Фурье түрлендіруі шеңберінде A_g q-деформацияланған Фурье көбейткішін енгіземіз және $1 \leq p < \infty$ үшін функционалды параметрді $L^p_q(\mathbb{R}_q)$ кеңістіктеріне кеңейтеміз. Біздің көзқарасымыз, әрине, Фурье көбейткіштерінің классикалық теориясын q-деформацияланған параметрге дейін кеңейтеді, бұл кванттық топтар мен коммутативті емес талдау контекстінде өте маңызды. Содан кейін біз q^2-Фурье түрлендіруі үшін гармоникалық талдаудың классикалық теңсіздіктерінің бірқатар негізгі q-аналогтарын белгілейміз, соның ішінде Палей, Хаусдорф–Янг, Хаусдорф–Янг–Пэйли және Харди-Литлвуд теңсіздіктері. Алынған нәтижелер олардың классикалық прототиптерін қорытып қана қоймай, q-деформацияланған кеңістіктерге талдаудың жаңа бағыттарын ашады. Маңызды қолданба ретінде біз көбейткіштер туралы Хёрмандер теоремасының q-деформацияланған нұсқасын дәлелдейміз, ол q-деформацияланған параметрде көбейткіштердің шектелгендігі үшін жеткілікті шарттарды береді. Бұл жұмыс q-деформацияланған гармоникалық талдауды одан әрі дамытуға негіз қалайды.
Түйін сөздер: q-Джексон интегралы, q-есептеу, Фурье көбейткіші, теңсіздік, мультипликатор, Хаусдорф–Янг теңсіздігі.

Н.С. Токмагамбетов
Карагандинский университет Букетова, Караганда, Казахстан
e-mail: nariman.tokmagambetov@gmail.com

О q-деформированной теореме Хёрмандера о мультипликаторах

Аннотация. Основные цели данной работы состоят в следующем: мы вводим q-деформированный мультипликатор Фурье A_g, определённый на пространстве $L^2_q(\mathbb{R}_q)$ в рамках q^2-преобразования Фурье, а также расширяем функциональную постановку до пространств $L^p_q(\mathbb{R}_q)$ при $1 \leq p < \infty$. Наш подход естественным образом продолжает классическую теорию фурье-мультипликаторов в q-деформированную постановку, что существенно в контексте квантовых групп и некоммутативного анализа. Далее мы устанавливаем ряд ключевых q-аналогов классических неравенств гармонического анализа для q^2-преобразования Фурье, включая неравенства Пэли, Хаусдорфа–Янга, Хаусдорфа–Янга–Пэли и Харди–Литтлвуда. Полученные результаты не только обобщают их классические прототипы, но и открывают новые направления анализа на q-деформированных пространствах. В качестве существенного приложения мы доказываем q-деформированный вариант теоремы Хёрмандера о мультипликаторах, дающий достаточные условия ограниченности мультипликаторов в q-деформированной постановке. Эта работа закладывает основу для дальнейшего развития q-деформированного гармонического анализа.
Ключевые слова: q-интеграл Джексона, q-исчисление, мультипликатор Фурье, неравенство, мультипликатор, неравенство Хаусдорфа–Янга.

1 Introduction

The history of quantum calculus (or q-deformation) started in the 18th century when L. Euler [9] investigated the infinite product in the following form:

$$(q;q)^{-1}_{\infty} = \prod_{k=0}^{\infty} \frac{1}{1-q^{k+1}}, \qquad |q| < 1.$$

It serves as a generating function for the partition function $p(n)$, which enumerates the number of distinct ways to express n as a sum of positive integers. In the early 20th century, F.H. Jackson introduced the q-derivative and the definite q-integral [6, 7], forming the basis of modern q-calculus. Over the past two decades, research on q-deformation has expanded rapidly. For instance, V. Kac and P. Cheung [8] studied its fundamental properties, while T. Ernst [10, 11] highlighted its importance in quantum computing models. Further developments include the work of N. Bettaibi and R.H. Bettaieb [4], who introduced a q-deformed Dunkl operator and analyzed its Fourier transform in [13, 14] (see also [16]). This operator is defined using Rubins q-differential operator ∂_q [17, 18]. For more details on the history and recent progress in q-calculus, see the monographs [1, 10–12, 15].

The q-difference calculus dates back to the early 20th century, with pioneering contributions by F. Jackson [6, 7] and R.D. Carmichael [5]. More recently, W. Al-Salam [3] and R.P. Agarwal [2] introduced the concept of fractional q-difference calculus. In recent years, fueled by the rapid growth of research in the q-partial dif equation, this theory has also undergone significant development (see, [25–27, 29–31]).

In this work, we establish some basic q-deformed integral inequalities for q^2-Fourier transform such as the Paley, Hausdorff-Young, Hausdorff-Young-Paley, and Hardy-Littlewood inequalities. The problem under consideration can be reformulated as proving the boundedness of an associated Fourier multiplier via an appropriate transformation. In this context, the *Hörmander multiplier theorem* is a fundamental result in Fourier analysis that provides conditions ensuring the boundedness of Fourier multiplier operators on L^p spaces. Specifically, it characterizes the regularity requirements for a multiplier function so that the associated operator, defined by multiplication in the Fourier domain, acts boundedly on

$L^p(\mathbb{R}^d$. Let σ be a function on $\mathbb{R}^d$, and define the Fourier multiplier operator A_σ by

$$A_\sigma f(x) = \mathcal{F}^{-1}\big[\sigma \cdot \hat{f}\big](x),$$

where $\mathcal{F}$ denotes the classical Fourier transform.

The theorem states that A_σ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq 2 \leq q < \infty$. if σ satisfies a condition, often expressed as

$$\sup_{\lambda>0} \lambda \Big(\int\limits_{|\sigma(s)|\geq\lambda} d_q s \Big)^{\frac{1}{p}-\frac{1}{q}}.$$

This statement generalizes earlier results by Mikhlin and provides a powerful framework for analyzing multipliers. It has important applications in partial differential equations, signal processing, and control theory, among others. Comprehensive discussions of these results and their further developments are available in the works of L. Hörmander [23], E.M. Stein [32], as well as in more recent texts like L. Grafakos [24]. Our formulation of q-deformed Fourier multiplier is more intuitive and aligns closely with the classical, commutative framework, which allows many of the same properties to carry over. Similar to the classical case, the key part of the proof depends on the Paley inequality and the Hausdorff–Young–Paley inequality for the classical Fourier transform, both of which are obtained through the Hausdorff–Young inequality. In the course of our work, we also derive q-analogue of several important inequalities such as the Paley, Hausdorff–Young–Paley, Hardy–Littlewood. Moreovere, we present a simple proof of the $L^p - L^q$ boundedness of Fourier multipliers that avoids using the Paley and Hausdorff–Young–Paley inequalities, drawing on the method introduced in [33].

2 Preliminaries

2.1 Basic notations on $\mathbb{R}_q$ space

Throughout this paper, we assume $0 < q < 1$. In this section, we will fix some notations and recall some preliminary results. We put $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and $\tilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}$. For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad n = 1, 2, \dots; \quad (a;q)_\infty = \prod_{k=0}^{\infty} \left(1 - aq^k\right).$$

We denote also

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

The q-analogue differential operator $D_q f(x)$ is

$$D_q f(x) := \frac{f(x) - f(qx)}{x(1-q)}.$$

The q-Jackson integrals are defined by (see, [6, 7])

$$\int_0^a f(x)\mathrm{d}_q x = (1-q)a\sum_{n=0}^{+\infty} q^n f\left(aq^n\right) \tag{1}$$

$$\int_a^b f(x)\mathrm{d}_q x = (1-q)\sum_{n=0}^{+\infty} q^n \left(bf\left(bq^n\right) - af\left(aq^n\right)\right) \tag{2}$$

and

$$\int_{\mathbb{R}_q} f(x)\mathrm{d}_q x = (1-q)\sum_{n=-\infty}^{+\infty} q^n \left\{f\left(q^n\right) + f\left(-q^n\right)\right\},$$

provided the sums converge absolutely.

In the following we denote by

- $L_q^p\left(\mathbb{R}_q\right) = \left\{ f : \|f\|_{L_q^p(\mathbb{R}_q)} = \left(\int_{\mathbb{R}_q} |f(x)|^p \, \mathrm{d}_q x \right)^{1/p} < \infty \right\}.$
- $L_q^\infty\left(\mathbb{R}_q\right) = \left\{ f : \|f\|_{L_q^\infty(\mathbb{R}_q)} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}.$

2.2 Fourier transform and Fourier multiplier on $\mathbb{R}_q$

The q^2-exponentials (see [18] and [17])

$$e(x;q^2) = \cos(-ix;q^2) + i\sin(-ix;q^2),$$

where the q^2-trigonometric functions

$$\cos(x;q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} x^{2k}}{[2k]_q!}$$

and

$$\sin(x;q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} x^{2k+1}}{[2k+1]_q!}.$$

Definition 2.1 *Let $f \in \mathcal{D}_q\left(\mathbb{R}_q\right)$. Then the q^2-Fourier transform of f is defined as follows*

$$\mathcal{F}(\xi;q^2) := \widehat{f}(\xi) \quad = \quad K\int_{\mathbb{R}_q} f(x)e(-ix\xi;q^2)d_q x \tag{3}$$

and its inverse

$$f(x) \quad = \quad K\int_{\mathbb{R}_q} e(ix\xi;q^2)\mathcal{F}(\xi;q^2)d_q\xi,$$

where $K = \frac{(1+q)^{1/2}}{2\Gamma_{q^2}(1/2)}$.

Moreover, we have the Plancherel (or Parseval) identity (see, [17])

$$\|f\|_{L^2_q(\mathbb{R}_q)} = \|\widehat{f}\|_{L^2_q(\mathbb{R}_q)}. \tag{4}$$

In [17], δ_y denote the weighted Dirac-measure at $y \in \mathbb{R}_q$ defined on $\mathbb{R}_q$ by

$$\delta_y(x) = \begin{cases} [(1-q)y]^{-1}, & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases}$$

and It satisfies the following properties:

1) for all $x, y \in \mathbb{R}_q$, we have the orthogonality relation

$$\delta_y(x) \ = \ K^2 \int\limits_{\mathbb{R}_q} e(ix\xi; q^2) e(-iy\xi; q^2) d_q\xi. \tag{5}$$

2) If $f \in L^1_q(\mathbb{R}_q)$, then we get that

$$f(y) \ = \ \int\limits_{\mathbb{R}_q} f(x)\delta_y(x) d_q x. \tag{6}$$

Definition 2.2 *We assume that the function $g : \mathbb{R}_q \to \mathbb{C}$ is bounded. Then, we introduce the q-deformated Fourier multiplier A_g on $L^2_q(\mathbb{R}_q)$ as follows*

$$A_g(f)(x) = K \int_{\mathbb{R}_q} g(\xi)\widehat{f}(\xi) e(ix\xi; q^2) d_q\xi. \tag{7}$$

Definition 2.3 *Let $1 \leq p, r \leq \infty$. Let $B : L^p_q(\mathbb{R}_q) \to L^r_q(\mathbb{R}_q)$ be a bounded linear operator. The, we define its adjoint operator $B^* : L^{r'}_q(\mathbb{R}_q) \to L^{p'}_q(\mathbb{R}_q)$ as follows*

$$(B(f_1), f_2) := \int_{\mathbb{R}_q} B(f_1)(\xi)\overline{f_2(\xi)} d_q\xi = \int_{\mathbb{R}_q} f_1(\xi)\overline{B(f_2)(\xi)} d_q\xi = (f_1, B^*(f_2)), \tag{8}$$

for all $f_1 \in L^p_q(\mathbb{R}_q)$ and $f_2 \in L^{r'}_q(\mathbb{R}_q)$.

2.3 The q-distribution function

In subsection, we state the distribution function $d_f(\lambda; q)$ on $\mathbb{R}_q$. Let Ω be a subset of $(0, \infty)$ and $z > 0$. Then, the definite q-integral with the function $\chi_\Omega(x)$ introduced as follows

$$\int\limits_{\mathbb{R}^+_q} \chi_{(0,z]}(x) f(x) \mathrm{d}_q x = (1-q) \sum_{q^n \leq z} q^n f(q^n) \tag{9}$$

and

$$\int\limits_{\mathbb{R}^+_q} \chi_{[z,\infty)}(x) f(x) \mathrm{d}_q x = (1-q) \sum_{z \leq q^n} q^n f(q^n), \tag{10}$$

where $\chi_\Omega(x)$ is the characteristic function of the set Ω (see, [20, formals 2.6-2.7] and [21]).

Definition 2.4 *(see, [28, Definition 2. p. 504]) The q-distribution function $d_f(\lambda;q)$ of $f:\mathbb{R}_q\to\mathbb{R}$ is a real-valued function, which expressed as*

$$d_f(\lambda;q)=\mu_q\{x\in\mathbb{R}_q:|f(x)|>\lambda\},\quad \lambda>0.$$

Moreover, we observe that

$$d_{f+g}(2\lambda;q)\le d_f(\lambda;q)+d_g(\lambda;q). \tag{11}$$

Using the distribution function, we present and demonstrate the following key characterization of the $L^p_q(\mathbb{R}_q)$ norm.

Proposition 2.5 *(see, [28, Proposition 4. p. 506])Let $0<p<\infty$ and $f\in L^p_q(\mathbb{R}_q)$. Then*

$$\|f\|^p_{L^p_q(\mathbb{R}_q)}=[p]_q\int\limits_{\mathbb{R}_q^+}\lambda^{p-1}d_f(\lambda;q)d_q\lambda. \tag{12}$$

Lemma 2.6 *(see, [28, Lemma 1. p. 506]) Let $f\in L^p_q(\mathbb{R}_q)$ for $0\le p<\infty$. Then*

a) We assume that $E_\lambda=\{x\in\mathbb{R}_q:|f(x)|>\lambda\}$

$$d_f(\lambda;q)\le\frac{1}{\lambda}\int\limits_{\mathbb{R}_q}\chi_{E_\lambda}(x)|f(x)|d_qx\le\frac{1}{\lambda}\int\limits_{\mathbb{R}_q}|f(x)|d_qx;$$

b) (The q-Chebyshev inequality).

$$d_f(\lambda;q)\le\frac{1}{\lambda^p}\int\limits_{\mathbb{R}_q}\chi_{E_\lambda}(x)|f(x)|^pd_qx,$$

3 A q-deformated interpolation theorem

In this section we establish a q-deformated interpolation theorem.

3.1 The q-deformated Marcinkiewicz Interpolation theorem

Definition 3.1 *(see, [28, Definition4. p. 507]) Assume that $0<p<\infty$. Then, we defined the space weak $L^{p,\infty}_q(\mathbb{R}_q)$ as follows*

$$\|f\|_{L^{p,\infty}_q(\mathbb{R}_q)}\ :=\ \left\{\inf_{\lambda>0}\left\{C_q>0:d_f(\lambda;q)\le\frac{C_q}{\lambda^p}\right\}=\sup_{\lambda>0}\left\{\lambda d_f^{1/p}(\lambda;q)\right\}<\infty\right\}. \tag{13}$$

The weak $L^{p,\infty}_q(\mathbb{R}_q)$ spaces are larger than the usual $L^p_q(\mathbb{R}_q)$ spaces.

For any $0<p<\infty$ and any f in $L^\infty_q(\mathbb{R}_q)$ we have

$$\|f\|_{L^{p,\infty}_q(\mathbb{R}_q)}\le\|f\|_{L^p_q(\mathbb{R}_q)}, \tag{14}$$

Hence, the embedding $L^{p,\infty}_q(\mathbb{R}_q)\hookrightarrow L^p_q(\mathbb{R}_q)$ holds.

Indeed, by (13) and the q-Chebyshev's inequality (see, Lemma 2.6 (b)), and we have

$$\|f\|_{L_q^{p,\infty}(\mathbb{R}_q)} = \sup_{\lambda>0}\{\lambda d_f^{1/p}(\lambda;q)\} = \sup_{\lambda>0}\Big\{\big(\int\limits_{E_\lambda}\chi_{E_\lambda}(x)|f(x)|^p d_qx\big)^{1/p}\Big\} \leq \|f\|^p_{L_q^p(\mathbb{R}_q)},$$

which implies that (14) holds.

Now, we can prove the following interpolation theorem, which will let us deduce $L_q^p(\mathbb{R}_q)$ boundedness from weak inequalities, since they measure the size of the distribution function.

Theorem 3.2 *(q-deformated Marcinkiewicz interpolation) Let $0 < p < s \leq \infty$ and T is a sublinear operator defined on $L_q^{p,\infty}(\mathbb{R}_q) + L_q^{s,\infty}(\mathbb{R}_q) := \left\{f_0 + f_1 : f_0 \in L_q^{p,\infty}(\mathbb{R}_q), f_1 \in L_q^{s,\infty}(\mathbb{R}_q)\right\}$. Assume that*

$$\|T(f)\|_{L_q^{p,\infty}(\mathbb{R}_q)} \leq C_0\|f\|_{L_q^{p,\infty}(\mathbb{R}_q)}, \quad \forall f \in L_q^{p,\infty}(\mathbb{R}_q), \tag{15}$$

$$\|T(f)\|_{L_q^{s,\infty}(\mathbb{R}_q)} \leq C_1\|f\|_{L_q^{s,\infty}(\mathbb{R}_q)}, \quad \forall f \in L_q^{s,\infty}(\mathbb{R}_q), \tag{16}$$

Then $\forall r \in (p,s)$ and $\forall f \in L_q^{r,\infty}(\mathbb{R})$ the following estimate holds

$$\|T(f)\|_{L_q^{r,\infty}(\mathbb{R}_q)} \leq C\|f\|_{L_q^{r,\infty}(\mathbb{R}_q)}, \tag{17}$$

where $C := 2[r]_q^{1/r}\left(\frac{1}{[r-p]_q} + \frac{1}{[s-r]_q}\right)^{1/r} C_0^\theta C_1^{1-\theta}$ and $\theta := \frac{1/r-1/s}{1/p-1/s}$.

Proof. For a fixed $\lambda > 0$ we suppose that the functions f_0 and f_1 by

$$f_0(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq C\lambda, \\ 0, & \text{if } |f(x)| > C\lambda, \end{cases} \qquad f_1(x) = \begin{cases} 0, & \text{if } |f(x)| \leq C\lambda, \\ f(x), & \text{if } |f(x)| > C\lambda, \end{cases}$$

for some $C > 0$ to be determined later.

Let $0 < p < r < s < \infty$. We assume that $E_0 := \{x : |f(x)| \leq C\lambda\}$ and $E_1 := \{x : |f(x)| > C\lambda\}$. Then it can then be easily verified that f_1 (the unbounded part of f) is an L_q^p function for $p < r$:

$$\begin{aligned}\int\limits_{\mathbb{R}_q^+}\lambda^{r-p-1}\|f_0\|^p_{L_q^p(\mathbb{R}_q)}d_q\lambda &= \int\limits_{\mathbb{R}_q^+}\lambda^{r-p-1}\int\limits_{\mathbb{R}_q}\chi_{E_0}(x)|f(x)|^p d_qx d_q\lambda \\ &= \int\limits_{\mathbb{R}_q}|f(x)|^p\int\limits_{\mathbb{R}_q^+}\lambda^{r-p-1}\chi_{E_1}(x)d_q\lambda d_qx \leq \frac{[C\lambda]^{p-r}\|f\|^r_{r,q}}{[r-p]_q},\end{aligned} \tag{18}$$

and that f_0 (the bounded part of f) is an $L_{s,q}^s(\mathbb{R}_q)$ function for $r < s$:

$$\begin{aligned}\int\limits_{\mathbb{R}_q^+}\lambda^{r-s-1}\|f_1\|^s_{s,q}d_q\lambda &= \int\limits_{\mathbb{R}_q^+}\lambda^{r-s-1}\int\limits_{\mathbb{R}_q}\chi_{E_1}(x)|f(x)|^s d_qx d_q\lambda \\ &= \int\limits_{\mathbb{R}_q}|f(x)|^s\int\limits_{\mathbb{R}_q^+}\lambda^{r-s-1}\chi_{E_0}(x)d_q\lambda d_qx \leq \frac{[C\lambda]^{s-r}\|f\|^r_{r,q}}{[s-r]_q}.\end{aligned} \tag{19}$$

the subadditivity property of T and Hypotheses (15) and (16) together with (11) now give

$$d_{Tf}(2\lambda;q) \overset{(11)}{\leq} d_{Tf_0}(\lambda;q)+d_{Tf_1}(\lambda;q) \overset{(15)(16)}{\leq} \frac{C_0^p}{\lambda^p}\|f_0\|_{p,q}^p+\frac{C_1^s}{\lambda^s}\|f_1\|_{s,q}^s \tag{20}$$

In view of the last estimates (18)-(20) and (12), we conclude that

$$\begin{aligned}
\|Af\|_{r,q}^r &\overset{(12)}{=} [r]_q\int\limits_{\mathbb{R}_q^+}[2\lambda]^{r-1}d_{Tf}(2\lambda;q)d_q2\lambda \\
&\overset{(20)}{=} 2^r[r]_q\left[C_0^p\int\limits_{\mathbb{R}_q^+}\lambda^{r-p-1}\|f_0\|_{p,q}^p d_q\lambda+C_1^s\int\limits_{\mathbb{R}_q^+}\lambda^{r-s-1}\|f_1\|_{s,q}^s d_q\lambda\right] \\
&\overset{(18)(19)}{\leq} 2^r[r]_q\left[\frac{C_0^pC^{r-p}}{[r-p]_q}+\frac{C_1^sC^{s-r}}{[s-r]_q}\right]\|f\|_{r,q}^r.
\end{aligned}$$

We assume that $C_0^pC^{r-p}=C_1^sC^{s-r}$, we get that

$$C=C_0^{\frac{p}{s-p}}C_1^{\frac{s}{s-p}} \quad\Rightarrow\quad C_0^pC^{r-p}=C_0^pC_0^{\frac{p(r-p)}{s-p}}C_1^{\frac{s(r-p)}{s-p}}=C_0^{\frac{p(r-p)}{s-p}}C_1^{\frac{s(r-p)}{s-p}}.$$

Therefore, we have shown (17), where

$$C^r=2^r[r]_qC_0^{\frac{p(r-p)}{s-p}}C_1^{\frac{s(r-p)}{s-p}}\left\{\frac{1}{r-p}+\frac{1}{s-r}\right\}.$$

This completes the proof.

We say that $\mathcal{A}\lesssim\mathcal{B}$ if there exists a positive constant $c>0$, which depends only on certain parameters of the spaces involved, such that $\mathcal{A}\leq c\mathcal{B}$. Similarly, we write $\mathcal{A}\asymp\mathcal{B}$ to indicate that both inequalities $\mathcal{A}\lesssim\mathcal{B}$ and $\mathcal{A}\gtrsim\mathcal{B}$ are satisfied, possibly with different constants in each inequality. In other words, $\mathcal{A}$ and $\mathcal{B}$ are equivalent up to multiplicative constants depending only on the space parameters.

4 The q-deformated Hausdorff-Young-Paley Inequality

Now, we start to prove q-deformated Hausdorff-Young-Paley inequality and its inverse inequality.

Theorem 4.1 *Let $1\leq p\leq 2$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then for any $f\in L_q^p(\mathbb{R}_q)$ we have*

$$\|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)}\leq\|f\|_{L_q^p(\mathbb{R}_q)}. \tag{21}$$

Proof. Let A is a linear operator such that $A(f)(\xi)=\widehat{f}(\xi)$ for $f\in L_q^p(\mathbb{R}_q)$, $1\leq p\leq 2$. Then, by using the Hölder inequality (see [19, Proposition 37.2]), we have

$$\begin{aligned}
\|A(f)\|_{L_q^\infty(\mathbb{R}_q)} &= \|\widehat{f}\|_{L_q^\infty(\mathbb{R}_q)}=\sup_{\xi\in\mathbb{R}_q}|\widehat{f}(\xi)| \\
&\leq \sup_{\xi\in\mathbb{R}_q}\|e(-i\cdot\xi;q^2)\|_{L_q^\infty(\mathbb{R}_q)}\|f\|_{L_q^1(\mathbb{R}_q)}\leq\|f\|_{L_q^1(\mathbb{R}_q)},
\end{aligned}$$

where $\sup_{\xi\in\mathbb{R}_q} \|e(-i\cdot\xi;q^2)\|_{L_q^\infty(\mathbb{R}_q)} \leq 1$. Moreover, by Plancherel's identity (4), we have

$$\|A(f)\|_{L_q^2(\mathbb{R}_q)} = \|\widehat{f}\|_{L_q^2(\mathbb{R}_q)} \stackrel{(4)}{=} \|f\|_{L_q^2(\mathbb{R}_q)}, \quad f \in L_q^2(\mathbb{R}_q).$$

Therefore, we derive that $A : L_q^1(\mathbb{R}_q) \to L_q^\infty(\mathbb{R}_q)$ and $A : L_q^2(\mathbb{R}_q) \to L_q^2(\mathbb{R}_q)$, with the operator norms at most 1. In the case, $\theta = 2/p'$, then $0 \leq \theta \leq 1$. Moreover, we have $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$ and $\frac{1}{p'} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$. Hence, It is follows from the Theorem 3.2 that the inequality (21) holds.

We now derive the reverse form of inequality (21) in the range $2 \leq p \leq \infty$.

Theorem 4.2 *Suppose that* $2 \leq p \leq \infty$ *and* $\widehat{f} \in L_q^{p'}(\mathbb{R}_q)$. *Then*

$$\|f\|_{L_q^p(\mathbb{R}_q)} \leq \|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)}, \tag{22}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let $f \in L_q^p(\mathbb{R}_q)$. then, from duality of $L_q^p(\mathbb{R}_q)$ we find that

$$\|f\|_{L_q^p(\mathbb{R}_q)} = \sup\left\{|(f,\overline{\varphi})| : \varphi \in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} = 1\right\}.$$

and using the Plancherel identity (4), we get

$$(f,\overline{\varphi}) = \int_{\mathbb{R}_q} \widehat{x}(s)\overline{\widehat{y}(s)}d_qs, \quad x,y \in \mathcal{S}(\mathbb{R}_q).$$

Therefore,

$$\begin{aligned}
\|f\|_{L_q^p(\mathbb{R}_q)} &= \sup\{|(f,\overline{\varphi})| : \varphi \in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} = 1\} \\
&= \sup\left\{\left|\int_{\mathbb{R}_q} \widehat{f}(s)\overline{\widehat{\varphi}(s)}d_qs\right| : \varphi \in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} = 1\right\} \\
&\leq \sup\left\{\int_{\mathbb{R}_q} |\widehat{f}(s)\overline{\widehat{\varphi}(s)}|d_qs : \varphi \in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} = 1\right\} \\
&\leq \sup\left\{\int_{\mathbb{R}_q} |\widehat{f}(s)||\widehat{\varphi}(s)|d_qs : \varphi \in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} = 1\right\} \\
&\leq \sup_{\substack{\varphi\in L_q^{p'}(\mathbb{R}_q) \\ \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1}} \left\{\left(\int_{\mathbb{R}_q} |\widehat{f}(s)|^{p'}d_qs\right)^{1/p'} \cdot \left(\int_{\mathbb{R}_q} |\widehat{\varphi}(s)|^p d_qs\right)^{1/p}\right\} \\
&= \sup_{\substack{\varphi\in L^{p'}(\mathbb{R}_q) \\ \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1}} \left\{\|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)} \cdot \|\widehat{\varphi}\|_{L_q^p(\mathbb{R}_q)}\right\}.
\end{aligned}$$

Here we used the inequality $|\widehat{f}(\xi)\overline{\widehat{\varphi}(\xi)}| \leq |\widehat{f}(\xi)||\overline{\widehat{\varphi}(\xi)}|$ for any $\xi \in \mathbb{R}_q$, applying the Hölder inequality (see [19, Proposition 37.2]) with respect to Fourier transforms of f and $\varphi \in L_q^{p'}(\mathbb{R}_q)$

with $\|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} = 1$. by using inequality (21) with respect to φ, we write that

$$\|f\|_{L_q^p(\mathbb{R}_q)} \overset{(21)}{\leq} \sup_{\substack{\varphi\in L_q^{p'}(\mathbb{R}_q) \\ \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1}} \left\{ \|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)} \cdot \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} \right\} = \|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)},$$

thereby completing the proof.

Next, we establish the q-deformated Hausdorff-Young-Paley inequality.

Theorem 4.3 *Assume that $1 < p \leq 2$ and let $\varphi : \mathbb{R}_q \to \mathbb{R}_+$ be a strictly positive function satisfyingthe following condition*

$$M_\varphi := \sup_{t>0} t \int\limits_{\varphi(\xi)\geq t} d_q\xi < \infty. \tag{23}$$

Then, we have the following inequality

$$\left(\int\limits_{\mathbb{R}_q} |\widehat{f}(\xi)|^p \varphi^{2-p}(\xi) d_q\xi \right)^{\frac{1}{p}} \leq c_p M_\varphi^{\frac{2-p}{p}} \|f\|_{L_q^p(\mathbb{R}_q)} \quad for \quad f \in L_q^p(\mathbb{R}_q), \tag{24}$$

where $c_p > 0$ is a constant independent of f.

Proof. We assume that ν be a measure on $\mathbb{R}_q$ by $\nu(\xi) := \varphi^2(\xi) d_q\xi > 0$. Detone a space $L_q^p(\mathbb{R}_q, \nu)$ as follows

$$\|f\|_{L_q^p(\mathbb{R}_q,\nu)} := \left\{ f : \left(\int\limits_{\mathbb{R}_q} |f(\xi)|^p \varphi^2(\xi) d_q\xi \right)^{\frac{1}{p}} < \infty \right\}.$$

One can readily verify that, endowed with the above norm, this space is Banach. We then introduce the operator $A : L_q^p(\mathbb{R}_q) \to L_q^p(\mathbb{R}_q, \nu)$ via the formula

$$(Af)(\xi) = \frac{\widehat{f(\xi)}}{\varphi(\xi)}.$$

It follows from $\widehat{f+\varphi}(\xi) \overset{(3)}{=} \widehat{f}(\xi) + \widehat{\varphi}(\xi), \quad f, \varphi \in L_q^2(\mathbb{R}_q)$, that A is a sub-liner (or quasi-linear) operator. Now, we will prove that A $A : L_q^p(\mathbb{R}_q) \to L_q^p(\mathbb{R}_q, \nu)$ is well-defined and bounded with $1 \leq p \leq 2$. Equivalently, we claim that (24) is valid under condition (23). We first verify that A is of weak types $(2, 2)$ and $(1, 1)$. The distribution function $d_{A(f)}(t)$, $t > 0$, with respect to $\varphi^2(s) > 0$, is defined by

$$d_{A(f)}(t) := \nu\{s > 0 : |A(f)| > t\} = \int\limits_{|A(f)|>t} \varphi^2(\xi) d_q\xi.$$

The next step is to show that

$$d_{A(f)}(t) \leq \left(\frac{c_2 \|f\|_{L^2_q(\mathbb{R}_q)}}{t} \right)^2 \quad \text{with} \quad c_2 = 1, \tag{25}$$

and

$$d_{A(f)}(t) \leq \frac{c_1 \|f|_{L^1_q(\mathbb{R}_q)}}{t} \quad \text{with} \quad c_1 = 2M_\varphi. \tag{26}$$

To begin with, we prove inequality (25). Using the q-Chebyshev inequality (see Lemma 2.6 (b)) together with (3), we obtain

$$t d_{A(f)}(t) \leq \|A(f)\|^2_{L^2_q(\mathbb{R}_q,\nu)} = \int\limits_{\mathbb{R}_q} |\widehat{f}(s)|^2 d_q s = \|\widehat{f}\|^2_{L^2_q(\mathbb{R}_q)} \overset{(3)}{=} \|f\|^2_{L^2_q(\mathbb{R}_q)}.$$

Therefore, the operator A is of weak type $(2,2)$ with its norm bounded above by $c_2 = 1$. Next, we proceed to prove inequality (26). Using Hцlder's inequality (cf. [19, Proposition 37.2]) for the exponents $p = 1$ and $p' = \infty$, we obtain

$$\begin{aligned} \frac{|\widehat{f}(\xi)|}{\varphi(\xi)} &\overset{(3)}{\leq} K \frac{\left| \int_{\mathbb{R}_q} f(x) e(-ix\xi; q^2) d_q x \right|}{\varphi(\xi)} \\ &\leq \frac{\|e(-i \cdot \xi; q^2)\|_{L^\infty_q(\mathbb{R}_q)} \|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)} \leq \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)}, \quad \xi \in \mathbb{R}_q. \end{aligned}$$

Therefore, we have

$$\{\xi \in \mathbb{R}_q : \frac{|\widehat{f}(\xi)|}{\varphi(\xi)} > t\} \subset \{\xi \in \mathbb{R}_q : \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)} > t\}$$

for any $t > 0$. Consequently,

$$\nu\{\xi \in \mathbb{R}_q : \frac{|\widehat{f}(\xi)|}{\varphi(\xi)} > t\} \leq \nu\{\xi \in \mathbb{R}_q : \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)} > t\}$$

for any $t > 0$. Setting $v := \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{t}$, we obtain

$$\nu\{\xi \in \mathbb{R}_q : \frac{|\widehat{f}(\xi)|}{\varphi(\xi)} > t\} \leq \nu\{\xi \in \mathbb{R}_q : \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)} > t\} = \int\limits_{\varphi(\xi) \leq v} \varphi^2(\xi) d_q \xi. \tag{27}$$

Let us estimate the right hand side. Now we claim that

$$\int\limits_{\varphi(\xi) \leq v} \varphi^2(\xi) d_q \xi \leq (1 + q^{1/2}) v \cdot M_\varphi. \tag{28}$$

Indeed, from this equality $\varphi^2(\xi)=(1-q)\sum\limits_{q^i\le\varphi^2(\xi)} q^i$, and (9)-(10) first we have

$$\begin{aligned}
\int\limits_{\varphi(\xi)\le v}\varphi^2(\xi)d_q\xi &= (1-q)\int\limits_{\varphi(\xi)\le v}\sum_{q^i\le\varphi^2(\xi)} q^i d_q\xi = (1-q)^2\sum_{\varphi(q^k)\le v} q^k \sum_{q^{i/2}\le\varphi(q^k)} q^i\\
&= (1-q)^2\sum_{q^{i/2}\le v} q^i \sum_{q^{i/2}\le\varphi(q^k)\le v} q^k\\
&\le (1+q^{1/2})(1-q^{1/2})\sum_{q^{i/2}\le v} q^i(1-q)\sum_{q^{i/2}\le\varphi(q^k)} q^k\\
&\le (1+q^{1/2})(1-q^{1/2})\sum_{q^{i/2}\le v} q^{i/2}q^{i/2}\int\limits_{q^{i/2}\le\varphi(\xi)} d_q\xi\\
&= (1+q^{1/2})\int\limits_0^v\left(t\int\limits_{t\le\varphi(\xi)} d_q\xi\right)d_{q^{1/2}}t. \qquad (29)
\end{aligned}$$

Since

$$t\int\limits_{t\le\varphi(\xi)} d_q\xi \le \sup_{t>0} t\int\limits_{t\le\varphi(\xi)} d_q\xi = M_\varphi$$

and $M_\varphi<\infty$ by assumption and (29), it follows that

$$\int\limits_{\varphi(\xi)\le v}\varphi^2(\xi)d_q\xi \overset{(29)}{\le} (1+q^{1/2})M_\varphi\int\limits_0^v d_{q^{1/2}}t \le (1+q^{1/2})v\cdot M_\varphi.$$

This establishes the claim (28). By combining (27) and (28), we derive (26), which confirms that A is of weak type $(1,1)$ with operator norm at most $c_1=2M_\varphi$. Applying Theorem 3.2 with parameters $p_1=1$, $p_2=2$, and $\frac{1}{p}=\frac{1-\eta}{1}+\frac{\eta}{2}$, we consequently obtain inequality (24). This completes the proof.

From the q-deformated Paley-type inequality stated in Theorem 4.3, we derive the following q-deformated Hardy–Littlewood inequality.

Theorem 4.4 *Assume that $1<p\le 2$ and $\varphi:\mathbb{R}_q\to\mathbb{R}_+$ be a strictly positive function satisfying the following condition*

$$\int\limits_{\mathbb{R}_q}\frac{1}{\varphi^\beta(s)}d_qs<\infty \quad for \quad some \quad \beta>0. \qquad (30)$$

Then, we have q-deformated Hardy–Littlewood inequality as follows

$$\left(\int\limits_{\mathbb{R}_q}|\widehat{f}(s)|^p\varphi^{\beta(p-2)}(s)d_qs\right)^{\frac{1}{p}} \le C_p\|f\|_{L^p_q(\mathbb{R}_q)} \quad for \quad f\in L^p_q(\mathbb{R}_q),$$

where $C_p>0$ is a constant independent of x.

Proof. It follows from the assumption 30 that

$$C_q = (1-q)\sum_{k\in\mathbb{Z}} q^k\varphi^{-\beta}(q^k) = (1-q)\sum_{\varphi^\beta(q^k)\leq\frac{1}{t}} q^k\varphi^{-\beta}(q^k)$$

$$\geq (1-q)t\sum_{\varphi^\beta(q^k)\leq\frac{1}{t}} q^k = t\int\limits_{\varphi^\beta(s)\leq\frac{1}{t}} d_qs = t\int\limits_{t\leq\frac{1}{\varphi^\beta(s)}} d_qs, \quad t>0.$$

Therefore, taking the supremum over all positive t, we obtain the bound

$$\sup_{t>0} t\int_{\{s\in\mathbb{R}_q: t\leq\frac{1}{\varphi^\beta(s)}\}} d_qs \leq C_q < \infty.$$

This shows that the integral expression is uniformly controlled by the constant C_q. Then, by applying Theorem 4.3 to the function defined by

$$h(s) = \frac{1}{\varphi^\beta(s)}, \quad s\in\mathbb{R}_q.$$

we derive the desired inequality.

Theorem 4.5 *Suppose that* $2\leq p<\infty$ *with* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\varphi:\mathbb{R}_q\to\mathbb{R}_+$ *be a strictly positive function satisfying the following condition*

$$\int\limits_{\mathbb{R}^d}\frac{1}{\varphi^\beta(s)}d_qs<\infty \quad for \quad some \quad \beta>0.$$

If

$$\int\limits_{\mathbb{R}^d}|\widehat{f}(s)|^p\varphi^{\frac{\beta p(2-p')}{p'}}(s)d_qs<\infty,$$

then

$$\|f\|^p_{L^p_q(\mathbb{R}_q)} \leq C_{p,q}\int\limits_{\mathbb{R}_q}|\widehat{f}(s)|^p\varphi^{\frac{\beta p(2-p')}{p'}}(s)d_qs, \quad f\in L^p_q(\mathbb{R}_q),$$

where $C_{p,q}>0$ *is a constant independent of* x.

Proof. For $L_p(\mathbb{R}_q)$ we have

$$\|f\|_{L^p_q(\mathbb{R}_q)} = \sup\Big\{\big|\langle f,g\rangle_{L^2_q(\mathbb{R}_q)}\big| : g\in L^{p'}(\mathbb{R}_q), \quad \|g\|_{L^{p'}(\mathbb{R}_q)}=1\Big\}.$$

It follows from (4) that

$$\langle f,g\rangle_{L^2_q(\mathbb{R}_q)} = \int\limits_{\mathbb{R}^d}\widehat{f}(s)\widehat{g}(s)d_qs, \quad f,g\in L^2_q(\mathbb{R}_q). \tag{31}$$

Using the Hölder inequality for any function $g \in L_q^{p'}(\mathbb{R}_q)$ with $\|g\|_{L_q^{p'}(\mathbb{R}_q)} = 1$, we deduce that

$$\begin{aligned}
\|f\|_{L_q^p(\mathbb{R}_q)} &= \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \{|\langle f, g\rangle_{L_q^2(\mathbb{R}_q)}| : g \in L_q^{p'}(\mathbb{R}_q)\} \\
&= \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\Big|\int_{\mathbb{R}_q} \widehat{f}(s)\widehat{g}(s) d_q s\Big| : g \in L_q^{p'}(\mathbb{R}_q)\Big\} \\
&\le \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\int_{\mathbb{R}_q} |\widehat{f}(s)\widehat{g}(s)| d_q s : g \in L_q^{p'}(\mathbb{R}_q)\Big\} \\
&\le \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\int_{\mathbb{R}_q} |\widehat{f}(s)||\widehat{g}(s)| d_q s : g \in L_q^{p'}(\mathbb{R}_q)\Big\} \\
&\le \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\int_{\mathbb{R}_q} \varphi^{\frac{\beta(2-p')}{p'}}(s)|\widehat{f}(s)| \cdot \varphi^{\frac{\beta(p'-2)}{p'}}(s)|\widehat{g}(s)| : g \in L_q^{p'}(\mathbb{R}_q)\Big\} \\
&\le \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\Big(\int_{\mathbb{R}_q} \varphi^{\frac{\beta p(2-p')}{p'}}(s)|\widehat{f}(s)|^p d_q s\Big)^{1/p} \cdot \Big(\int_{\mathbb{R}_q} \varphi^{\beta(p'-2)}(s)|\widehat{g}(s)|^{p'} d_q s\Big)^{1/p'}\Big\}.
\end{aligned}$$

Now applying Theorem 4.4 with respect to p', we get

$$\begin{aligned}
\|f\|_{L_q^p(\mathbb{R}_q)} &\le \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\Big(\int_{\mathbb{R}_q} \varphi^{\frac{\beta p(2-p')}{p'}}(s)|\widehat{f}(s)|^p d_q s\Big)^{1/p} \cdot \Big(\int_{\mathbb{R}^d} \varphi^{\beta(p'-2)}(s)|\widehat{g}(s)|^{p'} d_q s\Big)^{1/p'}\Big\} \\
&\lesssim \Big(\int_{\mathbb{R}_q} \varphi^{\frac{\beta p(2-p')}{p'}}(s)|\widehat{f}(s)|^p d_q s\Big)^{1/p} \cdot \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \|g\|_{L_q^{p'}(\mathbb{R}_q)}.
\end{aligned}$$

Since $\|g\|_{L_q^{p'}(\mathbb{R}_q)} = 1$, taking $C_{p,q} = c_{p',q}$, we complete the proof.

Remark 4.6 *Suppose $p = 2$, then the inequalities stated in Theorems 4.1 and 4.2 both simplify to the identity given by* (4).

The following result can be inferred from [22, Corollary 5.5.2, p. 120].

Proposition 4.7 *Let $d_q\nu_1(\xi) = \omega_1(\xi)d_q\xi$, $d\nu_2(\xi) = \omega_2(\xi)d_q\xi$, $\xi \in \mathbb{R}_q$. Suppose that $1 \le p, r_0, r_1 < \infty$. If a continuous linear operator A admits bounded extensions $A : L_q^p(\mathbb{R}_q) \to L_q^{r_0}(\mathbb{R}_q, \nu_1)$ and $A : L_q^p(\mathbb{R}_q) \to L_q^{r_1}(\mathbb{R}_q, \nu_2)$, then there exists a bounded extension $A : L_q^p(\mathbb{R}_q) \to L_q^r(\mathbb{R}_q, \nu)$ where $0 < \theta < 1$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ and $d_q\nu(\xi) = \omega(\xi)d_q\xi$, $\omega = \omega_1^{\frac{r}{r_0}(1-\theta)} \cdot \omega_2^{\frac{r}{r_1}\theta}$.*

Now, we obtain the q-deformated Hausdorff-Young-Paley inequality.

Theorem 4.8 *Suppose that $1 < p \le r \le p' < \infty$ for $\frac{1}{p} + \frac{1}{p'} = 1$. Let φ is given as in Theorem 4.3. Then*

$$\Big(\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^r \varphi(\xi)^{r(\frac{1}{r}-\frac{1}{p'})} d_q\xi\Big)^{\frac{1}{r}} \le c_{p,r,p'} M_\varphi^{\frac{1}{r}-\frac{1}{p'}} \|f\|_{L_q^p(\mathbb{R}_\theta^d)}.$$

where $c_{q,p,r,p'} > 0$ is a constant independent of f.

Proof. Let $A(x) := \widehat{f}$ be a linear operator acting on the space $L^p_q(\mathbb{R}_q)$. By using the inequality stated in (24) for $1 < p \le 2$, we then deduce that

$$\Big(\int\limits_{\mathbb{R}_q} |\widehat{f}(\xi)|^p \varphi^{2-p}(\xi) d_q\xi\Big)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p_q(\mathbb{R}_q)}.$$

In other words, $A : L^p_q(\mathbb{R}_q) \to L^p_q(\mathbb{R}_q, \nu_1)$ is a bounded map, where the weight is given by ?1(?):=f2-p(?)>0 $\omega_1(\xi) := \varphi^{2-p}(\xi) > 0$ with $\xi \in \mathbb{R}_q$. moreover, for $1 \le p \le 2$ with $\frac{1}{p} + \frac{1}{p'} = 1$, by applying the inequality (21), we obtain that

$$\Big(\int\limits_{\mathbb{R}_q} |\widehat{f}(\xi)|^{p'} d_q\xi\Big)^{1/p'} = \|\widehat{f}\|_{L^{p'}_q(\mathbb{R}_q)} \le \|f\|_{L^p_q(\mathbb{R}_q)},$$

which implyis that $A : L^p_q(\mathbb{R}_q) \to L^{p'}_q(\mathbb{R}_q, \nu_2)$, where $\nu_2(\xi) := 1 d_q\xi$ for all $\xi \in \mathbb{R}_q$. It follows from Proposition 4.7 that $A : L^p_q(\mathbb{R}_q) \to L^r_q(\mathbb{R}_q, \nu)$ with $d_q\nu = \omega(\xi) d_q\xi$, is bounded for any η such that $p \le \eta \le p'$, where the space $L^\eta_q(\mathbb{R}_q, \nu)$ is defined as

$$\|f\|_{L^\eta_q(\mathbb{R}_q,\nu)} := \left\{ f : \mathbb{R}_q \to \mathbb{R} : \left(\int\limits_{\mathbb{R}_q} |f(\xi)|^\eta \omega(\xi) d_q\xi\right)^{\frac{1}{\eta}} < \infty \right\},$$

where $\omega : \mathbb{R}_q \to \mathbb{R}$ is a positive function and will be defined later. Let us find the explicit form of ω. For fix $\theta \in (0,1)$ such that $\frac{1}{\eta} = \frac{1-\theta}{p} + \frac{\theta}{p'}$, we derive $\theta = \frac{p-\eta}{\eta(p-2)}$ and from Proposition 4.7 with respect to $r = \eta$, $r_0 = p$, and $r_1 = p'$, we have

$$\omega(\xi) = (\omega_1(\xi))^{\frac{\eta(1-\eta)}{r}} \cdot (\omega_2(\xi))^{\frac{\eta\eta}{r'}} = (\varphi^{2-r}(\xi))^{\frac{\eta(1-\eta)}{r}} \cdot 1^{\frac{\eta\eta}{r'}} = \varphi^{1-\frac{\eta}{r'}}(\xi) = \varphi^{\eta\left(\frac{1}{\eta} - \frac{1}{r'}\right)}(\xi)$$

for all $\xi \in \mathbb{R}_q$ and $\frac{2-r}{r} \cdot (1-\eta) = \frac{1}{\eta} - \frac{1}{r'}$. Hence, for $d_q\nu = \varphi^{\eta\left(\frac{1}{\eta} - \frac{1}{p'}\right)}(\xi) d_q\xi$ we obtain

$$\|A(x)\|_{L^\eta_q(\mathbb{R}_q,\nu)} \lesssim (M_\varphi^{\frac{2-r}{r}})^{1-\eta} \|x\|_{L^r_q(\mathbb{R}_q)} = M_\varphi^{\frac{1}{\eta} - \frac{1}{p'}} \|x\|_{L^p_q(\mathbb{R}_q)}, \quad x \in L^p_q(\mathbb{R}_q).$$

This completes the proof.

5 the q-deformated Hörmander multiplier theorem

First, we obtain the q^2-Fourier transform of the Fourier multiplier (7).

Lemma 5.1 *Let $g : \mathbb{R}_q \to \mathbb{C}$ be a bounded function. Then, we have*

$$\widehat{A_g(f)} = g \cdot \widehat{f}, \quad f \in L^p_q(\mathbb{R}_q). \tag{32}$$

for $f \in L^p_q(\mathbb{R}_q)$.

Proof. Let $f \in L^p_q(\mathbb{R}_q)$. Then, by (3), (5)-(6) and (7) we have

$$\begin{aligned}
\widehat{(A_g f)}(y) &\overset{(3)(7)}{=} K^2 \int\limits_{\mathbb{R}_q} \Big[\int\limits_{\mathbb{R}_q} g(\xi)\widehat{f}(\xi) e(ix\xi; q^2) d_q\xi \Big] e(-ixy; q^2) d_q x \\
&= \int\limits_{\mathbb{R}_q} g(\xi)\widehat{f}(\xi) \Big[K^2 \int\limits_{\mathbb{R}_q} e(ix\xi; q^2) e(-ixy; q^2) d_q x \Big] d_q\xi \\
&\overset{(5)}{=} \int\limits_{\mathbb{R}_q} g(\xi)\widehat{f}(\xi)\delta_y(\xi) d_q\xi \\
&\overset{(6)}{=} g(y)\widehat{f}(y),
\end{aligned}$$

for all $y \in L^p_q(\mathbb{R}_q)$.

Let us denote by $\overline{g}$ the complex conjugate of the function g, in Definition 2.3.

Lemma 5.2 *Suppose that $1 < p, q < \infty$. Let $A_g : L^p_q(\mathbb{R}^d_\theta) \to L^q_q(\mathbb{R}^d_\theta)$ be the Fourier multiplier defined by (7) with the symbol g. Then its adjoint $A^*_g = A_{\overline{g}}$ and $A_{\overline{g}} : L^{q'}_q(\mathbb{R}^d_\theta) \to L^{p'}_q(\mathbb{A}^d_\theta)$.*

Proof. For $h, f \in L^p_q(\mathbb{R}_q)$. Then, It follows frmm (8), (31), and (32) that

$$\begin{aligned}
(A_g f, h) &\overset{(31)}{=} \int\limits_{\mathbb{R}_q} \widehat{A_g f}(s)\widehat{h}(s) d_q s \\
&\overset{(32)}{=} \int\limits_{\mathbb{R}_q} g(s)\widehat{f}(s)\widehat{h}(s) d_q s = \int\limits_{\mathbb{R}_q} \widehat{f}(s) g(s)\widehat{h}(s) d_q s \\
&\overset{(32)}{=} \int\limits_{\mathbb{R}_q} \widehat{f}(s)\widehat{A_g(h)}(s) d_q s \overset{(31)}{=} (f, A_g h).
\end{aligned}$$

Since $L^\eta_q(\mathbb{R}_q)$ is dense in $L^p_q(\mathbb{R}_q)$, we have $A^*_g = A_g$.

Finally, we state the q-deformated Hörmander multiplier theorem.

Theorem 5.3 *Suppose that $1 < p \le 2 \le \eta < \infty$ and $g : \mathbb{R}_q \to \mathbb{R}$ be a bounded function. Then, the Fourier multiplier defined in (7) can be extended to act as a bounded linear operator from the space $L^p_q(\mathbb{R}_q)$ to the space $L^\eta_q(\mathbb{R}_q)$. Moreover, the following estimate holds*

$$\|A_g\|_{L^p_q(\mathbb{R}_q) \to L^\eta_q(\mathbb{R}_q)} \lesssim \sup_{\lambda > 0} \lambda \Big(\int\limits_{|g(s)| \ge \lambda} d_q s \Big)^{\frac{1}{p} - \frac{1}{q}}.$$

Proof. By duality it is sufficient to study two cases: $1 < p \le \eta' \le 2$ and $1 < \eta' \le p \le 2$, where $1 = \frac{1}{\eta} + \frac{1}{\eta'}$.

First, we consider the case $1 < p \le \eta' \le 2$, where $1 = \frac{1}{\eta} + \frac{1}{\eta'}$. By (32) we have

$$\widehat{A_g f} = g \cdot \widehat{f}, \quad f \in L^p_q(\mathbb{R}_q). \tag{33}$$

Then, it follows from it follows from Proposition 4.2 and (33) that

$$\|A_g f\|_{L^\eta_q(\mathbb{R}_q)} \overset{(22)}{\leq} \|\widehat{A_g f}\|_{L^{\eta'}_q(\mathbb{R}_q)} \overset{(33)}{=} \|g\widehat{f}\|_{L^{\eta'}_q(\mathbb{R}_q)}, \tag{34}$$

for all $f \in L^\eta_q(\mathbb{R}_q)$.

Thus, we denote $\eta' := r$ and $\frac{1}{s} := \frac{1}{p} - \frac{1}{\eta} = \frac{1}{\eta'} - \frac{1}{p'}$, then for $h(\xi) := |g(\xi)|^s, \xi \in \mathbb{R}_q$, then, by using the inequality in Theorem 4.8. In other words, we derive

$$\Big(\int\limits_{\mathbb{R}_q} \Big(|\widehat{f}(\xi)| \cdot |g(\xi)|\Big)^{\eta'} d_q\xi\Big)^{\frac{1}{\eta'}} \lesssim M^{\frac{1}{s}}_{|g|^s} \|x\|_{L^p_q(\mathbb{R}_q)} \tag{35}$$

for any $f \in L^p_q(\mathbb{R}_q)$. Let us study $M^{\frac{1}{s}}_{|g|^s}$ separately. Indeed, by definition

$$M^{\frac{1}{s}}_{|g|^s} := \Big(\sup_{\lambda>0} \lambda \int\limits_{|g(\xi)|^s \geq \lambda} d_q\xi\Big)^{\frac{1}{s}} = \Big(\sup_{\lambda>0} \lambda \int\limits_{|g(\xi)| \geq \lambda^{\frac{1}{s}}} d_q\xi\Big)^{\frac{1}{s}} = \Big(\sup_{\lambda>0} \lambda^s \int\limits_{|g(\xi)| \geq \lambda} d_q\xi\Big)^{\frac{1}{s}}.$$

Since $\frac{1}{s} := \frac{1}{p} - \frac{1}{\eta}$, it follows that

$$\begin{aligned} M^{\frac{1}{s}}_{|g|^s} &= \Big(\sup_{\lambda>0} \lambda^s \int\limits_{|g(\xi)| \geq \lambda} d_q\xi\Big)^{\frac{1}{p}-\frac{1}{\eta}} = \sup_{\lambda>0} \lambda^{s\left(\frac{1}{p}-\frac{1}{\eta}\right)} \Big(\int\limits_{|g(\xi)| \geq \lambda} d_q\xi\Big)^{\frac{1}{p}-\frac{1}{\eta}} \\ &= \sup_{\lambda>0} \lambda \Big(\int\limits_{|g(\xi)| \geq \lambda} d_q\xi\Big)^{\frac{1}{p}-\frac{1}{\eta}}. \end{aligned} \tag{36}$$

Hence, combining (34), (35), and (36) we obtain

$$\begin{aligned} \|A_g f\|_{L^\eta_q(\mathbb{R}^d_\theta)} &\overset{(34)}{\lesssim} \Big(\int\limits_{\mathbb{R}_q} \Big(|\widehat{x}(\xi)| \cdot |g(\xi)|\Big)^{\eta'} d_q\xi\Big)^{\frac{1}{\eta'}} \\ &\overset{(35)}{\lesssim} M^{\frac{1}{s}}_{|g|^s} \|x\|_{L^p_q(\mathbb{R}_q)} \overset{(36)}{=} \sup_{\lambda>0} \lambda \Big(\int\limits_{|g(\xi)| \geq \lambda} d_q\xi\Big)^{\frac{1}{p}-\frac{1}{\eta}} \|x\|_{L^\eta_q(\mathbb{R}_q)}, \end{aligned}$$

for $1 < p \leq \eta' \leq 2$ and $x \in L^p_q(\mathbb{R}^d_\theta)$.

Next, we consider the case $\eta' \leq p \leq 2$ so that $p' \leq (\eta')' = \eta$, where $1 = \frac{1}{\eta} + \frac{1}{\eta'}$ and $1 = \frac{1}{p} + \frac{1}{p'}$. Thus, the L^p_q-duality (see Lemma 5.2) yields that $A^*_g = A_g$ and

$$\|A_g\|_{L^p_q(\mathbb{R}_q) \to L^\eta_q(\mathbb{R}_q)} = \|A_g\|_{L^{\eta'}_q(\mathbb{R}_q) \to L^{p'}_q(\mathbb{R}_q)}.$$

Set $\frac{1}{p} - \frac{1}{\eta} = \frac{1}{s} = \frac{1}{\eta'} - \frac{1}{p'}$. Hence, by repeating the argument in the previous case we have

$$\begin{aligned} \|A_g(x)\|_{L^{p'}_q(\mathbb{R}_q)} &\lesssim \sup_{\lambda>0} \lambda \left(\int\limits_{\overline{|g(\xi)|} \geq \lambda} d_q\xi\right)^{\frac{1}{q'}-\frac{1}{\eta'}} \|x\|_{L^{\eta'}_q(\mathbb{R}_q)} \\ &= \sup_{\lambda>0} \lambda \left(\int\limits_{|g(\xi)| \geq \lambda} d_q\xi\right)^{\frac{1}{p}-\frac{1}{\eta}} \|x\|_{L^{\eta'}_q(\mathbb{R}_q)}. \end{aligned}$$

In other words, we have

$$\|A_g\|_{L_q^{\eta'}(\mathbb{R}_q)\to L_q^{p'}(\mathbb{R}_q)} \lesssim \sup_{\lambda>0} \lambda \left(\int\limits_{|g(\xi)|\geq\lambda} d_q\xi \right)^{\frac{1}{p}-\frac{1}{\eta}}.$$

Combining both cases, we obtain

$$\|A_g\|_{L_q^{p}(\mathbb{R}_q)\to L_q^{\eta}(\mathbb{R}_q)} \lesssim \sup_{\lambda>0} \lambda \left(\int\limits_{|g(\xi)|\geq\lambda} d_q\xi \right)^{\frac{1}{p}-\frac{1}{\eta}}$$

for all $1 < p \leq 2 \leq \eta < \infty$. This concludes the proof.

6 Acknowledgements

The work was supported by the grant from the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19579124).

References

[1] Annaby M.H., Mansour Z.S. *q-Fractional Calculus and Equations.* Springer, Heidelberg, 2012.

[2] Agarwal R.P. Certain fractional q-integrals and q-derivatives. *Proc. Camb. Philos. Soc.* 66. P.365-370. 1969.

[3] Al-Salam W.A. Some fractional q-integrals and q-derivatives. *Proc. Edinb. Math. Soc.* 15. P.135-140. 1966/67

[4] Bettaibi N., Bettaieb R.H. q-Analogue of the Dunkl transform on the real line. *Tamsui Oxf. J. Math. Sci.* 25:2. P.177-205. 2009.

[5] Carmichael R.D. The general theory of linear q-difference equations. *Amer. J. Math.* 34:2 P.147-168. 1912.

[6] Jackson F.H. On q-functions and a certain difference operator. *Trans. R. Soc. Edinb.* 46. P.253-281. 1908.

[7] Jackson F.H. On a q-definite integral. *Quart. J. Pure Appl. Math.* 41. P.193-203. 1910.

[8] Cheung P., Kac V. *Quantum Calculus.* Springer, New York, 2002.

[9] Euler L. *Introductio in analysin infinitorum.* Lausanne, Ch. VII. 1748.

[10] Ernst T. *A New Method of q-Calculus.* Ph.D. thesis, Uppsala Univ., 2002.

[11] Ernst T. *A Comprehensive Treatment of q-Calculus.* (Birkhauser, Basel, 2012).

[12] Exton H. *q-Hypergeometric Functions and Applications.* Ellis Horwood, Chichester, 1983.

[13] Fitouhi A., Bouzeffour F. q-cosine Fourier transform and q-heat equation. *Ramanujan J.* 28:3. P.443-461. 2012.

[14] Fitouhi A., Saoudi A. On q^2-analogue Sobolev-type spaces. *Le Matematiche* 70:2. P.63-77. 2015.

[15] Gasper G., Rahman M. *Basic Hypergeometric Series.* Cambridge Univ. Press, 1990.

[16] Koornwinder T.H., Swarttouw R.F. On q-analogues of Fourier and Hankel transforms. *Trans. Amer. Math. Soc.* 333:1. P. 445-461. 1992.

[17] Rubin R.L. Duhamel solutions of non-homogeneous q^2-analogue wave equations. *Proc. Amer. Math. Soc.* 135:3. P.777-785. 2007.

[18] Rubin R.L. A q^2-analogue operator for q^2-analogue Fourier analysis. *J. Math. Anal. Appl.* 212:2. P.571-582. 1997.

[19] Anastassiou G.A. *Intelligent Mathematics: Computational Analysis.* Springer, Berlin, 2011.

[20] Baiarystanov A.O., Persson L.E., Shaimardan S., Temirkhanova A. Some new Hardy-type inequalities in q-analysis. *J. Math. Inequal.* 10. P.761-781. 2016.

[21] Persson L.E., Shaimardan S. Some new Hardy-type inequalities for the Riemann–Liouville fractional q-integral operator. *J. Inequal. Appl.* 2015.

[22] Bergh J., Lofstrom J. *Interpolation Spaces: An Introduction.* Springer, Berlin, 1976.

[23] Hormander L. Estimates for translation-invariant operators in L^p spaces. *Acta Math.* 104. P.93-140. 1960.

[24] Grafakos L. *Classical Fourier Analysis.* 2nd ed. Springer, New York, 2014.

[25] Shaimardan S., Ruzhansky M. Direct and inverse source problems for heat equation in quantum calculus. *Filomat* 38:23. P.8141-8156. 2024.

[26] Shaimardan S., Karimov E., Ruzhansky M., Mamanazarov A. The Prabhakar fractional q-integral and q-differential operators, and their properties. *Filomat* 39:6. P.2003-2016. 2025.

[27] Shaimardan S., Persson L.E., Tokmagambetov N.S. On the heat and wave equations with the Sturm–Liouville operator in quantum calculus. *Abstr. Appl. Anal.* 2023. 2488165.

[28] Shaimardan S., Tokmagambetov N.S. q-deformed Hilbert transform and its related properties and inequalities. *Math. Inequal. Appl.* 19:3 (2016): 493-518.

[29] Shaimardan S., Tokmagambetov N.S. The Schrodinger equations generated by q-Bessel operator in quantum calculus. *Bull. Karaganda Univ. Math.* 105:1. P.102-108. 2022.

[30] Shaimardan S., Tokmagambetov N.S., Temirkhanova A.M. The Cauchy problems for q-difference equations with the Caputo fractional derivatives. *J. Math. Mech. Comput. Sci.* 113:1. P.43-57. 2022.

[31] Shaimardan S., Persson L.E., Tokmagambetov N.S. Well-posedness of heat and wave equations generated by Rubin's q-difference operator in Sobolev spaces. *Filomat* 37:17. P.5799-5812. 2023.

[32] Stein E.M. *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press, 1970.

[33] Ruzhansky M., Shaimardan S., Tulenov K. Fourier multipliers and their applications to PDE on the quantum Euclidean space. *Nonlinear Differ. Equ. Appl.* 32. 2025. 85.

Авторлар туралы мәлімет:

Токмагамбетов Нариман Сарсенұлы – PhD, ғылым департаментінің ғылыми қызметкері, академик Е.А. Бөкетов атындағы Қарағанды университеті (Қарағанды, Қазақстан, электронная пошта: nariman.tokmagambetov@gmail.com).

Информация об авторах:

Токмагамбетов Нариман Сарсенович – PhD, научный сотрудник департамента науки Карагандинского университета имени академика Е.А. Букетова(Караганда, Қазақстан, электронная почта: nariman.tokmagambetov@gmail.com).

Information about authors: *Tokmagambetov Nariman Sarsenovich – PhD, research fellow at the science department of academician E.A. Buketov Karaganda university(Karaganda, Kazakhstan, email: nariman.tokmagambetov@gmail.com).*

Received: August 11, 2025
Accepted: September 27, 2025