IRSTI 27.39.19 DOI: https://doi.org/10.26577/JMMCS202512842

N. Bokayev[1], **A. Gogatishvili**[2], **N. Kuzeubayeva**[1*]
[1] L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
[2] Institute of Mathematical of the Czech Academy of Sciences, Praha, Czech Republic
*e-mail: nurgul.kuzeubaeva@mail.ru

REDUCTION THEOREMS FOR DISCRETE HARDY OPERATOR ON MONOTONE SEQUENCE CONES ($0 < p < 1$)

In this work, we investigate the discrete Hardy and Copson operators acting on the cone of nonnegative monotone sequences. It is established that the weighted inequalities of type $l_p \to l_q$ for these operators, in the case $0 < q < \infty$, $0 < p < 1$, can be reduced to the corresponding inequalities defined on the cone of general nonnegative sequences. The latter possess a broader basis for proof, which significantly extends the possibilities for their analysis. Weighted inequalities for the integral Hardy operator (in the continuous setting) on the cone of nonnegative nonincreasing functions have been studied previously by many authors. Reduction theorems for inequalities involving Hardy-type integral operators on the cone of nonincreasing functions to inequalities on the cone of nonnegative functions are well established. In this paper, we provide several theorems that demonstrate the equivalence between inequalities for discrete Hardy and Copson operators on the cone of nonnegative nonincreasing sequences and the corresponding inequalities on the cone of nonnegative sequences. Our proofs differ substantially from those in the continuous case. Methods applicable in the continuous setting do not always work in the discrete setting. For the case $p > 1$, analogous results were obtained by the authors earlier.
Key words: reduction theorems, discrete Hardy operator, monotone sequences, weighted inequalities, Copson operator.

Н. Бокаев [1], А. Гогатишвили[2], Н. Кузеубаева[1*]
[1]Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Астана, Қазақстан
[2] Чехия Ғылым академиясының Математика институты, Прага, Чехия
*e-mail: nurgul.kuzeubaeva@mail.ru

Монотонды тізбектер конусындағы дискретті Харди операторы үшін редукциялық теоремалар ($0 < p < 1$)

Бұл жұмыста біз теріс емес монотонды тізбектер конусындағы дискретті Харди және Копсон операторларын қарастырамыз. $0 < q < \infty$, $0 < p < 1$ жағдайындағы монотонды тізбектер конусындағы дискретті Харди және Копсон операторлары үшін $l_p \to l_q$ түріндегі салмақты теңсіздіктерді теріс емес тізбектер конусындағы сәйкес теңсіздіктерге келтіруге болатыны көрсетілген. Соңғыларың дәлелдеу негізі кеңірек, бұл оларды талдау мүмкіндіктерін айтарлықтай кеңейтеді. Теріс емес өспейтін функциялар конусындағы интегралдық Харди операторы үшін (үзіліссіз жағдайда) салмақты теңсіздіктерді бұрын көптеген авторлар зерттеген. Бұл мақалада теріс емес өспейтін тізбектер конусындағы дискретті Харди және Копсон операторлары үшін теңсіздіктердің және теріс емес тізбектер конусындағы теңсіздіктерге эквиваленттілігіне қатысты әртүрлі теоремалар ұсынылған. ұсынылған дәлелдеулер үздіксіз жағдайдағылардан айтарлықтай ерекшеленеді. Үздіксіз параметрде қолданылатын әдістер дискретті параметрде әрқашан жұмыс істей бермейді. $p > 1$ жағдайына ұқсас нәтижелерді біз бұрын алғанбыз.
Түйін сөздер: редукция теоремалары, дискретті Харди операторы, монотонды тізбектер, салмақталған теңсіздіктер, Копсон операторы.

Н. Бокаев[1], А. Гогатишвили[2], Н. Кузеубаева[1*]

[1] Евразийский национальный университет им. Л.Н. Гумилёва, Астана, Казахстан
[2] Институт математики Чешской академии наук, Прага, Чешская Республика
*e-mail: nurgul.kuzeubaeva@mail.ru

Редукционные теоремы для дискретного оператора Харди на конусах монотонных последовательностей ($0 < p < 1$)

В данной статье мы рассматриваем дискретные операторы Харди и Копсона на конусе неотрицательных монотонных последовательностей. Показано, что весовые неравенства вида $l_p \to l_q$ для дискретных операторов Харди и Копсона на конусе монотонных последовательностей в случае $0 < q < \infty$, $0 < p < 1$ могут быть сведены к соответствующим неравенствам на конусе неотрицательных последовательностей. Последние обладают более широкой основой для доказательства, что существенно расширяет возможности их анализа. Весовые неравенства для интегрального оператора Харди (в непрерывном случае) на конусе неотрицательных невозрастающих функций ранее исследовались многими авторами. Хорошо известны также теоремы сведения неравенств для интегральных операторов типа Харди на конусе невозрастающих функций к неравенствам на конусе неотрицательных функций. В работе приводятся различные теоремы, касающиеся эквивалентности неравенств для дискретных операторов Харди и Копсона на конусе неотрицательных невозрастающих последовательностей и неравенств на конусе неотрицательных последовательностей. Представленные доказательства существенно отличаются от доказательств в непрерывном случае. Методы, применимые в непрерывной постановке, не всегда работают в дискретной. Для случая $p > 1$ аналогичные результаты были ранее получены нами.

Ключевые слова: редукционные теоремы, дискретный оператор Харди, монотонные последовательности, весовые неравенства, оператор Копсона.

1 Introduction

In this paper, we study weighted Hardy and Copson inequalities, focusing on their behavior on cones of nonnegative and monotone sequences. We establish conditions under which these inequalities hold and explore their relation to corresponding results in the continuous setting.

$$\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}x(k)\right)^{q}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}x(m)^{p}b(m)\right)^{\frac{1}{p}}, \tag{1}$$

$$\left(\sum_{m=1}^{\infty}\left(\sum_{k=m}^{\infty}x(k)\right)^{q}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}x(m)^{p}b(m)\right)^{\frac{1}{p}} \tag{2}$$

for non-negative, non-increasing sequences $x = \{x(m)\}$. Here $\{a(m)\}$ and $\{b(m)\}$ are given non-negative weight sequences, $p \in (0,1)$ and $q \in (0,\infty)$ are fixed parameters and the constant $C > 0$ is independent of x. The case $1 < p < \infty$ and $0 < q < \infty$ was considered in the recent paper [1], for discrete cases, Sawyer's duality theorem was obtained by R.Oinarov and S.Kh.Shalginbaeva [2], and using an effective method based on Sawyer's duality principle, which reduces inequalities (1) for non-negative, non-increasing sequences to modified inequalities for non-negative sequences, they give a characterization of the inequality of (1), in the cases $1 < p, q < \infty$, Sawyer's duality theorem cannot possible to use in the case when $q \in (0,1)$. The corresponding problem for unrestricted, non-negative sequences $\{x\}$ was solved by the authors in [3], [4] and ([5], Theorem 9.2). In the continuous setting, this problem has been extensively studied over the last twenty years (see [6], [7], [8], [11], and the references therein). Under the conditions $1 \le q < \infty$, and $0 < p < \infty$, an effective approach is provided by Sawyer's duality principle [12], which reduces inequalities (1) for nonnegative non increasing sequences to modified inequalities for the same class of sequences.

Our main results are the discrete analogues of the theorems of A.Gogatishvili and V.D.Stepanov [6].

The structure of the paper is as follows. In the next section, we formulate the main results of the study. Section 3 is devoted to preliminary results, while the final section contains the proofs of the principal statements concerning supremum operators with kernels. Throughout the paper, we shall adhere to the following notation. The set of all natural numbers will be denoted by $\mathbb{N}$. We write $A \lesssim B$ if there exists a positive constant C such that $A \le CB$. Furthermore, the notation $A \approx B$ ill indicate that both $A \lesssim B$ and $B \lesssim A$ hold simultaneously.

2 Main Results

We assume that $\{a(m)\}$ and $\{b(m)\}$ are sequences of non-negative terms throughout. Their partial sums are denoted by

$$A(m) = \sum_{k=1}^{m} a(k), \quad B(m) = \sum_{k=1}^{m} b(k), \quad m \in \mathbb{N}.$$

Theorem 1 *Let $0 < q \le \infty$, $0 < p < 1$. Suppose that $\{a(m)\}$ and $\{b(m)\}$ are given non-negative weight sequences.Then, the following six conditions are equivalent.*

(i) *Inequality (1) holds for every non-negative, non-increasing sequence $\{x(m)\}$.*

(ii) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\left(\sum_{k=i}^{\infty} y(k)\right)^{\frac{1}{p}}\right)^{q} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{3}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iii) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m} i^{p-1}\left(\sum_{k=i}^{\infty} y(k)\right)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{4}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iv) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{1\le i\le m} i^{p}\left(\sum_{i=k}^{\infty} y(k)\right)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{5}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(v) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\left(\sup_{i\le k<\infty} y(k)\right)^{\frac{1}{p}}\right)^{q} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{6}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(vi) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{1\le i\le m} i^{p}\left(\sup_{i\le k\le\infty} y(k)\right)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{7}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms.

Theorem 2 *Let $0<q\le\infty$, $0<p<1$. Suppose that $\{a(m)\}$ and $\{b(m)\}$ are given non-negative weight sequences. Then the following six conditions are equivalent:*

(i) *Inequality (1) holds for every non-negative, non-increasing sequence $\{x(m)\}$*

(ii) *For any $\alpha>0$, the following inequality is satisfied:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i} B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{q} a(i)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{8}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iii) *For any $\alpha>0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sup_{1\le k<i} B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{q} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{9}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iv) *For any $\alpha>0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p} B(i)^{\alpha+1}y(i)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{10}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(v) *For any $\alpha>0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{1\le i\le m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sum_{k=1}^{i} B(k)^{\alpha+1}y(k)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{11}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(vi) *For any $\alpha>0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{1\le i\le m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\le k\le i} B(k)^{\alpha+1}y(k)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}}, \tag{12}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms.

Theorem 3 *Let $0 < q \le \infty$, $0 < p < 1$. Suppose that $\{a(m)\}$ and $\{b(m)\}$ are prescribed non-negative weight sequences. Then the six conditions listed below are mutually equivalent:*
(i) *Inequality (2) is satisfied for every non-negative, non-increasing sequence$\{x(m)\}$.*

(ii) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{k=m}^{\infty}\left(\sum_{i=k}^{\infty}y(i)\right)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{13}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iii) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}(i-m)^{p}y(i)\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{14}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iv) *The following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{m\le i\le\infty}(i-m)^{p}\sum_{k=i}^{\infty}y(k)\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{15}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(v) *The following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{k=m}^{\infty}\sup_{k\le i\le\infty}y(i)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{16}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(vi) *The following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{m\le i\le\infty}(i-m)^{p}\sup_{i\le k\le\infty}y(k)\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{17}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms.

Theorem 4 *Let $0 < q \le \infty$, $0 < p < 1$. Suppose that $\{a(m)\}$ and $\{b(m)\}$ are given non-negative weight sequences. Then the following six conditions are equivalent:*

(i) *Inequality (2) holds for every non-negative, non-increasing sequence$\{x(m)\}$.*

(ii) *For any $\alpha > 0$, the inequality below is satisfied:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}B(i)^{-\frac{\alpha+1}{p}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{18}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iii) *For any $\alpha > 0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}B(i)^{-\frac{\alpha+1}{p}}\left(\sup_{1\le k<i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \quad (19)$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iv) *For any $\alpha > 0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}\left(\sum_{k=i}^{\infty}B(k)^{-\frac{\alpha+1}{p}}\right)^{p-1}B(i)^{-\frac{\alpha+1}{p}}\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{\alpha}{p}}a(m)\right) \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \quad (20)$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(v) *For any $\alpha > 0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{m\le i<\infty}B(i)^{-\frac{\alpha+1}{p}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \quad (21)$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(vi) *For any $\alpha > 0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{m\le i\le\infty}\left(\sum_{k=i}^{\infty}B(k)^{\frac{\alpha+1}{p}}\right)^{p}\left(\sup_{1\le k\le i}B(k)^{\alpha+1}y(k)\right)\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \quad (22)$$

for every sequence $\{y(m)\}$ consisting of non-negative terms.

3 Preliminaries

Lemma 1 *(Fubini, see [4])*

$$\sum_{m=1}^{\infty}a(m)\sum_{k=1}^{m}b(k)<\infty \quad \text{if and only if} \quad \sum_{m=1}^{\infty}b(m)\sum_{k=m}^{\infty}a(k)<\infty. \quad (23)$$

Moreover,

$$\sum_{m=1}^{\infty}a(m)\sum_{k=1}^{m}b(k)=\sum_{m=1}^{\infty}b(m)\sum_{k=m}^{\infty}a(k). \quad (24)$$

Lemma 2 *(Power Rule, see [15]) If $0<p<\infty$, then*

$$\min(p,1)\sum_{k=1}^{m} b(k)\left(\sum_{j=1}^{k} b(j)\right)^{p-1} \le \left(\sum_{k=1}^{m} b(k)\right)^{p} \le \max(p,1)\sum_{k=1}^{m} b(k)\left(\sum_{j=1}^{k} b(j)\right)^{p-1}. \quad (25)$$

Lemma 3 *(Power Rule for Tails, see [15]) If $0<p<\infty$, then*

$$\min(p,1)\sum_{k=1}^{m} b(k)\left(\sum_{j=k}^{m} b(j)\right)^{p-1} \le \left(\sum_{k=1}^{m} b(k)\right)^{p} \le \max(p,1)\sum_{k=1}^{m} b(k)\left(\sum_{j=k}^{m} b(j)\right)^{p-1}. \quad (26)$$

Lemma 4 *(Generalized Partial Sums lemma, see [13]) Let $N\in\mathbb{N}$ and $0<p\le 1$.*

(a) *If*

$$\left(\sum_{k=1}^{m} a(k)\right)^{p} \lesssim \sum_{k=1}^{m} b(k) \quad (m=1,2,\cdots,N)$$

then

$$\left(\sum_{k=1}^{N} a(k)x(k)\right)^{p} \lesssim \sup_{1\le m\le N}\frac{(\sum_{k=1}^{m} a(k))^{p}}{\sum_{k=1}^{m} b(k)}\sum_{k=1}^{N} b(k)x(k)^{p} \quad (m=1,2,\cdots,N)$$

and all non-negative non -ieasing sequences $(x(1),x(2),\cdots,x(N))$.

(b) *If*

$$\left(\sum_{k=m}^{N} a(k)\right)^{p} \lesssim \sum_{k=m}^{N} b(k) \quad (m=1,2,\cdots,N)$$

then

$$\left(\sum_{k=1}^{N} a(k)x(k)\right)^{p} \lesssim \sup_{1\le m\le N}\frac{\left(\sum_{k=m}^{N} a(k)\right)^{p}}{\sum_{k=m}^{N} b(k)}\sum_{k=1}^{N} b(k)x(k)^{p} \quad (m=1,2,\cdots,N)$$

and for all non-negative, non-increasing sequences $(x(1),x(2),\cdots,x(N))$;

4 Proofs of Main Results

Proof 1 Proof of Theorem 1. *We have the following estimates*

$$\left(\sup_{1\le i\le m} i^{p}\left(\sup_{i\le k\le\infty} y(k)\right)\right)^{\frac{1}{p}} \le \sup_{1\le i\le m} i\left(\sum_{k=i}^{\infty} y(k)\right)^{\frac{1}{p}} \le \sum_{i=1}^{m}\left(\sum_{k=i}^{\infty} y(k)\right)^{\frac{1}{p}}$$
$$\le \sup_{1<k<m} k\left(\sum_{i=1}^{k} i^{p-1}\right)^{-\frac{1}{p}}\left(\sum_{i=1}^{m} i^{p-1}\left(\sum_{k=i}^{\infty} y(k)\right)\right)^{\frac{1}{p}}. \quad (27)$$

The first two inequalities are trivial, and the last inequality follows from Lemma 4.

We also need the following trivial inequalities,

$$\left(\sup_{1\le i\le m} i^p\left(\sup_{i\le k\le\infty} y(k)\right)\right)^{\frac{1}{p}} \le \sum_{i=1}^{m}\left(\sup_{i\le k<\infty} y(k)\right)^{\frac{1}{p}} \le \sum_{i=1}^{m}\left(\sum_{k=i}^{\infty} y(k)\right)^{\frac{1}{p}}. \tag{28}$$

The equivalence

$(i) \Leftrightarrow (ii)$

easily follows by taking $x(k) = (\sum_{i=k}^{\infty} y(i))^{\frac{1}{p}}$ *in (1) and use Fubini Lemma 1.*

From the estimates (27) follow the following implications

$(iii) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (vi),$

and from the estimates (28)we have

$(ii) \Rightarrow (v) \Rightarrow (vi).$

The implication

$(vi) \Rightarrow (iii),$

follows from [[14], Theorem 2.3 and Theorem 2.4]. Therefore, we have all the implications.

Proof 2 Proof of Theorem 2.

$(i) \Rightarrow (ii)$.

In (1) consider decreasing sequence:

$$x(k) = \left(\frac{1}{\sum_{i=1}^{k} b(i)B(i)^{\frac{\alpha+1}{p}-1}} \sum_{i=1}^{k} B(i)^{\frac{\alpha+1}{p}-1} b(i) \sum_{j=i}^{\infty} y(j)\right)^{\frac{1}{p}}$$

. Using the following estimate,

$$x(k) \ge \left(\frac{1}{\sum_{i=1}^{k} b(i)B(i)^{\frac{\alpha+1}{p}-1}} \sum_{j=1}^{k}\left(\sum_{i=1}^{j} B(i)^{\frac{\alpha+1}{p}-1} b(i)\right) y(j)\right)^{\frac{1}{p}} \approx \left(\frac{1}{B(k)^{\frac{\alpha+1}{p}}} \sum_{j=1}^{k} B(j)^{\frac{\alpha+1}{p}} y(j)\right)^{\frac{1}{p}}$$

and (25) and (26) we obtain

$$\begin{aligned}
\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\left(\frac{1}{B(i)^{\frac{\alpha+1}{p}}} \sum_{k=1}^{i} B(k)^{\frac{\alpha+1}{p}} y(k)\right)\right)^{q} a(m)\right)^{\frac{1}{q}} &\lesssim \left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m} x(i)\right)^{q} a(m)\right)^{\frac{1}{q}} \\
&\le C\left(\sum_{m=1}^{\infty} x(m)^p b(m)\right)^{\frac{1}{p}} \\
&= C\left(\sum_{m=1}^{\infty}\left(\frac{1}{\sum_{i=1}^{k} b(i)B(i)^{\frac{\alpha+1}{p}-1}} \sum_{i=1}^{k} B(i)^{\frac{\alpha+1}{p}-1} b(i) \sum_{j=i}^{\infty} y(j)\right) b(m)\right)^{\frac{1}{p}} \\
&\lesssim C\left(\sum_{m=1}^{\infty}\left(\sum_{j=m}^{\infty} y(j)\right) b(m)\right)^{\frac{1}{p}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence (8) follows.
(ii) ⇒ (i).

$$x(k)=\frac{B(k)^{\frac{\alpha+1}{p}}}{B(k)^{\frac{\alpha+1}{p}}}x(k)\leq(\alpha+1)^{\frac{1}{p}}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\left(\sum_{j=1}^{k}B(j)^{\alpha}b(j)x(j)\right)^{\frac{1}{p}}.$$

Using this estimate, (8) for seguence $y(j)=B(j)^{-1}b(j)$, we get

$$\begin{aligned}&\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}x(k)\right)^{q}a(m)\right)^{\frac{1}{q}}\\&\leq(\alpha+1)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\left(\frac{1}{B(k)^{\alpha+1}}\sum_{j=1}^{k}B(j)^{\alpha}b(j)x(j)\right)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}}\\&\leq C(\alpha+1)^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}b(m)x(m)^{p}\right)^{\frac{1}{p}}.\end{aligned}$$

We have the following estimates:

$$\begin{aligned}&\left(\sup_{1\leq i\leq m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\leq k\leq i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\\&\leq\left(\sup_{1\leq i\leq m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}?\\&\leq\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\\&\leq\left(\sum_{i=1}^{m}\left(\sum_{k-i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}B(i)^{\alpha+1}y(i)\right)^{\frac{1}{p}}.\end{aligned}\tag{29}$$

The first two inequalities are trivial, and the last inequality follows from Lemma 4(b). We also have the following trivial estimates:

$$\begin{aligned}&\left(\sup_{1\leq i\leq m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\leq k\leq i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\\&\leq\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sup_{1\leq k\leq i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\leq\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}.\end{aligned}\tag{30}$$

From the estimates (29) follows following implications

$$(iv) \Rightarrow (ii) \Rightarrow (v) \Rightarrow (vi).$$

From the estimates (30) we have

$$(ii) \Rightarrow (iii) \Rightarrow (vi),$$

The implication

$$(vi) \Rightarrow (iv).$$

Follows from [[14],Theorem 3.2 and Theorem 2.4]. Therefore, we have all the implications.

Proof 3 Proof of Theorem 3. *We have the following estimates:*

$$\begin{aligned}
&\left(\sup_{m\le i\le\infty}(i-m)^p\left(\sup_{i\le k\le\infty}y(k)\right)\right)^{\frac{1}{p}} \le \left(\sup_{m\le i\le\infty}(i-m)^p\left(\sum_{k=i}^{\infty}y(k)\right)\right)^{\frac{1}{p}} \\
&\le \sum_{i=m}^{\infty}\left(\sum_{k=i}^{\infty}y(k)\right)^{\frac{1}{p}} \le C_p\left(\sum_{i=m}^{\infty}(i-m)^{p-1}\left(\sum_{k=i}^{\infty}y(k)\right)\right)^{\frac{1}{p}} \\
&\approx C_p\left(\sum_{i=m}^{\infty}(i-m)^p y(k)\right)^{\frac{1}{p}}. \qquad (31)
\end{aligned}$$

The first two inequalities are trivial, and the third inequality follows from Lemma 4(a), and the last equivalent follows by using (24).

We also need the following trivial inequalities,

$$\left(\sup_{m\le i\le\infty}\left(\sup_{i\le k\le\infty}y(k)\right)\right)^{\frac{1}{p}} \le \sum_{i=m}^{\infty}\left(\sup_{i\le k<\infty}y(k)\right)^{\frac{1}{p}} \le \sum_{i=m}^{\infty}\left(\sum_{k=i}^{\infty}y(k)\right)^{\frac{1}{p}}. \qquad (32)$$

The equivalence

$$(i) \Leftrightarrow (ii)$$

easily follows by taking $x(k) = (\sum_{i=k}^{\infty} y(i))^{\frac{1}{p}}$ *in (2).*

From the estimates (31) follows the implications

$$(iii) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (vi).$$

From the estimates (32) follows the implications

$$(ii) \Rightarrow (v) \Rightarrow (vi).$$

The implication

$$(vi) \Rightarrow (iii),$$

follows from [[14], Theorem 3.2 and Theorem 2.4]. Therefore, we have all the implications.

Proof 4 Proof of Theorem 4. *The equivalence (i) ⇔ (ii) follows same way as in the proof of Theorem 3. As in the proof of the Theorem 3, we have*

$$\left(\sup_{m\le i\le\infty}\left(\sum_{k=i}^{\infty}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\le k\le i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}$$
$$\le\left(\sup_{m\le i\le\infty}\left(\sum_{k=i}^{\infty}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}$$
$$\le\sum_{i=m}^{\infty}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}$$
$$\le\left(\sum_{i=m}^{\infty}\left(\sum_{k=i}^{\infty}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p-1}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}$$

which gives the implications

$$(iv)\Rightarrow(ii)\Rightarrow(v)\Rightarrow(vi).$$

Using the trivial estimates

$$\left(\sup_{m\le i\le\infty}\left(\sum_{k=i}^{\infty}\frac{1}{B_k^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\le k\le i}B_k^{\alpha+1}y_k\right)^{\frac{1}{p}}\le\sum_{i=m}^{\infty}\frac{1}{B_i^{\frac{\alpha+1}{p}}}\left(\sup_{1\le k\le i}B_k^{\alpha+1}y_k\right)^{\frac{1}{p}}$$
$$\le\sum_{i=m}^{\infty}\frac{1}{B_i^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i}B_k^{\alpha+1}y_k\right)^{\frac{1}{p}}.$$

We are obtaining the implications:

$$(ii)\Rightarrow(iii)\Rightarrow(vi).$$

The implication

$$(vi)\Rightarrow(iv),$$

follows from [14, Theorem 2.3 and Theorem 2.4].

5 Conclusion

In this paper, we establish that weighted inequalities of the type $l_p \to l_q$ for discrete Hardy and Copson operators on the cone of non-negative monotone sequences can, in the case $0 < q < \infty$, $0 < p < 1$, be reduced to the corresponding inequalities on the cone of non-negative sequences. This approach makes it possible to employ a broader range of proof techniques and significantly simplifies the analysis of the inequalities under consideration. The equivalence theorems relating inequalities for discrete Hardy and Copson operators on the cone of non-increasing sequences to those on the cone of non-negative sequences emphasize the intrinsic differences between the discrete and continuous cases. It is shown that methods successfully applied in the continuous case do not always prove effective in the discrete one. The results obtained complement previously known statements for the case $p > 1$ and broaden the theoretical framework for studying discrete Hardy and Copson operators in weighted spaces.

6 Declaration

The authors have no competing interests to declare. Relevant to the content of this article

7 Acknowledgements

The research was supported by the grant Ministry of Education and Science of the Republic of Kazakhstan (project no.AP26196065). The research of A.Gogatishvili was partially supported by the grant project 23-04720S of the Czech Science Foundation (GAČR), The Institute of Mathematics, CAS is supported by RVO:67985840, by Shota Rustaveli National Science Foundation (SRNSF), grant no: FR22-17770.

References

[1] N. Bokayev, A. Gogatishvili, G. Karshygina, N. Kuzeubayeva and T. Ünver, Reduction theorems for the discrete Hardy operator on the cones of monotone sequences. *Journal of Mathematical Sciences*, **2025** 291(2), 217–223.

[2] R. Oinarov and S. Kh. Shalginbaeva, Weighted Hardy inequalities on the cone of monotone sequences. *Izv. Minister. Nauki Akad. Nauk Resp Kaz. Ser. Fiz. -Mat.*, **1998**, 1, 33-42.

[3] G. Bennett, Some elementary inequalities. *Quart. J. Math. Oxford Ser.* **1987** 38(2),401-425.

[4] G. Bennett, Some elementary inequalities. *III. Quart. J. Math. Oxford Ser.* **1991**,42(2), 149–174.

[5] K. -G. Grosse-Erdmann, The blocking technique, weighted mean operators and Hardy's inequality. *Lecture Notes in Mathematics. Springer-Verlag, Berlin* **1998**, 1679.

[6] A. Gogatishvili and V. D. Stepanov, Reduction theorems for operators on the cones of monotone functions. *J. Math. Anal. Appl.* **2013**, 405(1), 156-172.

[7] A. Gogatishvili and V. D. Stepanov, Reduction theorems for weighted integral inequalities on the cone of monotone functions. *Uspekhi Mat. Nauk.* **2013** 68, 3-68. *Russian Math. Surveys.* **2013** 68(4), 597-664.

[8] G. Sinnamon, Hardy's inequality and monotonicity, Function Spaces and Nonlinear Analysis. *Mathematical Institute of the Academy of Sciences of the Czech Republic, Prague*, **2005**, 292—310.

[9] M. L. Goldman, Order-sharp estimates for Hardy-type operators on cones of quasimonotone functions. *Eurasian Math. J.*, **2011**, 2(3), 143–146.

[10] M. Johansson, V. D. Stepanov, E. P. Ushakova, Hardy inequality with three measures on monotone functions. *Math. Inequal. Appl.* **2008**,11(3), 393–413.

[11] H. P. Heinig, V. D. Stepanov, Weighted Hardy inequalities for increasing functions. *Canad. J. Math.* **1993**, 45(1), 104–116.

[12] E. Sawyer, Boundedness of classical operators on classical Lorentz spaces. *Studia Math.* **1990**,96(2),145–158.

[13] A. Gogatishvili, M. Krepela, R. Ol'hava, and L.Pick, Weighted inequalities for discrete iterated Hardy operators. *Mediterr. J. Math.* **2020**, 17(4, 132.

[14] A. Gogatishvili, L. Pick, and T. Ünver, Weighted inequalities for discrete iterated kernel operators. *Math. Nachr.* **2022**, 295(11), 2171-2196.

[15] G. Bennett and K. -G. Grosse-Erdmann, On series of positive terms. *Houston J. Math.* **2005**,31(2),541–586.

Авторлар туралы мәлімет:

Бокаев Нуржан – Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Іргелі математика кафедрасының профессоры (Астана, Қазақстан, электрондық пошта: bokayev2011@yandex.ru);

Амиран Гогатишвили – PhD, Чехия ғылым академиясының Математика институтында бас ғылыми қызметкері, (Прага, Чехия, электрондық пошта: gogatish@math.cas.cz);

Кузеубаева Нургуль (корреспондент автор) – Л.Н. Гумилев атындағы Еуразия ұлттық университеті PhD докторанты (Астана, Қазақстан, электрондық пошта: nurgul.kuzeubaeva@mail.ru).

Сведения об авторах:

Бокаев Нуржан – профессор кафедры Фундаментальной математики Евразийского национального университета имени Л.Н.Гумилева (Астана, Казахстан, электронная почта: bokayev2011@yandex.ru);

Амиран Гогатишвили – PhD, главный научный сотрудник Института математики Чешской Академии наук, (Прага, Чешская Республика, электронная почта: gogatish@math.cas.cz);

Кузеубаева Нургуль Касымхановна (корреспондент автор) – PhD докторант Евразийского национального университета имени Л.Н.Гумилева (Астана, Казахстан, электронная почта: nurgul.kuzeubaeva@mail.ru).

Information about authors:

Bokayev Nurzhan – Professor of the Department of Fundamental Mathematics at L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, email: bokayev2011@yandex.ru);

Gogatishvili Amiran — PhD, Chief Researcher at the Institute of Mathematics of the Czech Academy of Sciences (Praha, Czech Republic, email: gogatish@math.cas.cz);

Kuzeubayeva Nurgul (corresponding author) – PhD student of Gumilyov Eurasian National University, (Astana, Kazakhstan, email: nurgul.kuzeubaeva@mail.ru).

Received: August 12, 2025
Accepted: December 9, 2025