IRSTI 27.41.19 DOI: https://doi.org/10.26577/JMMCS202512844

G-H. Aimal Rasa[1*], M. Madiyarov[2], M. Mustafina[2]
[1]Kabul Education University, Kabul, Afghanistan
[2]Sarsen Amanzholov East Kazakhstan State University, Ust-Kamenogorsk, Kazakhstan
*e-mail: aimal.rasa14@gmail.com

ON THE UNIQUE SOLVABILITY OF NONLOCAL IN TIME PROBLEMS CONTAINING THE IONKIN OPERATOR IN THE SPATIAL VARIABLE

This paper studies a differential equation representing the differences of two operators. One of the operators is generated by linear differential expressions that depend on time. The second operator is the Ionkin operator with respect to the spatial variable. In this paper, the differential operator with respect to time is generated by two-point Birkhoff regular boundary conditions. At the same time, the elliptic operator with respect to the spatial variable does not satisfy the so-called Agmon conditions. Moreover, the operator with respect to the spatial variable is not self-adjoint. In the beginning, the solvability of the problem is proved. In the final part, the uniqueness of the solution is proved. Direct application of the methods of the authors' previous works to prove the uniqueness of the solution to the problem is quite problematic. However, the authors managed to modify the reasoning of previous works to prove the uniqueness of the solution to the problem.

Key words: elliptic operators, differential-operator equations, initial-boundary value problem, solvability of the problem, existence of a solution, uniqueness of a solution, operator eigenvalues, complete orthonormal systems.

Ғ-Х. Аймал Раса[1*], М. Мадияров [2], М. Мустафина [2]
[1]Кабул білім университеті, Кабул, Ауғанстан
[2]Сәрсен Аманжолов атындағы Шығыс Қазақстан университеті, Өскемен, Қазақстан
*e-mail: aimal.rasa14@gmail.com

Кеңістік айнымалылы Ионкин операторын қамтитын уақыт бойынша жергілікті емес есептерің бірегей шешімі туралы

Бұл жұмыста екі оператордың айырмасы болып табылатын дифференциалдық теңдеу зерттеледі. Операторлардың біріншісі уақытқа тәуелді сызықтық дифференциалдық өрнектер арқылы туындайды. Операторлардың екіншісі кеңістік айнымалыға тәуелді Ионкин операторын сипаттайды. Бұл жұмыста уақытқа қатысты дифференциалдық оператор екі нүктелік регулярлы Биркгоф шекаралық шарттары арқылы құрылған. Бұл жағыдайда кеңістік айнымалыға тәуелді оператор Агмон шарттарын қанағаттандырмайды. Сонымен қатар, кеңістік айнымалылы оператор түйіндес болмайды. Жұмыстың кіріспесіде есептің шешілетіндігі дәлелденеді. Қорытынды бөлімде шешімінің жалғыздығы дәлелденеді. Есептің шешімінің жалғыздығын дәлелдеу үшін авторлардың бұрынғы еңбектеріндегі әдістерді тікелей қолдану ыңғайсыз. Дегенмен, есептің шешімінің жалғыздығын дәлелдеу үшін бұрынғы еңбектеріндегі пайымдауларды қолдана алды.
Түйін сөздер: эллиптикалық операторлар, дифференциалдық-операторлық теңдеулер, бастапқы-шектік есеп, есептің шешімі, шешімінің болуы, шешімінің жалғыздығы, оператордың меншікті мәндері, толық ортонормаланған жүйелер.

Ғ-Х. Аймал Раса[1*], М. Мадияров [2], М. Мустафина [2]
[1]Кабульский образовательный университет , Кабул, Афганистан
[2]Восточно-Казахстанский университет имени С. Аманжолова, Усть-каменогорск, Казахстан
*e-mail: aimal.rasa14@gmail.com

Об однозначной разрешимости нелокальных по времени задач, содержащих оператор Ионкина по пространственной переменной

В данной работе исследуется дифференциальное уравнение, представляющее разно- сти двух операторов. Один из операторов, порождается линейными дифференциаль- ными выражениями, зависящими от времени. Второй из операторов представляет опе- ратор Ионкина по пространственной переменной. В настоящей работе дифференциальный оператор по времени порождается двухточечными регулярными по Биркгофу граничными условиями. В то же время эллиптический оператор по пространственной переменной не удовлетворяет так называемым условиям Агмона. Более того оператор по пространственной переменной не является самосопряженным. В начале доказывается разрешимость поставленной задачи. В заключительной части доказывается единственности решения. Непосредственное применение методов предыдущих работ авторов для доказательства единственности решения задачи достаточно проблемно. Однако авторам для доказательства единственности решения задачи удалось модифицировать рассуждения предыдущих работ.

Ключевые слова: эллиптические операторы, дифференциально-операторные уравнения, начально-краевая задача, разрешимость задачи, существование решения, единственность решения, собственные значения оператора, полные ортонормированные системы

1 Introduction

Let $0 < T < \infty$. We introduce the differential expression

$$l(t, \frac{d}{dt}) \equiv \frac{d^{2p}}{dt^{2p}} + \sum_{k=0}^{2p-2} p_k(t) \frac{d^k}{dt^k}, \quad t \in (0, T),$$

where $p_k(t) \in C^k[0, T]$, $k = 0, 1, \dots, 2p - 2$.

Let us consider in the domain $Q_T = (0, 1) \times (0, T)$ the differential equation

$$l(t, \frac{\partial}{\partial t})u(x,t) - u(x,t) + \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

with boundary conditions on x for fixed $t \in (0, T)$

$$u(0,t) = 0, \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x}, \tag{2}$$

with conditions on t for fixed $x \in \Omega$

$$U_{2\xi-1}(u(x,\cdot)) \equiv \frac{\partial^{j_\xi} u(x,t)}{\partial t^{j_\xi}}\Big|_{t=0} + \sum_{s=0}^{j_\xi-1} \left(\alpha_{2\xi-1,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=0} + \beta_{2\xi-1,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=T}\right) = 0,$$

$$U_{2\xi}(u(x,\cdot)) \equiv \frac{\partial^{j_\xi} u(x,t)}{\partial t^{j_\xi}}\Big|_{t=T} + \sum_{s=0}^{j_\xi-1} \left(\alpha_{2\xi,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=0} + \beta_{2\xi,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=T}\right) = 0, \ \xi = 1, \dots m,$$

$$U_{2m+\xi}(u(x,\cdot)) \equiv \alpha_{2m+\xi,\nu_\xi} \frac{\partial^{\nu_\xi} u(x,t)}{\partial t^{\nu_\xi}}\Big|_{t=0} + \beta_{2m+\xi,\nu_\xi} \frac{\partial^{\nu_\xi} u(x,t)}{\partial t^{\nu_\xi}}\Big|_{t=T} +$$

$$\sum_{s=0}^{\nu_\xi-1} \left(\alpha_{2m+\xi,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=0} + \beta_{2m+\xi,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=T}\right) = 0, \quad \xi = 1, \dots r. \tag{3}$$

Here $2m+r=2p$, and also in the next line $(0 \le j_1 < ... < j_m \le 2p-1,\ 0 \le \nu_1 < ... < \nu_r \le 2p-1)$ there are no identical natural numbers. For the coefficients of the condition on t the inequalities are satisfied

$$|\alpha_{2m+q,\nu_q}| + |\beta_{2m+q,\nu_q}| \ne 0, \quad q = 1, ..., r.$$

The right-hand side $f(x,t)$ is a given function. Note that conditions (3) are non-decomposable boundary conditions for $j = 1, ..., 2p$, i.e. they have the form

$$U_j(u(x,\cdot)) \equiv U_{j0}(u(x,\cdot)) + U_{jT}(u(x,\cdot)), \tag{4}$$

where for $a = 0, T$ the linear form $U_{ja}(u(x,\cdot))$ represents a differential expression depending on

$$u(x,a), \frac{\partial u(x,a)}{\partial t}, ..., \frac{\partial^{2p-1} u(x,a)}{\partial t^{2p-1}}.$$

According to the monograph by M.A. Naimark [1], boundary conditions of the form (3) are regular boundary conditions. In the work by G.M. Kesel'man [2] it is shown that regular in the sense of Birkhoff boundary conditions of the form (4) can be normalized and reduced to the form (3).

Boundary conditions (2) with respect to the variable x were first introduced and studied in the work of N.I. Ionkin [3]. An important distinctive feature of Ionkin's conditions is that the corresponding eigenvalue problem

$$-v''(x) = \lambda v(x), \quad 0 < x < 1$$

with boundary conditions (2) has infinitely many associated functions. The latter fact significantly influences the spectral expansions with respect to the system of eigen and associated functions of the Ionkin problem.

The purpose of this work is to find out what requirements the right-hand side of $f(x,t)$ must satisfy so that problem (1)–(2)–(3) is uniquely solvable?

We define the functional space of solutions $V_2^{1,2p}(Q_T)$ of problem (1)–(2)–(3) as the linear space of functions $u(x,t)$ belonging to the space $L_2(Q_T)$ and having a generalized derivative with respect to the spatial variable x and with respect to the variable t up to the order $2p$ inclusive, belonging to the same space, with a finite norm

$$\|u\|^2_{V_2^{1,2p}(Q_T)} = \int_0^T \left[\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} + |\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle| \right] dt,$$

where

$$\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle = \int_0^1 l(t,\frac{\partial}{\partial t})u(x,t)\,\overline{u(x,t)}\,dx.$$

It is obvious that the space $V_2^{1,2p}(Q_T)$ is a Banach space.

In the work of N.I. Ionkin [3] the eigenvalue problem was considered:

$$v''(x) + \lambda\, v(x) = 0, \quad 0 < x < 1 \tag{5}$$

$$v(0) = 0, \tag{6}$$

$$v'(0) = v'(1) \tag{7}$$

which is not self-adjoint. The adjoint to problem (5)–(7) has the form

$$v^{*\prime\prime}(x) + \lambda\, v^*(x) = 0, \quad 0 < x < 1, \tag{8}$$

$$v^{*\prime}(1) = 0, \tag{9}$$

$$v^*(0) = v^*(1). \tag{10}$$

It is known [3] that problem (4)–(6) has eigenvalues

$$\lambda_k = (2\pi\, k)^2, \quad k = 0, 1, ... \tag{11}$$

The system of eigenfunctions of problem (5)–(7) is calculated in the work [3] and it has the form:

$$v_0(x) = x,\ v_{2k-1}(x) = x\ \cos(2\pi\, k\, x),\ v_{2k}(x) = \sin(2\pi\, k\, x),\ k = 1, 2, ... \tag{12}$$

Note that each eigenvalue λ_k for $k > 0$ corresponds to an eigenfunction $v_{2k}(x)$ and an associated $v_{2k-1}(x)$. At the same time, for $k = 0$ the eigenvalue $\lambda_0 = 0$ is simple.

The system of eigen and associated functions of the adjoint problem to problem (4)–(6) is denoted by [3]:

$$v_0^*(x) = 2,\ v_{2k-1}^*(x) = 4\cos(2\pi\, k\, x),\ v_{2k}^*(x) = 4(1-x)\sin(2\pi\, k\, x),\ k = 1, 2, ... \tag{13}$$

In this case, each $\lambda_k = (2\pi\, k)^2$ with $k > 0$ corresponds to an eigenfunction $v_{2k-1}^*(x)$ and an associated $v_{2k}^*(x)$.

In the work [3] the following lemmas are proved.

Lemma 1 *[3]. Sequences of functions* (12) *and* (13) *form biorthogonal on the interval* $(0,1)$ *systems of functions, so that for any numbers i and j the following relation holds*

$$\langle v_i, v_j^* \rangle = \int_0^1 v_i(x)\, \overline{v_j^*(x)}\, dx = \delta_{ij},$$

here δ_{ij} is the Kronecker delta.

Lemma 2 *[3]. The sequence $v_0(x) = x$, $v_{2k-1}(x) = x\ \cos(2\pi\, k\, x)$, $v_{2k}(x) = \sin(2\pi\, k\, x)$, $k = 1, 2, ...$ forms a basis in the space of functions $L_2(0,1)$, and for any function $\varphi(x) \in L_2(0,1)$ the inequalities of F. Riesz [5] with some constants are valid r_1, r_2, R_1, R_2, :*

$$r_1\, \|\varphi\|_{L_2(0,1)} \leq \sum_{k=0}^{\infty} |\varphi_k|^2 \leq R_1\, \|\varphi\|_{L_2(0,1)}, \tag{14}$$

$$r_2\, \|\varphi\|_{L_2(0,1)} \leq \sum_{k=0}^{\infty} |\varphi_k^*|^2 \leq R_2\, \|\varphi\|_{L_2(0,1)}, \tag{15}$$

where $\varphi_k = \int_0^1 \varphi(x)\, \overline{v_k^*(x)} dx, \quad \varphi_k^* = \int_0^1 \varphi(x)\, \overline{v_k(x)} dx.$

In what follows we denote by $\rho_0 = 1$, $\rho_{2k-1} = \rho_{2k} = \sqrt[2p]{1 + (2\pi k)^2}$ for $k \geq 1$. Sequence of numbers We denote $\{\rho_k,\ k \geq 0\}$ by ρ.

If the right-hand side of the equation $f(x,t)$ belongs to the space $L_2(Q_T)$, then we introduce the sequences

$$f_k(t) = \int_0^1 f(x,t)\,\overline{v_k^*(x)}dx, \quad k \geq 0, \tag{16}$$

$$f_k^*(t) = \int_0^1 f(x,t)\,\overline{v_k(x)}dx, \quad k \geq 0. \tag{17}$$

Let $W_2^{0,1}(Q_T)$ denote the space of functions $f(x,t) \in L_2(Q_T)$ such that

$$\frac{\partial f(x,t)}{\partial t} \in L_2(Q_T),$$

$$\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,T)\|^2_{L_2(0,1)} + \int_0^T \|f(\cdot,t)\|^2_{L_2(0,1)}\,dt + \int_0^T \|\frac{\partial f(\cdot,t)}{\partial t}\|^2_{L_2(0,1)}\,dt < \infty.$$

The main result of this article is formulated in the following statement.

Theorem 1 *Let p be an arbitrary natural number. Let the condition $\Delta(\rho_k) \neq 0$ be satisfied for all $k \geq 0$ (the characteristic determinant $\Delta(\rho_k)$ is introduced by formula (23)).*

If the right-hand side $f(x,t)$ belongs to the space $W_2^{0,1}(Q_T)$, then there exists a unique solution $u(x,t) \in V_2^{1,2p}(Q_T)$ of problem (1)-(2)-(3), and the estimate

$$\int_0^T \Big[\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} + |\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle|\Big]\,dt \leq$$

$$M\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,T)\|^2_{L_2(0,1)} + \int_0^T \|f(\cdot,t)\|^2_{L_2(0,1)}\,dt + \int_0^T \|\frac{\partial f(\cdot,t)}{\partial t}\|^2_{L_2(0,1)}dt\Big], \tag{18}$$

where M is some constant independent of $f(x,t)$.

In the works of N.I. Ionkin [3, 4] a mixed problem for the heat equation was investigated. Theorem 1 generalizes the results of N.I. Ionkin [3, 4] when the differential part of the heat conductivity operator with respect to the variable t is replaced by a differential expression with respect to the variable t of order $2p$.

At the same time, in the works of A.I. Kozhanov [6, 7], R.R. Ashurov [8], K.B. Sabitov [9, 10] similar problems were studied, when the differential part of the heat conductivity operator with respect to the spatial variable x is replaced by more complex differential expressions with respect to the variable x.

Note that an equation of the form (1), according to the terminology of A.A. Dezin [11], refers to differential-operator equations. The issues of solvability of differential-operator equations were studied in the works [12–17]. V.V. Sheluchin [18, 19] studied the problem of forecasting ocean temperature based on average data for the previous period of time, which also belongs to the class of differential-operator equations.

There are various methods of proving uniqueness. Usually, an effective means of proving uniqueness is the maximum principle [20] and its various generalizations such as the Hopf [21] and Zaremba-Giraud [22] principles. For problem (1)–(2)–(3), the above principles are not satisfied. Therefore, when proving the uniqueness of a solution, we needed a toolkit other than the extremum principle.

In the work of V.A. Ilyin [23] a fairly universal method for proving the uniqueness of a solution for hyperbolic and parabolic equations is proposed. Under fairly general restrictions on the domain Ω, in the work [23] a theorem of uniqueness of a solution for hyperbolic and parabolic equations is proved. The meaning of the requirements of V.A. Ilyin's theorem [23] is that the elliptic part of a hyperbolic and parabolic operator has a complete system of eigenfunctions in the corresponding functional space.

We also note the work of I.V. Tikhonov [24], devoted to uniqueness theorems for linear nonlocal problems for abstract differential equations. This work is interesting because I.V. Tikhonov proposed a new method for proving uniqueness theorems. I.V. Tikhonov's method of proving uniqueness is based on the "method of quotients"for entire functions of exponential type. In the work of A.Yu. Popov, I.V. Tikhonov [25], the class of unique solvability of the heat equation with a nonlocal condition expressed by an integral over time on a fixed interval was determined. They managed to give a complete description of uniqueness classes in terms of the behavior of solutions for $|x| \to \infty$.

The differential equation (1) is the sum of two operators. One of the operators is generated by linear differential expressions depending on time. The second operator is the Ionkin operator with respect to the spatial variable. In this paper, the differential operator with respect to time is generated by two-point Birkhoff regular boundary conditions. At the same time, the elliptic operator with respect to the spatial variable does not satisfy the so-called Agmon conditions [26].

Moreover, the operator with respect to the spatial variable is not self-adjoint. Therefore, direct application of the methods of works [28–34] to prove the uniqueness of the solution to problem (1)–(2)–(3) is quite problematic. However, the authors managed to modify the reasoning of works [28–30] to prove the uniqueness of the solution to problem (1)–(2)–(3).

Recall that for unique solvability, the mutual arrangement of the spectra of the two operators indicated above plays an essential role. In the articles of the authors [28–30], the method for proving the uniqueness theorem is a hybrid of the method of guiding functionals of M.G. Crane [1, 27] and the method of V.A. Ilyin [23].

2 Formal representation of the solution to problem (1)–(2)–(3)

We seek the solution to problem (1)–(2)–(3) in the form

$$u(x,t) = y_0(t)\, v_0(x) + \sum_{k=1}^{\infty} \big(y_{2k-1}(t)\, v_{2k-1}(x) + y_{2k}(t)\, v_{2k}(x)\big), \tag{19}$$

Using the results of the monograph [1], we find the coefficients $y_k(t)$ using the formula

$$y_k(t) = \int_0^T \frac{H(t,\tau,\lambda_k)}{\Delta(\lambda_k)}\, f_k(\tau)\, d\tau, \tag{20}$$

where $f_k(\tau)$ are calculated using formula (16).
The formulas for the expressions $H(t,\tau,\rho_k^{2p})$ and $\Delta(\rho_k^{2p})$ are taken from the monograph [1] and will be given below.

For further calculations, it is convenient to denote by $\{\omega_\mu\}$ the roots of (-1) of degree $2p$. In this section, we assume that p is an even number. The results formulated for even p remain valid for odd p. In this case, minor changes are required in the course of proving the results.

If p is an even number, then the numbering of the numbers $\{\omega_1,...,\omega_{2p}\}$ can be subordinated to the inequalities

$$\mathrm{Re}\omega_1 \le \cdots \le \mathrm{Re}\omega_p < 0 < \mathrm{Re}\omega_{p+1} \le \cdots \le \mathrm{Re}\omega_{2p}. \tag{21}$$

Let $\rho_k = \sqrt[2p]{1+4\pi^2k^2}$. According to the monograph [1], we introduce a system of solutions $\{y_\mu(t,\rho_k)\}$ of the homogeneous equation $l(t,\frac{d}{dt})y_\mu(t)+\rho_k^{2p}\,y_\mu(t)=0$, which has an asymptotic representation

$$y_\mu(t,\rho_k) = e^{\omega_\mu\rho_k t}\cdot[1], \tag{22}$$

where $[1] = 1 + O(1/\rho_k)$ при $\rho_k \to \infty$.

Now we can enter the characteristic determinant

$$\Delta(\rho_k) = \det[U_j(y_\mu)]; \quad j,\ \mu = 1,\dots,2p. \tag{23}$$

From inequalities (21) and asymptotic representations (22) we obtain an asymptotic representation of the characteristic determinant for $\rho_k \to \infty$

$$\Delta(\rho_k) = \rho_k^{2(j_1+\dots+j_m)+\nu_1+\dots+\nu_r}\cdot e^{\rho_k(\omega_{p+1}+\dots+\omega_{2p})T}\cdot\Delta_0\cdot[1], \tag{24}$$

where

$$\Delta_0 = \begin{vmatrix} \omega_1^{j_1} & \dots & \omega_p^{j_1} & 0 & \dots & 0 \\ 0 & \dots & \dots & \omega_{p+1}^{j_1} & \dots & \omega_{2p}^{j_1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \omega_1^{j_m} & \dots & \omega_p^{j_m} & 0 & \dots & 0 \\ 0 & \dots & 0 & \omega_{p+1}^{j_m} & \dots & \omega_{2p}^{j_m} \\ \alpha_{2m+1,\nu_1}\omega_1^{\nu_1} & \dots & \alpha_{2m+1,\nu_1}\omega_1^{\nu_1} & \beta_{2m+1,\nu_1}\omega_{p+1}^{\nu_1} & \dots & \beta_{2m+1,\nu_1}\omega_{2p}^{\nu_1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{2m+r,\nu_r}\omega_1^{\nu_r} & \dots & \alpha_{2m+1,\nu_r}\omega_1^{\nu_r} & \beta_{2m+1,\nu_r}\omega_{p+1}^{\nu_r} & \dots & \beta_{2m+1,\nu_r}\omega_{2p}^{\nu_r} \end{vmatrix} \ne 0.$$

Let W_μ denote the algebraic complement of the element $y_\mu^{(2p-1)}(\tau,\rho_k)$ in the determinant

$$W(\tau) = \begin{vmatrix} y_1^{(2p-1)}(\tau,\rho_k) & \dots & y_{2p}^{(2p-1)}(\tau,\rho_k) \\ y_1^{(2p-2)}(\tau,\rho_k) & \dots & y_{2p}^{(2p-2)}(\tau,\rho_k) \\ \cdot & \dots & \cdot \\ y_1(\tau,\rho_k) & \dots & y_{2p}(\tau,\rho_k) \end{vmatrix}.$$

Let's define the function

$$g(t,\tau,\rho_k) = \pm\frac{1}{2}\sum_{\mu=1}^{2p} y_\mu(t,\rho_k)\,z_\mu(\tau,\rho_k),$$

where

$$z_\mu(\tau,\rho_k) = \frac{W_\mu(\tau)}{W(\tau)}.$$

Moreover, the sign $\ll + \gg$ is taken when $t > \tau$, and the sign $\ll - \gg$ when $t < \tau$.

From inequalities (21) and representations (22) it follows that

$$z_\mu(\tau,\rho_k) = \frac{e^{-\rho_k \omega_\mu \tau}}{2p \cdot \rho_k^{2p-1}} \cdot [1].$$

According to the monograph [1] we introduce the function

$$H(t,\tau,\rho_k) = \begin{vmatrix} y_1(t,\rho_k) & \dots & y_{2p}(t,\rho_k) & g(t,\tau,\rho_k) \\ U_1(y_1) & \dots & U_1(y_{2p}) & U_1(g)(\tau) \\ \cdot & \dots & \cdot & \cdot \\ U_{2p}(y_1) & \dots & U_{2p}(y_{2p}) & U_{2p}(g)(\tau) \end{vmatrix},$$

where

$$U_j(g) = -\frac{1}{2}\sum_{j=1}^{2p} U_{j0}(y_\mu)\, z_\mu(\tau) + \frac{1}{2}\sum_{j=1}^{2p} U_{jT}(y_\mu)\, z_\mu(\tau), \quad j = 1,\dots,2p.$$

Let $t > \tau$. In the determinant $H(t,\tau,\rho_k)$ we multiply the columns with numbers $1,\dots,p$ by the functions $\frac{1}{2}z_1(\tau),\dots,\frac{1}{2}z_p(\tau)$, and the columns $p+1,\dots,2p$ by the functions $-\frac{1}{2}z_{p+1}(\tau),\dots,-\frac{1}{2}z_{2p}(\tau)$ and add them with the last column. As a result, we obtain the relation

$$H(t,\tau,\rho_k) = \begin{vmatrix} y_1(t,\rho_k) & \dots & y_{2p}(t,\rho_k) & H_0(t,\tau) \\ U_1(y_1) & \dots & U_1(y_{2p}) & H_1(\tau,\rho_k) \\ \cdot & \dots & \cdot & \cdot \\ U_{2p}(y_1) & \dots & U_{2p}(y_{2p}) & H_{2p}(\tau,\rho_k) \end{vmatrix},$$

where

$$H_0(t,\tau,\rho_k) = \sum_{\mu=1}^{p} y_\mu(t)\, z_\mu(\tau),$$

$$H_j(\tau,\rho_k) = \sum_{\mu=1}^{p} U_{jT}(y_\mu)\, z_\mu(\tau) - \sum_{\mu=p+1}^{2p} U_{j0}(y_\mu)\, z_\mu(\tau), \quad j = 1,\dots,2p.$$

From inequalities (21) and asymptotic representations (22) we derive the asymptotic formula

$$H(t,\tau,\rho_k) = \frac{\rho_k^{2(j_1+\ldots+j_m)+\nu_1+\ldots+\nu_r} \cdot e^{\rho_k(\omega_{p+1}+\ldots+\omega_{2p})T}}{2p \cdot \rho_k^{2p-1}}\cdot$$

$$\begin{vmatrix} e^{\rho_k\omega_1 t} & \dots & e^{\rho_k\omega_1 t} & \widetilde{H}_0(t,\tau,\rho_k) \\ \cdot & \dots & \cdot & \widetilde{H}_1(\tau,\rho_k) \\ \cdot & \Delta_0 & \cdot & \cdot \\ \cdot & \dots & \cdot & \widetilde{H}_{2p}(\tau,\rho_k) \end{vmatrix} [1],$$

where

$$\widetilde{H}_0(t,\tau,\rho_k) = -\sum_{\mu=1}^{p} \omega_\mu e^{\rho_k \omega_\mu (t-\tau)}[1],$$

$$\widetilde{H}_{2\xi-1}(\tau,\rho_k) = -\sum_{\mu=1}^{p} \beta_{2\xi-1,j_\xi-1} \omega_\mu^{j_\xi+1} \rho_k^{-1} e^{\rho_k \omega_\mu (T-\tau)}[1] + \sum_{\mu=p+1}^{2p} \omega_\mu^{j_\xi+1} e^{-\rho_k \omega_\mu \tau}[1],$$

$$\widetilde{H}_{2\xi}(\tau,\rho_k) = -\sum_{\mu=1}^{p} \omega_\mu^{j_\xi+1} e^{\rho_k \omega_\mu (T-\tau)}[1] + \sum_{\mu=p+1}^{2p} \alpha_{2\xi,j_\xi-1} \omega_\mu^{j_\xi} \rho_k^{-1} e^{-\rho_k \omega_\mu \tau}[1], \xi = 1,\ldots,m,$$

$$\widetilde{H}_{2m+\xi}(\tau,\rho_k) = -\sum_{\mu=1}^{p} \beta_{2m+\xi,\nu_\xi} \omega_\mu^{\nu_\xi+1} e^{\rho_k \omega_\mu (T-\tau)}[1] + \sum_{\mu=p+1}^{2p} \alpha_{2m+\xi,\nu_\xi} \omega_\mu^{\nu_\xi+1} e^{-\rho_k \omega_\mu \tau}[1], \xi = 1,\ldots,r.$$

Therefore, for $0 < \tau < t < T$ the representation is correct

$$\frac{H(t,\tau,\rho_k)}{\Delta(\lambda_k)} = \frac{1}{2p\,\rho_k^{2p-1}} \cdot \widetilde{H}_0(t,\tau,\rho_k) - \sum_{s=1}^{2p} \frac{1}{\Delta_0\, 2p\, \rho_k^{2p-1}} h_s(t,\rho_k) \cdot \widetilde{H}_s(\tau,\rho_k), \tag{25}$$

where $h_s(t,\rho_k)$ is the determinant obtained from the characteristic determinant $\Delta(\lambda_k)$ by replacing its s-th row with the row

$$||e^{\omega_1 \rho_k t}[1], ..., e^{\omega_p \rho_k t}[1],\ e^{\omega_{p+1} \rho_k (t-T)}[1], ..., e^{\omega_{2p} \rho_k (t-T)}[1]||.$$

The expansion of the determinant $h_s(t,\rho_k)$ over the s-th row has the form

$$h_s(t,\rho_k) = \sum_{\mu=1}^{p} h_{s,\mu} e^{\rho_k \omega_\mu t}[1] + \sum_{\mu=p+1}^{2p} h_{s,\mu} e^{\rho_k \omega_\mu (t-T)}[1], s = 1, ..., 2p.$$

In this case, $h_{s,\mu}, s = 1,\ldots,2p$ are numbers representing the corresponding algebraic complements.

The asymptotic relation (25) remains valid for $0 < t < \tau < T$. As a result, equality (19), taking into account (20) and (25), will take the form

$$u(x,t) = \frac{1}{2p} v_0(x) \Big[-\int_0^T \widetilde{H}_0(t,\tau,\rho_0)\, f_0(\tau) d\tau - \frac{1}{\Delta_0} \sum_{s=1}^{2p} h_s(t,\rho_0) \int_0^T \widetilde{H}_s(\tau,\rho_0)\, f_0(\tau) d\tau \Big] +$$

$$\frac{1}{2p} \sum_{k=1}^{\infty} \frac{v_{2k-1}(x)}{\rho_{2k-1}^{2p-1}} \Big[-\int_0^T \widetilde{H}_0(t,\tau,\rho_{2k-1})\, f_{2k-1}(\tau) d\tau - \frac{1}{\Delta_0} \sum_{s=1}^{2p} h_s(t,\rho_{2k-1}) \int_0^T \widetilde{H}_s(\tau,\rho_{2k-1})\, f_{2k-1}(\tau) d\tau \Big] +$$

$$\frac{1}{2p} \sum_{k=1}^{\infty} \frac{v_{2k}(x)}{\rho_{2k}^{2p-1}} \Big[-\int_0^T \widetilde{H}_0(t,\tau,\rho_{2k})\, f_{2k}(\tau) d\tau - \frac{1}{\Delta_0} \sum_{s=1}^{2p} h_s(t,\rho_{2k}) \int_0^T \widetilde{H}_s(\tau,\rho_{2k})\, f_{2k}(\tau) d\tau \Big] =$$

$$\frac{1}{2p} \sum_{k=0}^{\infty} \frac{v_k(x)}{\rho_k^{2p-1}} \Big[-\int_0^T \widetilde{H}_0(t,\tau,\rho_k)\, f_k(\tau) d\tau - \frac{1}{\Delta_0} \sum_{s=1}^{2p} h_s(t,\rho_k) \int_0^T \widetilde{H}_s(\tau,\rho_k)\, f_k(\tau) d\tau \Big]. \tag{26}$$

Let us introduce the following notations

$$I_{k\mu}^{(1)}(t)=\rho_k\omega_\mu\int_0^t e^{\rho_k\omega_\mu(t-\tau)}\, f_k(\tau)d\tau,\quad I_{k\mu}^{(2)}(t)=\rho_k\omega_\mu\int_t^T e^{\rho_k\omega_\mu(t-\tau)}\, f_k(\tau)d\tau,$$

$$I_{k\mu s}^{(3)}(t)=e^{\rho_k\omega_\mu t}\rho_k\omega_s\int_0^T e^{\rho_k\omega_s(T-\tau)}\, f_k(\tau)d\tau,\quad I_{k\mu s}^{(4)}(t)=e^{\rho_k\omega_\mu t}\rho_k\omega_s\int_0^T e^{-\rho_k\omega_s\tau}\, f_k(\tau)d\tau,$$

$$I_{k\mu s}^{(5)}(t)=e^{\rho_k\omega_\mu (t-T)}\rho_k\omega_s\int_0^T e^{\rho_k\omega_s(T-\tau)}\, f_k(\tau)d\tau,\quad I_{k\mu s}^{(6)}(t)=e^{\rho_k\omega_\mu (t-T)}\rho_k\omega_s\int_0^T e^{-\rho_k\omega_s\tau}\, f_k(\tau)d\tau. \tag{27}$$

Now let us take into account the asymptotic representations of the functions $\widetilde{H}_0(t,\tau,\rho_k)$, $\widetilde{H}_s(\tau,\rho_k)$, $s=1,\ldots,2p$. As a result, from (26), using the notation (27), we have the following representation

$$u(x,t)=\frac{1}{2p}\sum_{k=0}^{\infty}\frac{v_k(x)}{\rho_k^{2p}}\cdot A_k(t), \tag{28}$$

where

$$A_k(t)=\Big[-\sum_{\mu=1}^{p} I_{k\mu}^{(1)}(t)-\sum_{\mu=p+1}^{2p} I_{k\mu}^{(2)}(t)+$$

$$\frac{1}{\Delta_0}\sum_{\xi=1}^{m}\Big(-\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2\xi-1,\mu}\beta_{2\xi-1,j_\xi-1}\frac{\omega_s^{j_\xi}}{\rho_k} I_{k\mu s}^{(3)}(t)+\sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2\xi-1,\mu}\omega_s^{j_\xi} I_{k\mu s}^{(4)}(t)-$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2\xi-1,\mu}\beta_{2\xi-1,j_\xi-1}\frac{\omega_s^{j_\xi}}{\rho_k} I_{k\mu s}^{(5)}(t)+\sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2\xi-1,\mu}\omega_s^{j_\xi} I_{k\mu s}^{(6)}(t)-$$

$$\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2\xi,\mu}\omega_s^{j_\xi} I_{k\mu s}^{(3)}(t)+\sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2\xi,\mu}\alpha_{2\xi,j_\xi-1}\frac{\omega_s^{j_\xi-1}}{\rho_k} I_{k\mu s}^{(4)}(t)-$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2\xi,\mu}\omega_s^{j_\xi} I_{k\mu s}^{(5)}(t)+\sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2\xi,\mu}\alpha_{2\xi,j_\xi-1}\frac{\omega_s^{j_\xi-1}}{\rho_k} I_{k\mu s}^{(6)}(t)\Big)-$$

$$\sum_{\xi=1}^{r}\Big(-\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2m+\xi,\mu}\beta_{2m+\xi,\nu_\xi}\omega_s^{j_\xi} I_{k\mu s}^{(3)}(t)+\sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2m+\xi,\mu}\alpha_{2m+\xi,\nu_\xi}\omega_s^{\nu_\xi} I_{k\mu s}^{(4)}(t)-$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2m+\xi,\mu}\beta_{2m+\xi,\nu_\xi}\omega_s^{\nu_\xi} I_{k\mu s}^{(5)}(t)+\sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2m+\xi,\mu}\alpha_{2m+\xi,\nu_\xi}\omega_s^{j_\xi} I_{k\mu s}^{(6)}(t)\Big)\Big].$$

Remark 1 *If, when solving problem (1)-(2)-(3), we use the expansion in the biorthogonal system $v_k^*(x)$, then we would obtain a representation of the solution $u(x,t)$, similar to the representation (28)*

$$u(x,t) = \frac{1}{2p}\sum_{k=0}^{\infty}\frac{v_k^*(x)}{\rho_k^{2p}} \cdot A_k^*(t), \tag{29}$$

where

$$A_k^*(t) = \Big[-\sum_{\mu=1}^{p} J_{k\mu}^{(1)}(t) - \sum_{\mu=p+1}^{2p} J_{k\mu}^{(2)}(t) +$$

$$\frac{1}{\Delta_0}\sum_{\xi=1}^{m}\Big(-\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2\xi-1,\mu}\beta_{2\xi-1,j_\xi-1}\frac{\omega_s^{j_\xi}}{\rho_k} J_{k\mu s}^{(3)}(t) + \sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2\xi-1,\mu}\omega_s^{j_\xi} J_{k\mu s}^{(4)}(t) -$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2\xi-1,\mu}\beta_{2\xi-1,j_\xi-1}\frac{\omega_s^{j_\xi}}{\rho_k} J_{k\mu s}^{(5)}(t) + \sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2\xi-1,\mu}\omega_s^{j_\xi} J_{k\mu s}^{(6)}(t) -$$

$$\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2\xi,\mu}\omega_s^{j_\xi} J_{k\mu s}^{(3)}(t) + \sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2\xi,\mu}\alpha_{2\xi,j_\xi-1}\frac{\omega_s^{j_\xi-1}}{\rho_k} J_{k\mu s}^{(4)}(t) -$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2\xi,\mu}\omega_s^{j_\xi} J_{k\mu s}^{(5)}(t) + \sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2\xi,\mu}\alpha_{2\xi,j_\xi-1}\frac{\omega_s^{j_\xi-1}}{\rho_k} J_{k\mu s}^{(6)}(t)\Big) -$$

$$\sum_{\xi=1}^{r}\Big(-\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2m+\xi,\mu}\beta_{2m+\xi,\nu_\xi}\omega_s^{j_\xi} J_{k\mu s}^{(3)}(t) + \sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2m+\xi,\mu}\alpha_{2m+\xi,\nu_\xi}\omega_s^{\nu_\xi} J_{k\mu s}^{(4)}(t) -$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2m+\xi,\mu}\beta_{2m+\xi,\nu_\xi}\omega_s^{\nu_\xi} J_{k\mu s}^{(5)}(t) + \sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2m+\xi,\mu}\alpha_{2m+\xi,\nu_\xi}\omega_s^{j_\xi} J_{k\mu s}^{(6)}(t)\Big)\Big][1].$$

Here we introduced the functions $J_{k\mu}^{(1)}(t), ..., J_{k\mu s}^{(6)}(t)$, which were obtained from $I_{k\mu}^{(1)}(t), ..., I_{k\mu s}^{(6)}(t)$ as a result of replacing $f_k(\tau)$ with $f_k^*(\tau)$, i.e.

$$J_{k\mu}^{(1)}(t) = \rho_k\omega_\mu \int_0^t e^{\rho_k\omega_\mu(t-\tau)} f_k^*(\tau)d\tau, \quad J_{k\mu}^{(2)}(t) = \rho_k\omega_\mu \int_t^T e^{\rho_k\omega_\mu(t-\tau)} f_k^*(\tau)d\tau,$$

$$J_{k\mu s}^{(3)}(t) = e^{\rho_k\omega_\mu t}\rho_k\omega_s \int_0^T e^{\rho_k\omega_s(T-\tau)} f_k^*(\tau)d\tau, \quad J_{k\mu s}^{(4)}(t) = e^{\rho_k\omega_\mu t}\rho_k\omega_s \int_0^T e^{-\rho_k\omega_s\tau} f_k^*(\tau)d\tau,$$

$$J_{k\mu s}^{(5)}(t) = e^{\rho_k\omega_\mu(t-T)}\rho_k\omega_s \int_0^T e^{\rho_k\omega_s(T-\tau)} f_k^*(\tau)d\tau, \quad J_{k\mu s}^{(6)}(t) = e^{\rho_k\omega_\mu(t-T)}\rho_k\omega_s \int_0^T e^{-\rho_k\omega_s\tau} f_k^*(\tau)d\tau.$$

3 Some auxiliary statements

In this section we will prove the following lemma.

Lemma 3 *Let $\gamma = \sin\frac{\pi}{2p}$ and p be an even number. If $f(x,t)$ is differentiable with respect to t, then for all $k \geq 1$ and $t \in (0,T)$ the following estimates hold:*

$$|I_{k\mu}^{(1)}(t)| \leq |f_k(t)| + |f_k(0)|e^{-\gamma\rho_k t} + \int_0^t |f_k'(\tau)|\, e^{-\gamma\rho_k(t-\tau)}\, d\tau \quad for \quad \mu = 1,\ldots,p;$$

$$|I_{k\mu}^{(2)}(t)| \leq |f_k(t)| + |f_k(T)|e^{-\gamma\rho_k(T-t)} + \int_t^T |f_k'(\tau)|\, e^{\gamma\rho_k(t-\tau)}\, d\tau \quad for \quad \mu = p+1,\ldots,2p;$$

$$|I_{k\mu s}^{(3)}(t)| \leq |f_k(T)|e^{-\gamma\rho_k t} + |f_k(0)|e^{-\gamma\rho_k(t+T)} + e^{-\gamma\rho_k t}\int_0^T |f_k'(\tau)|\, e^{-\gamma\rho_k(T-\tau)}\, d\tau,$$

$$for \quad s,\mu = 1,\ldots,p;$$

$$|I_{k\mu s}^{(4)}(t)| \leq |f_k(T)|e^{-\gamma\rho_k(t+T)} + |f_k(0)|e^{-\gamma\rho_k t} + e^{-\gamma\rho_k t}\int_0^T |f_k'(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau,$$

for $s = p+1,\ldots,2p, \quad \mu = 1,\ldots,p$;

$$|I_{k\mu s}^{(5)}(t)| \leq |f_k(T)|e^{\gamma\rho_k(t-T)} + |f_k(0)|e^{\gamma\rho_k(t-2T)} + e^{\gamma\rho_k(t-T)}\int_0^T |f_k'(\tau)|\, e^{-\gamma\rho_k(T-\tau)}\, d\tau,$$

$$for \quad s = 1,\ldots,p, \quad \mu = p+1,\ldots,2p;$$

$$|I_{k\mu s}^{(6)}(t)| \leq |f_k(T)|e^{\gamma\rho_k(t-2T)} + |f_k(0)|e^{\gamma\rho_k(t-T)} + e^{\gamma\rho_k(t-T)}\int_0^T |f_k'(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau,$$

$$for \quad s,\mu = p+1,\ldots,2p; \tag{30}$$

Similarly, similar estimates also hold for integrals $J_{k\mu}^{(1)}(t), J_{k\mu}^{(2)}(t), J_{k\mu s}^{(i)}(t), i = 3,4,5,6.$

The proof of Lemma 3 follows from the fact that for p – even the following inequalities hold:

$$\mathrm{Re}\omega_1 \leq \cdots \leq \mathrm{Re}\omega_p = -\gamma < 0 < \gamma = \mathrm{Re}\omega_{p+1} \leq \cdots \leq \mathrm{Re}\omega_{2p}.$$

For example, let us prove an estimate for $|I_{k\mu}^{(1)}(t)|$. The following identity holds

$$I_{k\mu}^{(1)}(t) = \rho_k\omega_\mu \int_0^t e^{\rho_k\omega_\mu(t-\tau)}\, f_k(\tau)d\tau = -\int_0^t f_k(\tau)\,\frac{d}{d\tau}e^{\rho_k\omega_\mu(t-\tau)}\, d\tau =$$

$$-f_k(t) + f_k(0)e^{\rho_k\omega_\mu t} + \int_0^t f_k'(\tau)\, e^{\rho_k\omega_\mu(t-\tau)}\, d\tau.$$

This implies the required estimate for $|I_{k\mu}^{(1)}(t)|$. The other statements in Lemma 3 are proved similarly.

Lemma 4 *There is an estimate for the integral*

$$|I^{(6)}_{k\mu s}(t)|^2 \leq |f_k(T)|^2 + |f_k(0)|^2 + \frac{1}{\gamma}\int_0^T |f'_k(\tau)|^2 d\tau. \tag{31}$$

For other quantities $|I^{(j)}_{k\mu s}(t)|^2$, $j = \overline{1,5}$, *similar estimates hold. Similar estimates also hold for quantities* $|J^{(j)}_{k\mu s}(t)|^2$, $j = \overline{1,6}$.

Proof 1 *For* $s > p$ *and* $\mu > p$ *we write out the value using relations* (30)

$$|I^{(6)}_{k\mu s}(t)|^2 \leq \Big(|f_k(T)|e^{\gamma\rho_k(t-2T)} + |f_k(0)|e^{\gamma\rho_k(t-T)} + e^{\gamma\rho_k(t-T)}\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau\Big)^2 \leq$$

$$\leq 3\Big(|f_k(T)|^2 e^{2\gamma\rho_k(t-2T)} + |f_k(0)|^2 e^{2\gamma\rho_k(t-T)} + e^{2\gamma\rho_k(t-T)}(\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau)^2\Big).$$

Since $0 < t < T$, *then* $(t-2T) < (t-T) < 0$. *Therefore, the following estimates are valid*

$$|f_k(T)|^2 e^{2\gamma\rho_k(t-2T)} \leq |f_k(T)|^2,$$

$$|f_k(0)|^2 e^{2\gamma\rho_k(t-T)} \leq |f_k(0)|^2,$$

$$e^{2\gamma\rho_k(t-T)}(\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau)^2 \leq (\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau)^2 \leq$$

$$\int_0^T |f'_k(\tau)|^2 d\tau \cdot \int_0^T e^{-2\gamma\rho_k\tau}\, d\tau = \frac{1-e^{-2\gamma\rho_k T}}{2\gamma\rho_k}\cdot\int_0^T |f'_k(\tau)|^2 d\tau \leq \frac{1}{\gamma}\int_0^T |f'_k(\tau)|^2 d\tau.$$

This implies (31). *Lemma 4 is proved. Other quantities* $|I^{(j)}_{k\mu s}(t)|^2$, $j = \overline{1,5}$ *are estimated similarly.*

4 Proof of solvability of problem (1)–(2)–(3)

In Section 3, formal representations of the solution to problem (1)–(2)–(3) are obtained. The solutions to problem (1)–(2)–(3) have representations (28) and (29). Now we will justify the convergence of these representations.

From Lemma 3 and representation (28) follow the necessary estimates for $\|u(\cdot,t)\|^2_{L_2(0,1)}$, $\|\frac{\partial u(\cdot,t)}{\partial x})\|^2_{L_2(0,1)}$.

First, let us evaluate the expression $\|u(\cdot,t)\|^2_{L_2(Q)} + \|\frac{\partial u(\cdot,t)}{\partial x})\|^2_{L_2(0,1)}$. To do this, let's consider

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} = \int_0^1 (u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2})\,\overline{u(x,t)}\, dx.$$

From (28) it follows that

$$u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{2p}\sum_{k=0}^{\infty} v_k(x)\cdot\, A_k(t).$$

We multiply the last expression by (29) as a scalar. As a result, we have

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} = \int_0^1 \frac{1}{2p}\Big(\sum_{k=0}^{\infty} v_k(x)\cdot\, A_k(t)\Big)\frac{1}{2p}\,\Big(\sum_{j=0}^{\infty}\frac{1}{\rho_j^{2p}}\cdot\overline{v_j^*(x)\,A_j^*(t)}\Big)\,dx =$$

$$\frac{1}{4p^2}\,\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{\rho_j^{2p}}A_k(t)\cdot\overline{A_j^*(t)}\cdot\int_0^1 v_k(x)\cdot\,\overline{v_j^*(x)}\,dx = \frac{1}{4p^2}\,\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}\,A_k(t)\cdot\overline{A_k^*(t)}.$$

Thus, we get

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x})\|^2_{L_2(0,1)} = \frac{1}{4p^2}\,\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}\,A_k(t)\cdot\overline{A_k^*(t)}. \tag{32}$$

From (32) we obtain the following upper bound

$$\frac{1}{4p^2}\,\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}\,|A_k(t)\cdot\overline{A_k^*(t)}| \le \frac{1}{4p^2}\,\sum_{k=0}^{\infty}\frac{1}{2}\Big(|\frac{A_k(t)}{\rho_k^p}|^2 + |\frac{\overline{A_k^*(t)}}{\rho_k^p}|^2\Big) \le$$

$$\frac{1}{8p^2}\,\sum_{k=0}^{\infty}\,\Big(|A_k(t)|^2 + |\overline{A_k^*(t)}|^2\Big). \tag{33}$$

Taking into account (28), we estimate the expression from above

$$|A_k(t)|^2 \le M\cdot\Big[\sum_{\mu=1}^{p}|I^{(1)}_{k\mu}(t)|^2 + \sum_{\mu=p+1}^{2p}|I^{(2)}_{k\mu}(t)|^2 + \sum_{\mu=1}^{p}\sum_{s=1}^{p}|I^{(3)}_{k\mu s}(t)|^2 +$$

$$\sum_{\mu=1}^{p}\sum_{s=p+1}^{2p}|I^{(4)}_{k\mu s}(t)|^2 + \sum_{\mu=p+1}^{2p}\sum_{s=1}^{p}|I^{(5)}_{k\mu s}(t)|^2 + \sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p}|I^{(6)}_{k\mu s}(t)|^2\Big]. \tag{34}$$

From Lemma 3, as well as from Lemma 4, the following inequality follows

$$|A_k(t)|^2 \le M\Big[|f_k(0)|^2 + |f_k(t)|^2 + |f_k(T)|^2 + \frac{1}{\gamma}\int_0^T |f_k'(\tau)|^2 d\tau\Big]. \tag{35}$$

Summing both parts of inequality (35) over k, we have

$$\sum_{k=0}^{\infty}|A_k(t)|^2 \le M\,p^2\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)} +$$

$$\|f(\cdot,T)\|^2_{L_2(0,1)} + \frac{1}{\gamma}\int_0^T \|\frac{\partial f(\cdot,\tau)}{\partial\tau}\|^2_{L_2(0,1)} d\tau\Big]. \tag{36}$$

Here we use the inequalities from Lemma 2.
Using the statements of the second part of Lemma 3, we have a similar estimate for the expression

$$\sum_{k=0}^{\infty} |\overline{A_k^*(t)}|^2 \leq M\,p^2\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)} +$$

$$\|f(\cdot,T)\|^2_{L_2(0,1)} + \frac{1}{\gamma}\int_0^T \|\frac{\partial f(\cdot,\tau)}{\partial \tau}\|^2_{L_2(0,1)} d\tau\Big]. \tag{37}$$

As a result, from inequalities (32), (34)–(35)–(36)–(37) we obtain the estimate

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x})\|^2_{L_2(0,1)} \leq M_1 \cdot \Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)} +$$

$$\|f(\cdot,T)\|^2_{L_2(0,1)} + \frac{1}{\gamma}\int_0^T \|\frac{\partial f(\cdot,\tau)}{\partial \tau}\|^2_{L_2(0,1)} d\tau\Big]. \tag{38}$$

Scalar multiplication of the expression $l(t,\frac{\partial}{\partial t})u(\cdot,t)$ by $u(\cdot,t)$ we get

$$\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle = \int_0^1 \Big[f(x,t) - \frac{1}{2p}\sum_{k=0}^{\infty} v_k(x)\cdot A_k(t),\, \overline{u(x,t)}\Big]\, dx =$$

$$\int_0^1 f(x,t)\,\overline{u(x,t)}\,dx - \frac{1}{2p}\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}\,A_k(t)\cdot\overline{A_k^*(t)}. \tag{39}$$

Next, using the above estimates, we obtain

$$|\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle| \leq \|u(\cdot,t)\|_{L_2(0,1)}\cdot\|f(\cdot,t)\|_{L_2(0,1)} + \frac{1}{2p}\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}\,|A_k(t)\cdot\overline{A_k^*(t)}| \leq$$

$$\frac{1}{2}\Big(\|u(\cdot,t)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)}\Big) + M\cdot\sum_{k=0}^{\infty}\Big(|A_k(t)|^2 + \overline{|A_k^*(t)|}^2\Big). \tag{40}$$

Summing up inequalities (38) and (40), we get as a result

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} + |\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle| \leq$$

$$M_1\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)} + \|f(\cdot,T)\|^2_{L_2(0,1)} + \frac{1}{\gamma}\int_0^T \|\frac{\partial f(\cdot,\tau)}{\partial \tau}\|^2_{L_2(0,1)} d\tau\Big]. \tag{41}$$

Now it remains to integrate both parts of inequality (41) over t from 0 to T. As a result, we have

$$\int_0^T \Big[\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} + |\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle|\Big]\, dt \leq$$

$$M_2\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,T)\|^2_{L_2(0,1)} + \int_0^T \|f(\cdot,t)\|^2_{L_2(0,1)}\, dt + \int_0^T \|\frac{\partial f(\cdot,t)}{\partial t}\|^2_{L_2(0,1)} dt\Big]. \tag{42}$$

As a result, for the solution $u(x,t)$ we have the required estimate.

5 Proof of the uniqueness of the solution of problem (1)–(2)–(3)

In the previous section, sufficient conditions for solvability on the right-hand side $f(x,t)$ of equation (1) were found.

Now we will prove the uniqueness of the solution $u(x,t)$ of problem (1)–(2)–(3). To do this, we denote by A the operator corresponding to Ionkin's problem (5)–(6)–(7). We also introduce the operator B, defined by the differential expression $l(t,\frac{d}{dt})$ according to the formula

$$Bw(t) = l(t,\frac{d}{dt})w(t), \quad 0 < t < T$$

on the domain of definition

$$D(B) = \{w(t) \in W_1^{2p}(0,T) : U_j(w) = 0, j = 1, 2, ..., 2p\}.$$

Let us introduce for $s = 1,\dots,2p$ solutions $\kappa_s(t,\mu)$ of the homogeneous equation

$$l^{+}(t,\frac{d}{dt})\,\ae_s(t,\mu) = \mu \cdot \kappa_s(t,\mu), \quad 0 < t < T$$

with inhomogeneous conditions

$$U_j^*(\ae_s(\cdot,\mu)) = \delta_{j,s-1} \cdot \Delta^*(\mu), \quad j = 1,\dots,2p.$$

Here the expression $l^{+}(t,\frac{d}{dt})$ is the adjoint differential expression to the expression $l(t,\frac{d}{dt})$. The set of linear forms $U_1^*, U_2^*, ..., U_{2p}^*$ defines the domain of the adjoint operator B^*, that is

$$D(B^*) = \{w(t) \in W_1^{2p}(0,T) : \ U_j^*(w) = 0, j = 1, 2, ..., 2p\}.$$

Here $\Delta^*(\mu)$ is the characteristic determinant of the operator B^*. Note that all solutions $\ae_s(t,\mu)$ represent entire functions of μ.

Let μ_0 be the zero of the characteristic determinant $\Delta^*(\mu)$ and its multiplicity be m_0. Then for any $s = 1,\dots,2p$ in the ordered row

$$\left[\kappa_s(t,\mu_0), \frac{1}{1!}\frac{\partial}{\partial\mu}\kappa_s(t,\mu_0), \cdots, \frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial\mu^{m_0-1}}\kappa_s(t,\mu_0)\right] \tag{43}$$

the first non-zero function represents the eigenfunction of the operator B^*, and the subsequent members of the row give a chain of associated functions generated by it.

Finally, for the sake of completeness, we present the Lagrange formula [1]. For any two functions $w(t)$ and $R(t)$ from $W_2^{2p}(0,T)$, the identity holds

$$\int_0^T l(t,\frac{d}{dt})w(t)\overline{R(t)}dt - \int_0^T w(t)\overline{l^{+}(t,\frac{d}{dt})R(t)}dt =$$

$$\sum_{j=1}^{2p}\left[U_j(w) \cdot \overline{U_{4p-j+1}^*(R)} - U_{j+2p}(w) \cdot \overline{U_j^*(R)}\right], \tag{44}$$

where the linear forms $U_{2p+1}(\cdot),...,U_{4p}(\cdot)$ are chosen so that the system of $4p$ forms $U_1(\cdot),...,U_{4p}(\cdot)$ is a linearly independent system of linear forms. According to the results of the monograph [1], the set of linear forms $U_1^*(\cdot),...,U_{4p}^*(\cdot)$ is determined uniquely by the system of forms $U_1(\cdot),...,U_{4p}(\cdot)$.

Now we proceed to the proof of the uniqueness of the solution to problem (1)–(2)–(3). Consider $u(x,t)$ the solution to the homogeneous boundary value problem (1)–(2)–(3) for $f \equiv 0$ and show that $u(x,t) \equiv 0$ for $(x,t) \in Q_T$. For a fixed $x \in (0,1)$ we introduce the function

$$F_s(x,\mu) = \int_0^T u(x,t)\,\overline{\kappa_s(t,\overline{\mu})}dt. \tag{45}$$

t is not difficult to see that

$$F_s(x,\mu) - \frac{d^2}{dx^2}F_s(x,\mu) = \int_0^T (u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2})\,\overline{\kappa_s(t,\overline{\mu})}dt =$$

$$\int_0^T l(t,\frac{\partial}{\partial t})u(x,t)\cdot\overline{\kappa_s(t,\overline{\mu})}dt. \tag{46}$$

According to the Lagrange formula (44) we have

$$F_s(x,\mu) - \frac{d^2}{dx^2}F_s(x,\mu) = \int_0^T u(x,t)\,\overline{l^+(t,\frac{d}{dt})\kappa_s(t,\overline{\mu})}dt - U_{s+2p}(u(x,\cdot))\,\overline{\Delta^*(\overline{\mu})}$$

$$= \mu\int_0^T u(x,t)\,\overline{\kappa_s(t,\overline{\mu})}dt - U_{s+2p}(u(x,\cdot))\,\overline{\Delta^*(\overline{\mu})}$$

$$= \mu F_s(x,\mu) - U_{s+2p}(u(x,\cdot))\,\overline{\Delta^*(\overline{\mu})}. \tag{47}$$

Note the connection between $\Delta(\lambda)$ and $\Delta^*(\lambda)$. For all complex λ the identity holds

$$\Delta(\lambda) = \overline{\Delta^*(\overline{\lambda})}.$$

Therefore, for $s = 1,\ldots,2p$, equality (47) can be rewritten as

$$F_s(x,\mu) - \frac{d^2}{dx^2}F_s(x,\mu) = \mu F_s(x,\mu) - U_{s+2p}(u(x,\cdot))\,\Delta^*(\mu). \tag{48}$$

which is valid for all $x \in (0,1)$ and all complex μ.

If μ_0 is zero of the characteristic determinant $\Delta(\lambda)$ of multiplicity m_0, then for all $s = 1,\ldots,2p$ and all $x \in (0,1)$ from (48) the equalities follow

$$F_s(x,\mu_0) - \frac{d^2}{dx^2}F_s(x,\mu_0) = \mu_0 F_s(x,\mu_0),$$

$$\frac{1}{1!}\frac{\partial}{\partial\mu}\big(F_s(x,\mu) - \frac{\partial^2}{\partial x^2}F_s(x,\mu)\big)\big|_{\mu=\mu_0} = \mu_0\frac{1}{1!}\frac{\partial}{\partial\mu}(F_s(x,\mu)\big|_{\mu=\mu_0} + F_s(x,\mu_0),$$

$$\dots$$

$$\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial\mu^{m_0-1}}\big(F_s(x,\mu)-\frac{\partial^2}{\partial x^2}F_s(x,\mu)\big)\big|_{\mu=\mu_0}$$

$$=\mu_0\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}F_s(x,\mu)}{\partial\mu^{m_0-1}}\big|_{\mu=\mu_0}+\frac{1}{(m_0-2)!}\frac{\partial^{m_0-2}F_s(x,\mu)}{\partial\mu^{m_0-2}}\big|_{\mu=\mu_0}. \tag{49}$$

Since, according to the condition of the theorem, $\Delta(\lambda_k)\neq 0$ for all $k\geq 0$, then no λ_k can coincide with the eigenvalue μ_0. Therefore, from relations (49) it follows that for all $x\in(0,1)$ the equalities are satisfied

$$F_s(x,\mu_0)=0,\ \frac{1}{1!}\frac{\partial F_s(x,\mu)}{\partial\mu}\big|_{\mu=\mu_0}=0,\dots,\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}F_s(x,\mu)}{\partial\mu^{m_0-1}}\big|_{\mu=\mu_0}=0. \tag{50}$$

Thus, for all $x\in(0,1)$, the complex number μ_0 is a zero of the function $F_s(x,\mu)$ of multiplicity no less than m_0.

Since $F_s(x,\mu)$ is an entire function of μ and each zero μ_0 of the characteristic determinant $\Delta(\mu)$ of multiplicity m_0 is also a zero of $F_s(x,\mu)$ of multiplicity not less than m_0, then the ratio $\frac{F_s(x,\mu)}{\Delta(\mu)}$ is also an entire function of μ. According to the methodology of V. A. Ilyin [23] we multiply the function $F_s(x,\mu)$ scalarly by the root function $v_k(x)$, $k\geq 0$ of the operator A and denote them by

$$G_{sk}(\mu)=\int_0^1 F_s(x,\mu)\overline{v_k(x)}\,dx,\quad k\geq 0,\quad 1\leq s\leq 2p. \tag{51}$$

The multiplicities of zeros in μ of the functional $G_{sk}(\mu)$ are not less than the multiplicities of zeros of the functions $F_s(x,\mu)$. We also introduce the functions

$$Q_{sk}(\mu)\equiv\frac{G_{sk}(\mu)}{\Delta(\mu)}=\int_0^1\overline{v_k(x)}\int_0^T u(x,t)\,\frac{\overline{æ_s(t,\overline{\mu})}}{\Delta(\mu)}\,dt\,dx, \tag{52}$$

which also represent entire functions of μ.

Further analysis of entire functions $Q_{sk}(\mu)$ is based on the technique of estimating the orders of growth and types of entire functions. Note that the entire function $Q_{sk}(\mu)$ does not depend on the choice of the fundamental system of solutions of the homogeneous equation

$$l^+(t,\frac{d}{dt})\,R(t)=\mu\cdot R(t),\quad 0<t<T.$$

Let $\mu=\rho^{2p}$. Let ρ be an arbitrary complex number from the sector $S_0=\{\rho\in\mathbb{C}|\,0<\arg\rho<\frac{\pi}{2p}\}$. Let us enumerate the numbers $\omega_1,\omega_2,\dots,\omega_{2p}$ in the following order

$$\mathrm{Re}(\rho\omega_1)\leq\mathrm{Re}(\rho\omega_2)\leq\dots\leq\mathrm{Re}(\rho\omega_p)<0<\mathrm{Re}(\rho\omega_{p+1})\leq\dots\leq\mathrm{Re}(\rho\omega_{2p}), \tag{53}$$

when ρ lies strictly inside the sector S_0.

Let us choose a fundamental system of solutions of the homogeneous adjoint equation

$$l^+(t,\frac{d}{dt})\,h(t)=-\rho^{2p}\cdot h(t),\quad 0<t<T,$$

so that the asymptotic relations are satisfied

$$h_1(t,\rho) = e^{\rho\omega_1 t}[1+O(1/\rho)], \dots, h_n(t,\rho) = e^{\rho\omega_{2p} t}[1+O(1/\rho)],\ \rho\in S_0,\ \rho\to\infty. \tag{54}$$

As a result [1] for any ρ from the sector S_0 we have an asymptotic representation of the characteristic determinant $\widetilde{\Delta}(\rho)$ for $\rho\to\infty$, written through the fundamental system of solutions $\{h_1(t,\rho),\dots,h_{2p}(t,\rho)\}$.

In the work [2] the conjugate linear forms $U^*_{2p}(\cdot),\dots,U^*_1(\cdot)$ are written out explicitly. Taking into account their representation for $\rho\in S_0$, $\rho\to\infty$ we have for $j\le p$

$$U^*_{2p}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_1)}[1],$$

$$U^*_{2p-1}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_1)}[0],$$

$$\dots$$

$$U^*_{2p-2m+2}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_m)}[1],$$

$$U^*_{2p-2m+1}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_m)}[0],$$

$$U^*_{r}(h_j) = (\rho\omega_j)^{(2p-1-\nu_1)}[\overline{\alpha}_1],$$

$$\dots$$

$$U^*_{1}(h_j) = (\rho\omega_j)^{(2p-1-\nu_r)}[\overline{\alpha}_r].$$

Similarly, when $j>p$ for $\rho\in S_0$, $\rho\to\infty$ we have

$$U^*_{2p}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_1)}e^{\rho\omega_j T}[0],$$

$$U^*_{2p-1}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_1)}e^{\rho\omega_j T}[1],$$

$$\dots$$

$$U^*_{2p-2m+2}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_m)}e^{\rho\omega_j T}[0],$$

$$U^*_{2p-2m+1}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_m)}e^{\rho\omega_j T}[1],$$

$$U^*_{r}(h_j) = (\rho\omega_j)^{(2p-1-\nu_1)}e^{\rho\omega_j T}[\overline{\beta}_1],$$

$$\dots$$

$$U^*_{1}(h_j) = (\rho\omega_j)^{(2p-1-\nu_r)}e^{\rho\omega_j T}[\overline{\beta}_r].$$

Here it is designated for brevity $[a] = a + O(1/\rho)$.

We substitute all these expressions into the characteristic determinant

$$\widetilde{\Delta}^*(\overline{\mu}) = \det(U^*_\nu(h_j)) = \rho^{\widehat{\alpha}}\, e^{\rho(\omega_{p+1}+\ldots+\omega_{2p})T}\,\Delta^*_0, \tag{55}$$

where

$$\widehat{\alpha} = 2[2p-1-\gamma_1+\ldots+2p-1-\gamma_m] + (2p-1-\nu_1)+\ldots+(2p-1-\nu_r),$$

$$\Delta^*_0 = [\theta^*_0].$$

The number θ_0^* is nonzero, since according to the work of [2] the conjugate linear forms $U_{2p}^*(\cdot), \ldots, U_1^*(\cdot)$ are also Birchhoff regular.

For any ρ from the sector S_0 the asymptotic representation of $\overline{\widetilde{\kappa_1}}(t,\rho)$ for $\rho \to \infty$ has the following form, written in terms of the fundamental system of solutions

$$\overline{\widetilde{\kappa_1}}(t,\rho) = \frac{1}{(\rho\omega_p)^{2p-1-\gamma_1}} \rho^{\widehat{\alpha}} e^{\rho(\omega_{p+1}+\cdots+\omega_{2p})T} [\xi_0^*], \tag{56}$$

where ξ_0^* is some numerical determinant.

We obtain similar asymptotic representations for $\overline{\widetilde{\kappa_s}}(t,\rho)$ for $s > 1$.

From this it follows that

$$Q_{1k}(\mu) = \int_0^1 \left(\int_0^T \frac{\overline{\widetilde{\kappa_1}}(t,\overline{\mu})}{\widetilde{\Delta^*}(\rho)} u(x,t) dt \right) v_k(x)\, dx =$$

$$\int_0^1 \int_0^T \frac{[\xi_0^*]}{(\rho\omega_p)^{2p-1-\gamma_1} [\theta_0^*]} u(x,t)\, v_k(x)\, dt\, dx. \tag{57}$$

Using Riemann's lemma ([19], p. 496), we easily obtain

$$\lim_{|\rho| \to \infty} Q_{1k}(\mu) = 0, \quad \rho \in S_0.$$

It immediately follows from this

$$\lim_{\rho \to \infty} Q_{1k}(\mu) = 0, \quad \rho \in S_0.$$

Thus, along all rays $\rho \in S_0$ and $\rho \to \infty$ we have the limit equality

$$\lim_{\rho \to \infty} Q_{1k}(\mu) = 0.$$

We obtain similar asymptotic representations for $Q_{sk}(\mu)$ for $s > 1$ and for all $k = 0, 1, 2, \ldots$.

Exactly the same analysis can be carried out for the sector $\rho \in S_1$, where $S_1 = \{\rho \in \mathbb{C} |\ \frac{\pi}{2p} < \arg \rho < \frac{\pi}{p}\}$.

Therefore, according to the Phragmen-Lindelof and Liouville theorems ([20], p. 203) for functions of finite order we obtain that

$$Q_{sk}(\mu) \equiv 0 \quad \text{при всех} \quad \mu \in \mathbb{C}.$$

From here for any $k = 0, 1, 2, \ldots$ and for any $s = 1, \ldots, 2p$ we have

$$\int_0^1 v_k(x) F_s(x,\mu)\, dx \equiv 0, \quad \forall \mu \in \mathbb{C}.$$

Then from the completeness of the system $\{v_k(x),\, k = 0, 1, 2, \ldots\}$ in $L_2(0,1)$ it follows that

$$F_s(x,\mu) \equiv 0, \quad \forall x \in (0,1), \quad \forall \mu \in \mathbb{C}, \quad s = 1, \ldots, 2p.$$

Therefore, we have

$$\int_0^T \overline{\widetilde{æ_s}(t,\overline{\mu})}\, u(x,t)\, dt \equiv 0, \quad \forall x \in (0,1), \quad \forall \mu \in \mathbb{C}, \quad s = 1, \dots, 2p.$$

It follows from this

$$\frac{1}{\nu!}\frac{\partial^\nu}{\partial \mu^\nu}\int_0^T \overline{\widetilde{\kappa_s}(t,\overline{\mu})}\, u(x,t)\, dt \equiv 0, \quad \forall x \in (0,1), \quad \forall \mu \in \mathbb{C}, \quad s = 1, \dots, 2p, \quad \forall \nu \geq 0. \tag{58}$$

Now, instead of μ in equality (58), we substitute μ_τ – an arbitrary eigenvalue of the operator B. The multiplicity of the eigenvalue μ_τ is considered equal to m_τ. Let the parameter ν in formula (58) take the values $1, 2, \dots, m_\tau - 1$. Then, by virtue of (43), from (58) we obtain that for any fixed $x \in (0,1)$ the function $u(x,t)$ is orthogonal to all eigenfunctions of the operator B^*. Since the system of eigenfunctions of the operator B^* is a complete system in $L_2(0,T)$, it follows from this that

$$u(x,t) \equiv 0, \quad \forall t \in (0,T), \quad x \in (0,1).$$

Thus, the uniqueness of the solution to problem (1)–(2)–(3) is completely proven.

References

[1] Naimark M.A., *Linear differential operators*, Nauka, 1969.

[2] Kesel'man G.M., On the unconditional convergence of expansions in terms of eigenfunctions of some differential operators, *Izv. Vuz. Math.*, 82-93. (1964).

[3] Ionkin, N.I., Solution of a boundary value problem of heat conduction theory with non-classical boundary condition, *Differ. Equ.*, **13** (1977): 294-304. (in Russian)

[4] Ionkin N.I., On the stability of one problem of heat conduction theory with a non-classical boundary condition, *Differ. Equ.*, **15** (1979): 1279-1283. (in Russian)

[5] Ilyin, V.A., On another generalization of the Bessel inequality and the Riesz-Fischer theorem for a Fourier series in a uniformly bounded orthonormal system, *Trans. of the Steklov Math. Inst.*, **219** (1997): 211-219.

[6] Kozhanov, A.I., Koshanov, B.D., Sultangazieva, Zh.B., New Boundary Value Problems for the Fourth-Order Quasi-Hyperbolic Type, *Sib. Elec. Math. Rep.*, **16** (2019): 1410-1436. https:doi.org/10.33048/semi.2019.16.098

[7] Kozhanov A.I., Pinigina, N.R., Boundary Value Problems for High-Order Nonclassical Differential Quantities, *Math. Notes.*, **101** (2017): 403-412. https:doi.org/10.4213/mzm11172

[8] Ashurov R.R., Mukhiddinova, A.T., Initial boundary value problems for hyperbolic equations with an elliptic operator of arbitrary order, *Bulletin of KRAUNC, Phys. and Math. Scien.*, **30** (2020): 8-19. https:doi.org/10.26117/2079-6641-2020-30-1-8-19

[9] Sabitov K.B., Vibrations of a beam with fixed ends, *Vest. Sam. state tech. univ., Ser. Phys.-math. scien.*, **19** (2015): 311-324. https://doi.org/10.14498/vsgtu1406

[10] Sabitov K.B., On the theory of initial-boundary value problems for the equation of rods and beams, *Differ. Equ.*, **53** (2017): 89-100. https://doi.org/10.1134/S0012266117010086

[11] Dezin A.A., Operator differential equations. Method of model operators in the theory of boundary value problems, *Proc. of the Steklov Math. Inst.*, **229** (2000): 175.

[12] Grisvard P., Equations operaationnelles abstraites et problemes aux limites *Ann. Scuola norm. super. Pisa.*, **21** (1967): 308-347.

[13] Dubinscii, Y.A., On an abstract theorem and its applications to boundary conditions problems for non-classical equations *Math. collection.*, **79** (1969): 91-117. http:dx.doi.org/10.1070/SM1969v008n01ABEH001111

[14] Romanko, V.K., Boundary value problems for one class of differential operators, *Differ. Equ.*, **10** (1974): 117-131.

[15] Orynbasarov, M., On the solvability of boundary value problems for parabolic and polyparabolic equations in a non-cylindrical region with non-smooth lateral boundaries, *Differ. Equ.*, **30** (1994): 151-161. https://doi.org/10.1137/0108027

[16] Kanguzhin B.E.; Koshanov B.D., Necessary and sufficient conditions for the solvability of boundary value problems for the polyharmonic equation, *Ufa Math. J.*, **2** (2010): 41-52.

[17] Aitzhanov S., Koshanov B., Kuntuarova, A., Solvability of Some Elliptic Equations with Nonlocal Boundary, *Mathematics*, **12** (2024): 4010. https://doi.org/10.3390/math12244010

[18] Shelukhin, V.V., Problem with time-averaged data for nonlinear parabolic equations, *Sib. Math. J.*, **32** (1991): 154-165. https:doi.org/10.1007/BF00972778

[19] Shelukhin, V.V., The problem of predicting ocean temperature from average data for the previous period of time, *Report RAN.*, **324** (1991): 760-764. https:doi.org/10.1007/BF00972778

[20] Fridman A., *Equations with Partial Derivatives of Parabolic Type*, Prentice-Hall: Englewood Cliffs, NJ, USA, 1964.

[21] Miranda K., *Equations with Partial Derivatives of Elliptic Type*, Springer: Berlin/Heidelberg, 1970.

[22] Bitsadze A.V., *Mixed Type Equations*, SAcademy of Sciences of the USSR Publ.: Moscow, Russia, 1959.

[23] Ilin V.A., On the solvability of mixed problems for hyperbolic and parabolic equations, *Russ. Math. Surv.*, **15** (1960): 97-154. https://doi.org/10.1070/RM1960v015n02ABEH004217

[24] Tikhonov I.V., Uniqueness theorems in linear nonlocal problems for abstract differential equations, *Izv. Math.*, **67** (2003): 133-166. https:doi.org/10.4213/im429

[25] Popov A.Y., Tikhonov I.V., Uniqueness classes in a time-nonlocal problem for the heat equation and the complex eigenfunctions of the Laplace operator, *Differ. Equ.*, **40** (2004): 396-405.

[26] Agmon S., On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, *Comm. Pure and Appl. Math.*, **15** (1962): 119-143. https://doi.org/10.1002/cpa.3160150203

[27] Crane M.G., On Hermitian operators with directional functionals, *Collection of works of the Inst. of Math. of the Acad. of Scien. of the Ukr. SSR.*, **10** (1948): 83-105.

[28] Kanguzhin B.E., Koshanov B.D., The Uniqueness Criterion for a Solution to a Boundary Value Problem for the Operator $\frac{\partial^{2p}}{\partial t^{2p}} - A$ with an Elliptic Operator A of Arbitrary Order, *Sib. Math. J.*, **63** (2022): 1083-1090. https://doi.org/10.1134/S0037446622060088

[29] Kanguzhin B.E., Koshanov B.D., Uniqueness Criteria for Solving a Time Nonlocal Problem for a High-Order Differential Operator Equation $l(\cdot) - A$ with a Wave Operator with Displacement, *Symmetry*, **14**:6 (2022). https://doi.org/10.3390/sym14061239

[30] Kanguzhin B.E., Koshanov B.D., Criteria for the uniqueness of a solution to a time-nonlocal problem for an operator-differential equation $l(\cdot) - A$ with the Tricomi operator A, *Differ. Equ.*, **59** (2023): 4-14. https://doi.org/10.1134/S0012266123010019

[31] Kanguzhin B., Kaiyrbek Z., Uaissov B. In one scenario, the development of a defect in the attachment of the rod . Journal of Mathematics, Mechanics and Computer Science, (2024), 121(1), 46-51. https://doi.org/10.26577/JMMCS202412115

[32] Kanguzhin, B. E., Auzerkhan G., Tastanov M.G. The method of variation of arbitrary constants in the case of a system of linear differential equations of different orders. Journal of Mathematics, Mechanics and Computer Science, (2021), 111(3), 16-27. https://doi.org/10.26577/JMMCS.2021.v111.i3.02

[33] Kanguzhin, B., Akanbay E., Madibayuly, Z. Jointing of thin elastic rods and generalized Kirchhoff conditions at nodes. Journal of Mathematics, Mechanics and Computer Science, (2020), 106(2), 58-68. https://doi.org/10.26577/JMMCS.2020.v106.i2.06

[34] Kanguzhin B., Dosmagulova K., Akanbay Y. On the laplace-beltrami operator in stratified sets composed of punctured circles and segments. Journal of Mathematics, Mechanics and Computer Science, (2025), 125(1). https://doi.org/10.26577/JMMCS2025125103

Авторалар туралы мәлімет:

Гулам Хазрат Аймал Раса (корреспондент автор) – Кабул білім университетінің оқытушысы (Кабул, Ауганстан, электрондық пошта: aimal.rasa14@gmail.com);

Мадияров Мураткан Набенович – Сәрсен Аманжолов атындағы Шығыс Қазақстан университетінің профессоры (Өскемен, Қазақстан, электрондық пошта: muratkanmadiyarov@gmail.com).

Мустафина Мергуль Оралбековна – Сәрсен Аманжолов атындағы Шығыс Қазақстан университетінің 3-курс PhD докторанты (Өскемен, Қазақстан, электрондық пошта: mergulm011@gmail.com).

Сведения об авторах:

Гулам Хазрат Аймал Раса (корреспондент автор) – преподаватель Кабульского образовательного университета (Кабул, Афганистан, электронная почта: aimal.rasa14@gmail.com);

Мадияров Мураткан Набенович – профессор Восточно-Казахстанского университета имени Сарсена Аманжолова (Усть-Каменогорск, Казахстан, электронная почта: muratkanmadiyarov@gmail.com).

Мустафина Мергуль Оралбековна – PhD докторант 3-курса Восточно-Казахстанского университета имени Сарсена Аманжолова (Усть-Каменогорск, Казахстан, электронная почта: mergulm011@gmail.com).

Information about authors:

Ghulam Hazrat Aimal Rasa (corresponding author) – lecturer at Kabul University of Education (Kabul, Afghanistan, email: aimal.rasa14@gmail.com);

Madiyarov Muratkan Nabenovich – Professor at the Sarsen Amanzholov East Kazakhstan University (Ust-Kamenogorsk, Kazakhstan, email: muratkanmadiyarov@gmail.com).

Mustafina Mergul Oralbekovna – 3rd year PhD student, Sarsen Amanzholov East Kazakhstan University (Ust-Kamenogorsk, Kazakhstan, email: mergulm011@gmail.com).