IRSTI 27.25.19 DOI: https://doi.org/10.26577/JMMCS202512845

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ON THE OPTIMAL RECOVERY OF FUNCTIONS FROM THE CLASS $W_2^{\mathrm{r},\alpha}$

In this paper, the problem of optimal recovery of functions from the anisotropic Sobolev class $W_2^{\mathrm{r},\alpha}$ in the power-logarithmic scale by the values of linear functionals is solved in the Hilbert metric, and the limiting error of the optimal computing unit is found. Thus, the following results are obtained here: 1) The exact order of error of optimal recovery of functions $f \in W_2^{\mathrm{r},\alpha}$ by computing units constructed based on the values of linear functionals defined on the class under consideration has been established;2) The computing unit that realizes the established exact order is written out in explicit form; 3) The limiting error of the specified optimal computing unit is found, which preserves its optimality and can not be improved in order. The actuality of the problem studied here is that, firstly, the class $W_2^{\mathrm{r},\alpha}$ is a finer scale of classifications of periodic functions by the rate of decrease of their trigonometric Fourier coefficients than the anisotropic Sobolev class in the power scale W_2^{r}, secondly, the set of computing units $(l^{(N)}, \varphi_N)$ with linear functionals is a fairly wide set containing all partial sums of Fourier series over all possible orthonormal systems, all possible finite convolutions with special kernels, as well as all finite sums of approximation used in orthowidths, linear widths and greedy algorithms.

Key words: optimal recovery, optimal computing unit, limiting error, exact order, anisotropic Sobolev class, trigonometric Fourier coefficients, linear functionals

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$W_2^{\mathrm{r};\alpha}$ класы функцияларының оптималды қалыптастыруы туралы

Бұл жұмыста гильберттік метрикада дәреже – логарифмдік шкаладағы анизотропты Соболев $W_2^{\mathrm{r},\alpha}$ класы функцияларын оптималды қалыптастыру есебі шешілген және оптималды есептеу агрегатының шектік қателігі табылған. Сонымен, мұнда келесі нәтижелер алынған: 1) $f \in W_2^{\mathrm{r},\alpha}$ функцияларын қарастырып отырған класта анықталған сызы́тық функционалдар мәндері арқылы ќұрылған есептеу агрегаттарымен оптималды қалыптастырудағы қателіктің дәл реті тағайындалған; 2) Тағайындалған дәл реті жүзеге асыратын есептеу агрегаты айқын түрде жазылып келтірілген; 3) Көрсетілген оптималды есептеу агрегатының оның оптималдығын сақтайтын және реті бойынша жақсармайтын шектік қателігі табылған. Осында зерттеліп отырылған есептің өзектілігі мына жайттар арқылы түсіндіріледі: біріншіден, периодты функцияларды олардың тригонометриялық Фурье коэффициенттерінің кему жылдамдығы бойынша классификациялап сипаттауда $W_2^{\mathrm{r},\alpha}$ класы дәрежелік шкаладағы анизотропты W_2^{r} класымен салыстырғанда дәл әрі терең шкаладағы класс болады, екіншіден, сызықтық функционалдарға сәйкес $(l^{(N)}, \varphi_N)$ есептеу агрегаттарының жиыны жеткілікті кең жиын болып табылады, өйткені, бұл жиын құрамына барлық мүмкін ортонормаланған жүйелерге сәйкес Фурье қатарларының барлық дербес қосындылары, арнайы өзегі бар ақырлы конволюциялар, сонымен бірге, ортодиаметрлерде, сызықтық диаметрлерде және гриди алгоритмдердегі жуықтауларда қолданылатын барлық ақырлы қосындылар кіреді.

Түйін сөздер: оптималды қалыптастыру, оптималды есептеу агрегаты, шектік қателік, дәл рет, анизотропты Соболев класы, тригонометриялық Фурье коэффициенттері, сызықтық функционал

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Об оптимальном восстановлении функций из класса $W_2^{\mathrm{r};\alpha}$

В данной работе в гильбертовой метрике решена задача оптимального восстановления функций из анизотропного класса Соболева в степенно – логарифмической шкале $W_2^{\mathrm{r},\alpha}$ по значениям линейных функционалов и найдена предельная погрешность оптимального вычислительного агрегата. Тем самым, здесь получены следующие результаты: 1) Установлен точный порядок погрешности оптимального восстановления функций $f \in W_2^{\mathrm{r},\alpha}$ вычислительными агрегатами, построенными по значениям линейных функционалов, определенных на рассматриваемом классе; 2) В явном виде выписан вычислительный агрегат, реализующий установленный точный порядок; 3) Найдена предельная погрешность указанного оптимального вычислительного агрегата, сохраняющая его оптимальность и неулучшаемая по порядку. Актуальность изучаемой здесь задачи заключается в том, что во – первых, класс $W_2^{\mathrm{r},\alpha}$ является более тонкой шкалой классификаций периодических функций по скорости убывания их тригонометрических коэффициентов Фурье, чем анизотропный класс Соболева W_2^{r} в степенной шкале, во – вторых, множество вычислительных агрегатов $(l^{(N)}, \varphi_N)$ с линейными функционалами является достаточно широким множеством, содержащим все частичные суммы рядов Фурье по всевозможным ортонормированным системам, всевозможные конечные свертки со специальными ядрами, а также все конечные суммы приближения, использующиеся в ортопоперечниках, линейных поперечниках и жадных алгоритмах.

Ключевые слова: оптимальное восстановление, оптимальный вычислительный агрегат, предельная погрешность, точный порядок, анизотропный класс Соболева, тригонометрические коэффициенты Фурье, линейный функционал

1 Introduction

The problem of optimal recovery of functions of class $F = \{f(x) : x \in \Omega\}$ by the values $l_N^{(1)}(f), \ldots, l_N^{(N)}(f)$ of linear functionals $l_N^{(1)} : F \to \mathbb{C}, \ldots, l_N^{(N)} : F \to \mathbb{C}$ consist in establishing the exact order of quantity

$$\delta_N(L_N, F, L^q) = \inf_{(l^{(N)}, \varphi_N) \in L_N} \delta_N((l^{(N)}, \varphi_N), F, L^q), \tag{1}$$

where $L^q \equiv L^q(\Omega)$ is a set of measurable functions on Ω with finite norm

$$\|f\|_{L^q} = \begin{cases} \left(\int\limits_{\Omega} |f(x)|^q dx\right)^{1/q}, & \text{if} \quad 1 \le q < \infty; \\ \sup\limits_{x \in \Omega} \mathrm{vrai}|f(x)|, & \text{if} \quad q = \infty, \end{cases}$$

$$\delta_N((l^{(N)}, \varphi_N), F, L^q) = \sup_{f \in F} \left\| f(\cdot) - \varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); \cdot) \right\|_{L^q},$$

φ_N – an arbitrary function such that $\varphi_N(\tau_1,\ldots,\tau_N;x) \in L^q$ as a function of x for any $\tau_1,\ldots,\tau_N, l^{(N)} \equiv l^{(N)}(f) = (l_N^{(1)}(f),\ldots,l_N^{(N)}(f)), L_N = \{l^{(N)}\} \times \{\varphi_N\}$.

Here we note that by 'establishing the exact order of quantity (1)' we should understand finding a positive sequence $\{\psi_N\}_{N\geq 1}$ such that for some constants $A_1 > 0$ and $A_2 > 0$ independent of N, the inequalities $A_1\psi_N \leq \delta_N(L_N, F, L^q) \leq A_2\psi_N$ are satisfied for all N.

Everywhere below, the pair $(l^{(N)},\varphi_N) \in L_N$ will be called a computing unit, and the pair $(\tilde{l}^{(N)},\widetilde{\varphi}_N) \in L_N$ that realizes the exact order of quantity (1), an optimal computing unit.

The quantity (1) was first considered in [1] when recovering analytical functions in the L^2 metric. Then the research of the paper [1] was continued in [2]. In [3], the relationships of the quantity (1) with the known quantities - the Kolmogorov width, the Gelfand width, and the linear width - were studied. Further, in [4] - [8] the exact orders of (1) were established with an indication of the optimal computing units. Moreover, in [6] and [7], in addition to establishing the exact order of (1), the limiting errors of the optimal computing units were found (the definition of the limiting error is given below).

In [8], the Sobolev class $W_2^{r;\alpha}$ in the power-logarithmic scale was considered for the first time, and under certain conditions on the vector $\alpha = (\alpha_1,\ldots,\alpha_s)$, a theorem on the optimal recovery of functions from the considered class in the metric of the space $L^q, 2 \leq q \leq \infty$ was formulated. Here we managed to remove these conditions on the optimal recovery in the metric of L^2 and to find the limiting error of the optimal computing unit.

2 The definitions of the class $W_2^{\mathrm{r},\alpha}$ and of the limiting error

First, we agree on the notation used. As usual, $[a]$ denotes the integer part of a number $a, m = (m_1,\ldots,m_s) \in Z^s, |J|$ denote the amount of elements of a finite set J. Here and throughout the text, for each vector $\mathrm{r} = (r_1,\ldots,r_s)$ with positive components , we set

$$\lambda \equiv \lambda(r_1,\ldots,r_s) = (1/r_1 + \cdots + 1/r_s)^{-1}.$$

For positive functions $f(x), x \geq 1$ and $g(x), x \geq 1$ the notation $f(x) \ll g(x)$ will mean the existence of some quantity $C > 0$, independent of the variable x, such that $f(x) \leq Cg(x)$ holds for all $x \geq 1$. And the simultaneous fulfillment of inequalities $f(x) \leq Cg(x)$ and $g(x) \leq Cf(x)$ is written as $f(x) \succ\prec g(x)$. Everywhere below, the symbol $\square$ will denote the end of the proof.

Let be given an integer number $s \geq 2$, vectors $r = (r_1,\ldots,r_s)$ and $\alpha = (\alpha_1,\ldots,\alpha_s)$ such that $r_i > 0$ and $\alpha_i \in \mathbb{R}$ for each $i = 2,3,\ldots,s$. The anisotropic Sobolev class $W_2^{r;\alpha} \equiv W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ in a power – logarithmic scale, by definition, consists of all summable on a cube $[0,1]^s$ and 1 – periodic by each variable functions $f(x) = f(x_1,\ldots,x_s)$, that satisfy the condition

$$\sum_{m\in Z^s} \left|\hat{f}(m)\right|^2 (\overline{m}_1^{2r_1}\ln^{2\alpha_1}(\overline{m}_1+1) + \cdots + \overline{m}_s^{2r_s}\ln^{2\alpha_s}(\overline{m}_s+1)) \leq 1,$$

where $\hat{f}(m) = \int\limits_{[0,1]^s} f(x)e^{-2\pi i(m,x)}dx, m \in Z^s$ are the trigonometric Fourier – Lebesgue coefficients of the function $f, \overline{m}_j = \max\{1;|m_j|\}$ for each $j = 1,\ldots,s$.

For each $f \in F$ calculating the values $l_N^{(1)}(f), \ldots, l_N^{(N)}(f)$ of the functionals $l_N^{(1)} : F \to \mathbb{C}, \ldots, l_N^{(N)} : F \to \mathbb{C}$, with few exceptions, cannot be exact. Therefore, for the optimal computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$, the problem arises of finding the error $\widetilde{\varepsilon}_N$ in calculating the values $\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f)$ functionals $\tilde{l}_N^{(1)} : F \to \mathbb{C}, \ldots, \tilde{l}_N^{(N)} : F \to \mathbb{C}$, that preserves optimality $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ and cannot be improved in order. The definition of $\widetilde{\varepsilon}_N$ was first given in [9] within the framework of the study «Computational(Numerical) Diameter». There, the error $\widetilde{\varepsilon}_N$ was called the limiting error of the optimal computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$. Further, in [10], an equivalent definition of $\widetilde{\varepsilon}_N$ was formulated within the general formulation of the problem of recovering the operator $T : F \to Y$, where F is a functional class, Y is a normed space. Now we present from [7] the definition of the limiting error, stated at $Tf = f$ and $Y = L^q, q \in [2, +\infty]$. The limiting error of an optimal computing $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ is defined as a sequence $\widetilde{\varepsilon}_N > 0$ such that

$$\triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), F, L^q) \succ\prec \delta_N(D_N, F, L^q) \quad \text{and}$$

$$\overline{\lim_{N\to\infty}} \frac{\triangle_N(\eta_N\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), F, L^q)}{\delta_N(D_N, F, L^q)} = +\infty$$

for any arbitrary slowly increasing to $+\infty$ positive sequence $\{\eta_N\}_{N\geq 1}$, where

$$\triangle_N(\varepsilon_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), F, L^q) =$$

$$= \sup_{f\in F} \sup_{|\gamma_N^{(1)}|\leq 1, \ldots, |\gamma_N^{(N)}|\leq 1} \left\| f(\cdot) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f) + \gamma_N^{(1)}\varepsilon_N, \ldots, \tilde{l}_N^{(N)}(f) + \gamma_N^{(N)}\varepsilon_N; \cdot) \right\|_{L^q}$$

for each positive sequence ε_N.

3 Formulations of the obtained results

Further, for the sake of brevity, we will use the following notations

$$\delta_N(L_N) \equiv \delta_N(L_N, W_2^{\mathrm{r};\alpha}, L^2), \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \equiv \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N), W_2^{\mathrm{r};\alpha}, L^2),$$

$$\triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \equiv \triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), W_2^{\mathrm{r};\alpha}, L^2).$$

We have proved the following two theorems.

Theorem 1. *Let the number $s \in \mathbb{N}\backslash 1$, vectors $\mathrm{r} = (r_1, \ldots, r_s), r_1 > 0, \ldots, r_s > 0$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$ be given, and let $\lambda \equiv (1/r_1 + \cdots + 1/r_s)^{-1} > 1/2$,*

$$N_i \equiv N_i(K) = [K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}], K \geq 2$$

for each $i \in 1, \ldots, s, N \equiv N(K) = \prod_{i=1}^{s}(2N_i + 1)$. Then there exists a quantity $C_0 > 0$ such that for all integers $K \geq C_0$ the relations

$$\delta_N(L_N) \succ\prec \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \succ\prec \frac{1}{N^\lambda(\ln N)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}}, \tag{2}$$

hold, here, the computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ is defined by the equalities

$$\tilde{l}_N^{(1)}(f) = \hat{f}(\widetilde{m}^{(1)}), \ldots, \tilde{l}_N^{(N)}(f) = \hat{f}(\widetilde{m}^{(N)}), \widetilde{\varphi}_N(z_1, \ldots, z_N; x) = \sum_{\tau=1}^{N} z_\tau e^{2\pi i(\widetilde{m}^{(\tau)}, x)},$$

where $\{\widetilde{m}^{(1)}, \widetilde{m}^{(2)}, \ldots, \widetilde{m}^{(N)}\}$ *is some ordering of the set*

$$A_K = \{m \in Z^s : |m_1| \le N_1, \ldots, |m_s| \le N_s\}.$$

Theorem 2. *The quantity* $\widetilde{\varepsilon}_N = \frac{1}{N^{\lambda+1/2}(\ln N)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}}$ *is the limiting error of the optimal computing unit*

$$(\tilde{l}^{(N)}, \widetilde{\varphi}_N) \equiv \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); x) = \sum_{\tau=1}^{N} f(\widetilde{m}^{(\tau)}) e^{2\pi i(\widetilde{m}^{(\tau)}, x)},$$

i.e.

$$\triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \succ\prec \delta_N(L_N) \tag{3}$$

and for any arbitrarily slowly increasing to $+\infty$ *positive sequence* $\{\eta_{N(K)}\}_{K\ge1}$ *the equality*

$$\overline{\lim_{K\to\infty}} \frac{\triangle_N(\eta_N\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N))}{\delta_N(L_N)} = +\infty \tag{4}$$

holds.

Remark 1. *Relations (2) under the conditions* $1/\min\{r_1, r_1+\alpha_1\}+\cdots+\min\{r_s, r_s+\alpha_s\} < 2$ *and* $r_i + \alpha_i > 0 (i = 1, \ldots, s)$ *on the vector* $\alpha = (\alpha_1, \ldots, \alpha_s)$ *obtained in [8].*

Remark 2. *In the case* $\alpha_1 = \alpha_2 = \cdots = \alpha_s = 0$ *the main results of the article [6] follow from Teorems 1 and 2.*

4 Auxiliary statements

Lemma 1 (see, Lemma 1.2.4 from [11]). *For each* $\gamma \in \mathbb{R}$ *there exists a quantity* $C_1(\gamma) \ge 2$ *such that for all integers* $K \ge C_1(\gamma)$ *the relation* $\ln(K \ln^\gamma K) \succ\prec \ln K$ *holds.*

Lemma 2. *Let a number* $B \ge 2$, *vectors* $\mathrm{r} = (r_1, \ldots, r_s), r_1 > 0, \ldots, r_s > 0$ *and* $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$ *be given, and let* $\gamma_i = \alpha_1/r_1 + \cdots + \alpha_s/r_s - \alpha_i/\lambda$ *for each* $i \in \{1, \ldots, s\}$. *Then there is an integer* $K_0 \ge 2$ *such that for all integers* $K \ge \max\{C_1(\gamma_i), K_0\}$ *the relations*

$$\ln(BN_i) \succ\prec \ln K \succ\prec \ln N (i = 1, \ldots, s), \tag{5}$$

hold, where $N_i \equiv N_i(K) = [K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}]$ *for each* $i = 1, \ldots, s, N \equiv N(K) = \prod_{i=1}^{s}(2N_i + 1)$.

Proof. Since for each $i \in \{1, \ldots, s\}$ the equality

$$\lim_{K\to\infty} K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i} = +\infty$$

is true, there exists some integer $K_0 \ge 2$ such for all integers $K \ge K_0$ the inequality $K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i} \ge B$ holds. Therefore,

$$\ln(BN_i) \le \ln\left(BK^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}\right) \le$$

$$\leq 2\ln\left(K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}\right) \leq \frac{2\lambda}{r_i}\ln(K\ln^{\gamma_i}K) \tag{6}$$

It is clear that

$$\ln(BN_i) \geq \ln\left(K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}\right) \geq \frac{\lambda}{r_i}\ln(K\ln^{\gamma_i}K). \tag{7}$$

According to Lemma 1, for all integers $K \geq C_1(\gamma_i)$ the relation $\ln(K\ln^{\gamma_i}K) \succ\prec \ln K$ is valid. Therefore, due to (6) and (7), for all integers $K \geq \max\{C_1(\gamma_i), K_0\}$ the relation

$$\ln(BN_i) \succ\prec \ln K, i \in \{1,\ldots,s\} \tag{8}$$

holds. From the definition of number N_i and the equality $N = \prod_{i=1}^{s}(2N_i+1)$ the inequalities $K \leq N \leq 2^{2s}K$ easily follow, hence, taking $K \geq 2$ into account, we obtain $\ln K \leq \ln N \leq (2s+1)\ln K$, which together with (8) will lead to the relations (5). $\square$

Lemma 3. *Let be given numbers $r > 0, \beta \neq 0$ and a positive, strictly increasing on the interval $(C_2, +\infty)$ function $\varphi(\delta)$, where $C_2 \geq 1$. If for some $C_3 > 0$ and $C_4 > 0$ the inequalities*

$$C_3\delta^r\ln^\beta(2\delta) \leq \varphi(\delta) \leq C_4\delta^r\ln^\beta(2\delta), \tag{9}$$

are true for each $\delta \in (C_2, +\infty)$, then there exist positive quantities $C_5 \geq 1, C_6 > 0$ and $C_7 > 0$ such that the inequalities

$$C_6\delta^{1/r}\ln^{-\beta/r}(2\delta) \leq \varphi^*(\delta) \leq C_7\delta^{1/r}\ln^{-\beta/r}(2\delta), \tag{10}$$

are satisfied for all $\delta \in (C_5, +\infty)$, where φ^* is the function inverse to φ.

Proof. Let us prove the lemma for case $\beta > 0$ (in the case $\beta < 0$, analogous arguments are carried out). Let

$$C_6 = (r^\beta C_4^{-1})^{1/r} \quad \text{and} \quad C_7 = (r^\beta C_3^{-1}2^\beta)^{1/r}. \tag{11}$$

Since

$$\lim_{\delta\to+\infty}\delta^{1/r}\ln^{-\beta/r}(2\delta) = +\infty, \tag{12}$$

there exists a positive quantity $C_8 > C_2$ such that the inequality $C_6\delta^{1/r}\ln^{-\beta/r}(2\delta) > C_2$ is satisfied for all $\delta \in (C_8, +\infty)$. Therefore, by virtue of the right-hand side of (9), we obtain

$$\varphi(C_6\delta^{1/r}\ln^{-\beta/r}(2\delta)) \leq \frac{C_4C_6^r\delta\ln^{-\beta}(2\delta)}{r^\beta}\ln^\beta(C_6^r\delta\ln^{-\beta}(2\delta)). \tag{13}$$

Since $\lim\limits_{\delta\to+\infty}\ln^{-\beta}(2\delta) = 0$, then for some positive $C_9 \geq C_8$ the inequality $\ln^{-\beta}(2\delta) \leq C_6^{-r}$ is satisfied for all $\delta \in (C_9, +\infty)$. Consequently, continuing (13), taking into account the first equality from (11), for all $\delta \in (C_9, +\infty)$ we have

$$C_6\delta^{1/r}\ln^{-\beta/r}(2\delta) \leq \varphi^*(\delta). \tag{14}$$

According to (12), there is a positive quantity $C_{10} > C_2$ such that the inequality $C_7\delta^{1/r}\ln^{-\beta/r}(2\delta) > C_2$ holds for all $\delta \in (C_{10}, +\infty)$. Therefore, by virtue of the left part of (9), we obtain

$$\varphi_N(C_7\delta^{1/r}\ln^{-\beta/r}(2\delta)) \geq \frac{C_3C_7\delta\ln^{-\beta}(2\delta)}{r^\beta}\ln^\beta(C_7^r\delta\ln^{-\beta}(2\delta)). \tag{15}$$

Since $\lim\limits_{\delta\to+\infty} C_7^r\sqrt{\delta}\ln^{-\beta}(2\delta) = +\infty$, then there is a quantity $C_{11} > C_{10}$ such that for all $\delta \in (C_{11}, +\infty)$ the inequality $C_7^r\sqrt{\delta}\ln^{-\beta}\delta \geq 1$ is satisfied. Therefore, using (15) and second equality from (11) we arrive to the inequality

$$C_7\delta^{1/r}\ln^{-\beta/r}(2\delta) \geq \varphi^*(\delta). \tag{16}$$

Now, if we take $C_5 = \max\{C_9, C_{11}\}$, then by virtue of inequalities (14) and (16), the inequalities (10) are satisfied for all $\delta \in (C_5, +\infty)$. $\square$

Lemma 4. *If $\lambda > 1/2$, then the trigonometric Fourier series of each function $f \in W_2^{\mathrm{r};\alpha}$ converges absolutely.*

Proof. It is clear that for all $\delta \in [1, +\infty)$ the relations

$$\ln(1+\delta) \succ\prec \ln(2\delta) \succ\prec \ln(A\delta) \tag{17}$$

hold, where $A > \max\{2, e^{-\alpha_1/r_1}, \ldots, e^{-\alpha_s/r_s}\}$. Using Hölder inequality and the definition of class $W_2^{\mathrm{r};\alpha}$ for every $x \in [0,1]^s$ we have

$$\sum_{m\in Z^s}|\hat{f}(m)e^{2\pi i(m,x)}| \leq \sum_{m\in Z^s}|\hat{f}(m)| \leq$$

$$\leq \left(\sum_{m\in Z^s}\frac{1}{\overline{m}_1^{2r_1}\ln^{2\alpha_1}(\overline{m}_1+1)+\cdots+\overline{m}_s^{2r_s}\ln^{2\alpha_s}(\overline{m}_s+1)}\right)^{1/2}. \tag{18}$$

Let $\omega_1(x) = (1/\ln^{\alpha_1}A)x^{r_1}\ln^{\alpha_1}(Ax), \ldots, \omega_s(x) = (1/\ln^{\alpha_s}A)x^{r_s}\ln^{\alpha_s}(Ax)$ for all $x \geq 1$. Then by virtue of (17) and (18) we obtain

$$\sum_{m\in Z^s}|\hat{f}(m)| \ll \left(\sum_{m\in Z^s}\frac{1}{\omega_1^2(\overline{m}_1)+\cdots+\omega_s^2(\overline{m}_s)}\right)^{1/2}. \tag{19}$$

The functions $\omega_1, \ldots, \omega_s$ strictly increase on the interval $[1, +\infty)$. Indeed, for each $i \in \{1, \ldots, s\}$ we have

$$\omega_i'(x) = (1/\ln^{\alpha_i}A)(r_i\ln(Ax)+\alpha_i)x^{r_i-1}\ln^{\alpha_i-1}(Ax) >$$

$$> (1/\ln^{\alpha_i}A)(r_i\ln e^{-\alpha_i/r_i}+\alpha_i)x^{r_i-1}\ln^{\alpha_i-1}(Ax) = 0.$$

Further, denoting by ω_i^* the inverse of the function ω_i, for each $p = 1, 2, \ldots$, we define the set $X_p = \{m \in Z^s : |m_1| \leq \omega_1^*(\omega(2p)), \ldots, |m_s| \leq \omega_s^*(\omega(2p))\}$, where ω is the inverse of the function $\omega^*(x) = \omega_1^*(x) \times \cdots \times \omega_s^*(x)$ strictly increasing on $[1, +\infty)$.

For $m = (m_1, \ldots, m_s) \in X_{p+1} \backslash X_p$ there is an index $\theta \in \{1, \ldots, s\}$ such that

$$|m_\theta| \geq \omega_\theta^*(\omega(2^p)) \Leftrightarrow \omega_\theta(|m_\theta|) \geq \omega(2^p).$$

Therefore, taking into account the inequality $\omega_1^2(\overline{m}_1)+\cdots+\omega_s^2(\overline{m}_s) \geq \omega_\theta^2(\overline{m}_\theta)$ and the relation $|X_{p+1} \backslash X_p| \succ\prec 2^p$, for all $p = 1, 2, \ldots$ we obtain

$$\sum_{m \in X_{p+1} \backslash X_p} \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)} \ll \frac{2^p}{\omega^2(2^p)}. \tag{20}$$

If we put $X_0 = \emptyset$, then according to equality $X_p = \bigcup\limits_{K=0}^{p-1} (X_{K+1} \backslash X_K)$ we have

$$\sum_{m \in Z^s} \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)} =$$

$$= \lim_{p \to \infty} \sum_{K=0}^{p-1} \sum_{m \in X_{K+1} \backslash X_K} \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)}.$$

Therefore , by virtue of (20) the inequality

$$\sum_{m \in Z^s} \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)} \ll \sum_{K=0}^{\infty} \frac{2^K}{\omega^2(2^K)} \tag{21}$$

is true. According to Lemma 3, there exists a quantity $C_{12} > 0$ such that for all integers $K > C_{12}$ the inequalities

$$\omega(2^K) \gg 2^{\lambda K} \ln^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}(3 \cdot 2^K) \geq 2^{\lambda K}(K+1)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}$$

are satisfied, whence,

$$\sum_{\{K: K > C_{12}\}} \frac{2^K}{\omega^2(2^K)} \ll \sum_{\{K: K > C_{12}\}} \frac{1}{2^{(2\lambda-1)K}(K+1)^{2\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}} < \infty, \tag{22}$$

since for $\lambda > 1/2$ the last series converges according to D'Alembert's criterion. As a result of (22), the series $\sum\limits_{K=0}^{\infty} \frac{2^K}{\omega^2(2^K)}$ converges. Therefore, due to (21) and (19) for $\lambda > 1/2$ the trigonometric Fourier series of each function $f \in W_2^{\mathrm{r};\alpha}$ converges absolutely. $\square$

Lemma 5 (see, Lemma A from [12]). *Suppose we are given an integer $s \geq 1$. Then, for each integer $N \geq 1$, the following assertion holds: for any set $G \equiv \{m^{(1)}, \ldots, m^{(N')}\} \subset Z^s$ such that $N' = |G| \geq 2N$ and $|G| \succ\prec N$, and for arbitrary linear functionals $l_1, \ldots, l_N$ defined at least on the set of all trigonometric polynomials with spectrum in G, there exist complex numbers $\{c_k\}_{k=1}^{N'}$ satisfying the conditions $\sum\limits_{k=1}^{N'} |c_k| \geq N$, $\sum\limits_{k=1}^{N'} |c_k|^2 = N$, moreover, if $\chi(x) = \sum\limits_{k=1}^{N'} c_k e^{2\pi i(m^{(k)},x)}$, then $l_1(\chi) = 0, \ldots, l_N(\chi) = 0$ and $\|\chi\|_{L^\infty} \geq N, \|\chi\|_{L^2} = \sqrt{N}$.*

5 Proof of theorem 1

Next, we will consider integers $K > C_0 = \max\{C_1(\gamma_1), \dots, C_1(\gamma_s), K_0\}$. Since

$$\widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f); x\right) = \sum_{\tau=1}^{N} \hat{f}(\overline{m}^{(\tau)}) e^{2\pi i(\widetilde{m}^{(\tau)}, x)} = \sum_{m \in A_K} \hat{f}(m) e^{2\pi i(m,x)},$$

then according to Lemma 4 the equality

$$f(x) - \widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f); x\right) = \sum_{m \in Z^s \setminus A_K} \hat{f}(m) e^{2\pi i(m,x)} \tag{23}$$

holds. For any $m \in Z^s \setminus A_K$ there is a number $\nu \in \{1, \dots, s\}$ such that $|m_\nu| > N_\nu$. Therefore, taking into account the monotony of the function ω_ν, we have

$$\frac{1}{\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \dots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1)} \ll$$

$$\ll \frac{1}{\omega_1^2(\overline{m}_1) + \dots + \omega_s^2(\overline{m}_s)} \ll \frac{1}{\omega_\nu^2(\overline{m}_\nu)} \ll \frac{1}{N_\nu^{2r_\nu} \ln^{2\alpha_\nu}(AN_\nu)}. \tag{24}$$

Due to the definition of the number N_ν and the relation (5) the inequality

$$N_\nu^{r_\nu} \ln^{\alpha_\nu}(AN_\nu) \gg N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)}$$

is true. Therefore, continuing the inequality (24) we obtain

$$\frac{1}{\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \dots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1)} \ll$$

$$\ll \frac{1}{N^{2\lambda} (\ln N)^{2\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)}}. \tag{25}$$

From (23), by virtue Parseval equality, the definition of the class under consideration and the equality (25), follows

$$\left\| f(\cdot) - \widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f); \cdot\right) \right\|_{L^2} \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)}}, \tag{26}$$

wherefrom we have

$$\delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)}}. \tag{27}$$

Let the function $\varphi_N(z_1, \dots, z_N; y) : \mathbb{C}^N \times [0,1]^s \mapsto \mathbb{C}$ and linear functionals $l_N^{(1)}, \dots, l_N^{(N)}$ defined on class $W_2^{\mathrm{r};\alpha}$ be given and let $G_N = \{m \in Z^s : |m_1| \le [M_1], \dots, |m_s| \le [M_s]\}$, where $M_i = N^{\lambda/r_i} (\ln N)^{\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)/r_i} (\ln N)^{-\alpha_i/r_i}$ for each $i = 1, 2, \dots, s$. Since the inequality

$|G_N| > 2N$ and the relation $|G_N| \succ\prec N$ are valid for G_N, then by virtue of Lemma 5 for linear functionals $l_N^{(1)}, \dots, l_N^{(N)}$ there are complex numbers $c_m, m \in G_N$ such that

$$\sum_{m\in G_N} |c_m|^2 = N, \tag{28}$$

moreover, if $\prod_N(x) = \sum\limits_{m\in G_N} c_m e^{2\pi i(m,x)}$, then $l_N^{(1)}(\prod_N) = 0, \dots, l_N^{(N)}(\prod_N) = 0$ and

$$\left\|\prod\nolimits_N\right\|_{L^2} = \sqrt{N}. \tag{29}$$

Let's consider the function

$$g_N(x) = \frac{1}{N^{\lambda+1/2}(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}} \prod\nolimits_N(x).$$

Taking into account the definitions of the numbers $M_i, i \in \{1, \dots, s\}$ and equality (28) we obtain

$$\sum_{m\in G_N} |\hat{g}_N(m)|^2 \left(\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1+1) + \dots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s+1)\right) \ll$$

$$\ll \frac{1}{N} \sum_{m\in G_N} |c_m|^2 \ll 1.$$

Therefore, for some $C_{13} > 0$ the function $f_N(x) = C_{13} g_N(x)$ belongs to class $W_2^{\mathrm{r};\alpha}$. Further, using the equality (29), we find

$$\|f_N\|_{L^2} \gg \frac{1}{N^\lambda(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}}. \tag{30}$$

Since $f_N \in W_2^{\mathrm{r};\alpha}$ and $l_N^{(1)}(f_N) = 0, \dots, l_N^{(N)}(f_N) = 0$, then

$$\sup_{f\in W_2^{\mathrm{r};\alpha}} \left\| f(\cdot) - \varphi_N(l_N^{(1)}(f), \dots, l_N^{(N)}(f); \cdot)\right\|_{L^2} \geq$$

$$\geq \frac{1}{2}(\|f_N(\cdot) - \varphi_N(0, \dots, 0; \cdot)\|_{L^2} + \|(-f_N)(\cdot) - \varphi_N(0, \dots, 0; \cdot)\|_{L^2}) \geq \|f_N\|_{L^2},$$

wherefrom, taking into account (30), we obtain

$$\delta_N(L_N) \gg \frac{1}{N^\lambda(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}}. \tag{31}$$

Consequently, by virtue of the inequalities $\delta_N(L_N) \leq \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N))$ and (27) the relations (2) hold. $\square$

6 Proof of theorem 2

For arbitrarily given numbers $|\gamma_N^{(\tau)}| \leq 1(\tau = 1, \ldots, N)$ the inequality

$$\left\| f(\cdot) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f) + \gamma_N^{(1)}\widetilde{\varepsilon}_N, \ldots, \tilde{l}_N^{(N)}(f) + \gamma_N^{(N)}\widetilde{\varepsilon}_N; \cdot)\right\|_{L^2} \leq$$

$$\leq \left\| f(\cdot) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); \cdot)\right\|_{L^2} + \left\| \sum_{\tau=1}^{N} (-\gamma_N^{(\tau)})\widetilde{\varepsilon}_N e^{2\pi i(\widetilde{m}^{(\tau)}, \cdot)} \right\|_{L^2} \tag{32}$$

holds. It is clear, that

$$\left\| \sum_{\tau=1}^{N} (-\gamma_N^{(\tau)})\widetilde{\varepsilon}_N e^{2\pi i(\widetilde{m}^{(\tau)}, \cdot)} \right\|_{L^2} \ll \frac{1}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)}}.$$

Therefore, according to inequalities (26) and (32), we have

$$\left\| f(\cdot) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f) + \gamma_N^{(1)}\widetilde{\varepsilon}_N, \ldots, \tilde{l}_N^{(N)}(f) + \gamma_N^{(N)}\widetilde{\varepsilon}_N; \cdot)\right\|_{L^2} \ll$$

$$\ll \frac{1}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)}},$$

wherefrom, due to the arbitrariness of the numbers $\gamma_N^{(\tau)}(\tau = 1, \ldots, N)$ and the function f follows

$$\triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \ll \frac{1}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)}}. \tag{33}$$

Since $\delta_N(L_N) \leq \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \leq \triangle_N(\widetilde{\varepsilon}_N, (\bar{l}^{(N)}, \overline{\varphi}_N))$, then using (31) and (33) we arrive to (3).

Further, for each integer $K > C_0$ we take $\beta_K = \min\{\eta_N, \ln N\}$ and define the set

$$H_K = \{m \in Z^s : [J_1] \leq |m_1| \leq 2[J_1], \ldots, [J_s] \leq |m_s| \leq 2[J_s]\},$$

where $N = N(K), \eta_N \equiv \{\eta_{N(K)}\}_{K \geq 1}$ is a positive sequence arbitrarily slowly increasing to $+\infty$,

$$J_i = N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)/r_i}(\ln N)^{-\alpha_i/r_i}\beta_K^{-1/r_i}(i = 1, \ldots, s).$$

Since $\lim\limits_{K \to +\infty} \beta_K = +\infty$, there exist some quantity $C_{14} > C_0$ such that for all integers $K > C_{14}$ the inequality

$$\beta_K \geq 1 \tag{34}$$

holds. For an arbitrary vector $m = (m_1, \ldots, m_s) \in H_K$ the inequality

$$\ln^{\alpha_i}(A\overline{m}_i) \ll \ln^{\alpha_i} N, \tag{35}$$

is true, where $\alpha_i \in \mathbb{R}$ and $\overline{m}_i = \max\{1; |m_i|\}$ for each $i \in \{1, 2, \ldots, s\}$. The validity of the last inequality is established using Lemma 1 and inequalities $\ln^{-1/r_i} N \leq \beta_K^{-1/r_i} \leq 1$. Using (35) for any vector $m = (m_1, \ldots, m_s) \in H_K$, we easily obtain

$$\overline{m}_i^{r_i} \ln^{\alpha_i}(A\overline{m}_i) \ll N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}\beta_K^{-1}. \tag{36}$$

Now, for each $K > C_{14}$ we consider the function $h_K(x) = \beta_K \widetilde{\varepsilon}_N \sum\limits_{m \in H_K} e^{2\pi i(m,x)}$. Due to inequalities $|H_K| \ll N \cdot \beta_K^{-1/\lambda}, 1/\lambda > 0$, (36) and (34) we have

$$\sum_{m \in H_K} |\hat{h}_K(m)|^2 (\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \cdots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1)) \ll$$

$$\ll \frac{1}{N} \sum_{m \in H_K} 1 \ll \frac{1}{\beta_K^{1/\lambda}} \ll 1.$$

Therefore, for some $C_{15} > 0$ the function $t_K(x) = C_{15} h_K(x)$ belongs to class $W_2^{\mathrm{r};\alpha}$. It is clear that

$$\|t_K\|_{L^2} \gg \frac{1}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}} \beta_K^{1-1/(2\lambda)}. \tag{37}$$

Let

$$\widetilde{\gamma}_N^{(\tau)} = -\frac{\hat{t}_k(m^{(\tau)})}{\widetilde{\varepsilon}_N \eta_N}, \overline{\gamma}_N^{(\tau)} = -\frac{(-\hat{t}_K)(m^{(\tau)})}{\widetilde{\varepsilon}_N \eta_N} (\tau = 1, \ldots, N \equiv N(K)),$$

$$\Phi_N = \{(l^{(N)}, \varphi_N) : l_N^{(1)}(f) = \hat{f}(m^{(1)}), \ldots, l_N^{(N)}(f) = \hat{f}(m^{(N)})\}$$

for each integer $K > C_{14}$. Since

$$|\widetilde{\gamma}_N^{(\tau)}| \leq 1, |\overline{\gamma}_N^{(\tau)}| \leq 1, \hat{t}_K(m^{(\tau)}) + \eta_N \widetilde{\gamma}_N^{(\tau)} \widetilde{\varepsilon}_N = 0, (-\hat{t}_K)(m^{(\tau)}) + \eta_N \overline{\gamma}_N^{(\tau)} \widetilde{\varepsilon}_N = 0$$

for any $\tau \in \{1, \ldots, N\}$, then for every computing unit $(l^{(N)}, \varphi_N) \in \Phi_N$ we have

$$\sup_{f \in W_2^{\mathrm{r};\alpha}} \sup_{|\gamma_N^{(1)}| \leq 1, \ldots, |\gamma_N^{(N)}| \leq 1} \Big\| f(\cdot) - \varphi_N(\hat{f}(m^{(1)}) + \gamma_N^{(1)} \eta_N \widetilde{\varepsilon}_N, \ldots,$$

$$\hat{f}(m^{(N)}) + \gamma_N^{(N)} \eta_N \widetilde{\varepsilon}_N; \cdot) \Big\|_{L^2} \geq$$

$$\geq \max\Big\{ \Big\| t_K(\cdot) - \varphi_N(\hat{t}_K(m^{(1)}) + \widetilde{\gamma}_N^{(1)} \eta_N \widetilde{\varepsilon}_N, \ldots, \hat{t}_K(m^{(N)}) + \widetilde{\gamma}_N \eta_N \widetilde{\varepsilon}_N; \cdot) \Big\|_{L^2},$$

$$\Big\| (-t_K)(\cdot) - \varphi_N((-\hat{t}_K)(m^{(1)}) + \overline{\gamma}_N^{(1)} \eta_N \widetilde{\varepsilon}_N, \ldots, (-\hat{t}_K)(m^{(N)}) + \overline{\gamma}_N^{(N)} \eta_N \widetilde{\varepsilon}_N; \cdot) \Big\|_{L^2} \Big\} =$$

$$= \max\{\|t_K(\cdot) - \varphi_N(0, \ldots, 0; \cdot)\|_{L^2}, \|(-t_K)(\cdot) - \varphi_N(0, \ldots, 0; \cdot)\|_{L^2}\} \geq \|t_K\|_{L^2}. \tag{38}$$

Comparing inequalities (37) and (38), for any computing unit we obtain

$$\triangle_N(\eta_N \widetilde{\varepsilon}_N, (l^{(N)}, \varphi_N)) \gg \delta_N(L_N) \cdot \beta_K^{1-1/(2\lambda)}. \tag{39}$$

Since $1 - 1/(2\lambda) = \frac{2\lambda-1}{2\lambda} > 0$, then $\overline{\lim\limits_{K \to \infty}} \beta_K^{1-1/(2\lambda)} = +\infty$. Therefore, due to (39), for each $(l^{(N)}, \varphi_N) \in \Phi_N$ we have

$$\overline{\lim_{K \to \infty}} \frac{\triangle_N(\eta_N \widetilde{\varepsilon}_N, (l^{(N)}, \varphi_N))}{\delta_N(L_N)} = +\infty. \tag{40}$$

Since $(\tilde{l}^{(N)}, \widetilde{\varphi}_N) \in \Phi_N$, then from (40) the equality (4) follows. $\square$

7 Conclusion

When considering the class $W_2^{\mathrm{r};\alpha}$ in problems of optimal recovery of functions and finding limiting errors, in contrast to the multidimensional Sobolev classes with a dominant mixed derivative SW, Korobov E and the isotropic Sobolev class W both the exact order ψ_N and the limiting error $\widetilde{\varepsilon}_N$ do not depend on the number s of the variable functions $f(x) = f(x_1, \ldots, x_s) \in W_2^{\mathrm{r};\alpha}$ (see, for example, [4], [13], [14]). Therefore, the theorems formulated and proven here are important results in approximation theory, numerical analysis and computational mathematics.

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Received: August 22, 2025
Accepted: November 16, 2025