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EXACT SOLUTIONS OF EQUATIONS OF THE TWO-PRIMARY-BODY PROBLEM IN THE RESTRICTED THREE-BODY PROBLEM WITH VARIABLE MASSES

This study investigates the translational-rotational motion of a non-stationary, axisymmetric body of variable mass in the Newtonian gravitational field of two primary spherical bodies with variable masses, formulated within the framework of the restricted three-body problem with variable masses in a barycentric coordinate system. The masses of the bodies vary isotropically. The small axisymmetric body may change its size and shape while remaining axially symmetric throughout the process. The restricted formulation implies that the small body does not influence the motion of the two primary spherical bodies with variable masses. The study focuses on the secular perturbations of translational-rotational motion in the considered three-body system. Since the exact solutions for the translational-rotational motion of the two primary spherical bodies with variable masses in the barycentric coordinate system are unknown, the differential equations of the two-body problem and those of the non-stationary small body are investigated jointly. Due to the complexity of the problem, the translational-rotational motion of the three-body system is studied using perturbation theory in analogues of Delaunay-Andoyer variables. Exact analytical solutions of the differential equations for the secular perturbations of the translational-rotational motion in the problem of two primary spherical bodies in terms of Delaunay-Andoyer variable analogues are obtained. These exact solutions open the possibility of further investigating the secular perturbations of the translational-rotational motion of a non-stationary, axisymmetric body within the restricted three-body problem with variable masses.
Key words: Restricted three-body problem, variable mass, translational-rotational motion, analogues of the Delaunay-Andoyer variables, perturbation theory.

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Шектелген үш дене мәселесі аясындағы массалары айнымалы екі негізгі дене мәселесі теңдеулерінің нақты шешімдері

Бұл жұмыста шектелген үш дене есебінің аясында массалары айнымалы екі негізгі сфералық дененің ньютондық тартылыс өрісіндегі массасы айнымалы бейстационар өстік симметриялы кіші дененің барицентрлік координаттар жүйесіндегі ілгерілемелі-айналмалы қозғалысы қарастырылады. Денелердің массалары уақыт бойынша изотропты өзгереді, сондықтан реактивті күштер мен реактивті моменттер пайда болмайды. Кіші өстік симметриялы дененің өлшемі мен пішіні уақыт бойынша өзгереді, бірақ әрқашан өстік симметриялы күйін сақтайды. Шектелген есептің қойылымы - кіші массалы дене екі негізгі сфералық дененің қозғалысына әсер етпейдігін сипаттайды. Бұл жұмыста үш дененің де ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқулары зерттеледі. Массалары айнымалы екі негізгі сфералық дененің барицентрлік координаттар жүйедегі ілгерілемелі-айналмалы қозғалысының дәл шешімдері белгісіз болғандықтан, екі негізгі дененің және бейстационар кіші дененің дифференциалдық теңдеулері бірлесіп қарастырылады. Мәселенің күрделілігіне байланысты үш денелі жүйенің ілгерілемелі-айналмалы қозғалысы Делоне-Андуайе айнымалыларының аналогтарындағы ұйытқу теориясы әдістерімен зерттелді. Нәтижесінде екі негізгі сфералық дененің ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқу теңдеулерінің дәл аналитикалық шешімдері алынды.

Алынған шешімдер шектелген үш дене есебінің аясында массасы айнымалы бейстационар өстік симметриялы дененің ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқуын әрі қарай зерттеуге мүмкіндік береді.
Түйін сөздер: шектелген үш дене мәселесі, айнымалы масса, айналмалы-ілгерілемелі қозғалысы, Делоне-Андуайе айнымалыларының аналогтары, ұйытқу теориясы.

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Точные решения уравнений задачи двух основных тел в ограниченной задаче трех тел с переменными массами

В работе рассмотрено поступательно-вращательное движение нестационарного малого осесимметричного тела переменной массы в ньютоновском поле притяжения двух основных сферических тел с переменными массами в рамках ограниченной задачи трех тел с переменными массами в барицентрической системе координат. Массы тел меняются со временем изотропно, поэтому не появляются реактивные силы и реактивные моменты. Малое осесимметричное тело может менять размеры и формы при этом все время остается осесимметричным. Ограниченная постановка задачи характеризуется тем, что малое тело не влияет на движение двух основных сферических тел с переменными массами. Исследуется вековые возмущения поступательно-вращательного движения в рассматриваемой проблеме всех трех тел. Решения поступательно-вращательного движения двух основных сферических тел с переменными массами в барицентрической системе координат неизвестна, поэтому дифференциальные уравнения задачи двух основных тел и дифференциальные уравнения нестационарного малого тела исследуется совместно. Проблема сложная, поэтому поступательно-вращательное движение системы трех тел исследуется методами теорий возмущений в аналогах переменных Делоне-Андуайе. Были получены точные аналитические решения дифференциальных уравнений вековых возмущений поступательно-вращательного движения задачи двух основных сферических тел в аналогах переменных Делоне-Андуайе. Эти решения открывают возможность дальнейшего исследования вековых возмущений поступательно-вращательного движения нестационарного малого осесимметричного тела в рамках ограниченной задачи трёх тел с переменными массами.

Ключевые слова: ограниченная задача трех тел, переменная масса, поступательно-вращательное движение, аналоги переменных Делоне-Андуайе, теория возмущения.

1 Introduction

The investigation of the impact of the variability of celestial bodies' masses on the dynamic evolution of gravitational systems is a relevant problem in modern astronomy and astrophysics. In this paper, we investigate the celestial-mechanical formulation of the problem of the translational-rotational motion of a non-stationary axisymmetric small body in the Newtonian gravitational field of two primary spherically symmetric bodies with variable masses and radius in a restricted formulation.

The masses of the bodies are variable, with the laws of mass variation being arbitrarily prescribed functions of time - $m_1 = m_1(t), m_2 = m_2(t), m_3 = m_3(t)$. In the general case, the mass changes occur isotropically but at different rates

$$\frac{\dot{m}_1(t)}{m_1(t)} \neq \frac{\dot{m}_2(t)}{m_2(t)} \neq \frac{\dot{m}_3(t)}{m_3(t)}$$

no reactive forces or reactive moments appear. A small axisymmetric body can change its size and shape, has three mutually perpendicular planes of symmetry, and remains axisymmetric at all times. In this work, the restricted form of the problem is characterised by the fact that the small body does not affect the motion of the two primary spherical bodies.

A brief overview of studies related to the present work is provided. The formation and dynamical evolution of planetary systems is a central theme of modern astronomy. The influence of single and binary stars on planet formation is of great importance, since nearly half of all stars are found in binary or multiple stellar systems. One of the most effective ways to assess this influence is to obtain an accurate picture of the population of binary stars. In [1], an extensive database was created as a result of a comprehensive literature survey, in order to carry out a complete census of all known binary stars hosting planets to date. The database includes the characteristics (orbit or separation, stellar masses, dynamical stability, etc.) of 759 systems (including 31 circumbinary systems), which is nine times larger than the previous complete census of binary stars with planets. Among the 728 S-type systems, 651 are binaries, 73 are triples, and 4 are quadruples.

Binary stars are considered key natural laboratories for the study of stellar physics, which explains their inclusion in photometric space observations starting from the very first orbital telescope launched in 1968. The review [2] follows the history of binary star observations and the scientific insights gained, beginning with the early ultraviolet missions, moving through the phase of mission diversification with various satellite projects, and reaching the present stage of large-scale surveys focused on planetary transits. Over this time, detached, semi-detached, and contact binaries have been studied, comprising stars at different evolutionary stages—dwarfs, subgiants, giants, supergiants—as well as compact objects such as white dwarfs and neutron stars, often accompanied by planets or accretion disks. Modern surveys have uncovered a wide range of phenomena, including pulsating stars in eclipsing binaries and systems that host transiting planets. Particular emphasis is placed on eclipsing binaries due to their high scientific value, and on the most recent missions, which, owing to their extensive sky coverage, provide unique opportunities for comprehensive astrophysical research.

A group of researchers from NASA's Eclipsing Binary Patrol citizen science project [3] has published a catalogue containing 10,001 eclipsing binary star systems. This discovery significantly expands our knowledge of stellar physics and formation processes, and opens up new opportunities for the search for exoplanets.

In [4], an interesting object was discovered 70 light years from Earth: the amazing world of the ν Octantis system, an exoplanetary system consisting of binary stars and one planet. The main star there is slightly more massive than the Sun, and its companion is a white dwarf. The planet, squeezed into a narrow space between the binary stars, not only exists in a complex gravitational environment, but also has a retrograde orbit. The main star is a subgiant, 1.6 times more massive than the Sun, and the second is an object with a mass of about half that of the Sun. They orbit each other with a period of 1,050 days.

In [5], a unique planet was discovered in the 2M1510 system. The planet's orbit is almost perpendicular to the plane of the binary stars' orbits. The shape of the planet itself is normal, but it has an unusual orbit. There are three brown dwarfs in the system — too large to be planets, too small to be full-fledged stars. Two dwarfs revolve around each other, while the third, located further away, revolves around both of its companions. Planet 2M1510 b, which has attracted the interest of scientists, is in a polar orbit around the two central brown dwarfs.

In [6], the classical dynamics of binary stars undergoing mass exchange between them is studied. Assuming that one of the stars is more massive than the other, the dynamics of the lighter star is analysed as a function of its mass change over time. Within the framework of approximations and mass transfer models, a general result is obtained which establishes that if the lighter star loses mass, its period increases. If the lighter star gains mass, its period decreases. Such non-stationarity in the dynamics of binary stars can significantly affect the dynamic evolution of planetary systems around binary stars. We also note a two-volume fundamental monograph devoted to close binary stars [7]. In the book [8], the evolution of the rotational motion of a rigid body about its center of mass is examined under the assumption that the body's mass and dimensions remain constant. The rotational motion of a triaxial satellite about its center of mass with moments of inertia close to one another is analyzed, and several interesting results are obtained.

This review shows that the development of celestial-mechanical models of non-stationary binary stars and planets is a relevant topic.

This work is structured as follows. Section 2 presents the formulation of the problem of translational-rotational motion of a non-stationary axisymmetric small body in the Newtonian gravitational field of two massive spherically symmetric bodies with variable masses and radius in a restricted formulation and the equations of motion. Section 3 provides exact analytical solutions for the rotational motion of two primary bodies in variable Eulers. Section 4 provides exact analytical solutions for the equation of secular perturbations of the translational motion of the centre of mass of two primary spherical bodies with variable masses in a barycentric coordinate system. In Section 5, analytical expressions for the coordinates and velocities of two primary spherical bodies with variable masses are obtained based on exact solutions of the differential equations of secular perturbations. The conclusion highlights the main result of the work and further prospects for research.

2 Formulation of the problem

In this work considered translational-rotational motion of three non-stationary celestial bodies with variable masses. There, P_1, P_2 are the primary spherically symmetric bodies with variable masses and variable dimensions, whose motion is determined by the problem of these two bodies with variable masses.

The third body P_3 is axisymmetric small body, and does not affect the motion of the first two bodies. The body P_3 has three mutually perpendicular planes of symmetry. The principal axes of inertia of the own coordinate system are directed along the line of intersection of the three mutually perpendicular planes, and this position is preserved during evolution.

Assumptions and differential equations of the problem in the barycentric coordinate system were obtained in [9]. The equations of translational motion of two spherically symmetric bodies in the barycentric coordinate system are as follows

$$\ddot{\vec{r}}_i = -f\tilde{m}_i\frac{\vec{r}_i}{r_i^3} + \tilde{A}_i\dot{\vec{r}}_i + \tilde{B}_i\vec{r}_i, \tag{1}$$

where, f - gravitational constant, $\tilde{m}_i = \tilde{m}_i(t) = m\nu_j^3 = m_j^3(t)/m^2, \nu_j = m_j/m, m = m_1 + m_2, \tilde{A}_i = 2\dot{\nu}_j/\nu_j, \tilde{B}_i = \ddot{\nu}_j/\nu_j - 2\dot{\nu}_j^2/\nu_j^2, i \neq j, i, j = 1, 2.$

The translational- rotational motion of two non-stationary bodies can be conveniently studied in analogues of Delaunay-Andoyer variables [10], [11].

Rotational motion of two spherical bodies with variable masses will first be considered in analogues of Euler's variables. Euler's dynamic equations are greatly simplified due to the spherical symmetry of the bodies and take the form [12]

$$\frac{d}{dt}(A_i p_i) = 0, \qquad \frac{d}{dt}(B_i q_i) = 0, \qquad \frac{d}{dt}(C_i r_i) = 0, \tag{2}$$

Accordingly, Euler's kinematic equations can be written as

$$p_i = \dot{\psi}_i \sin\theta_i \sin\varphi_i + \dot{\theta}_i \cos\varphi_i, \;\; q_i = \dot{\psi}_i \sin\theta_i \cos\varphi_i - \dot{\theta}_i \sin\varphi_i, \;\; r_i = \dot{\psi}_i \cos\theta_i + \dot{\varphi}_i, \tag{3}$$

The equations of translational-rotational motion of a small non-stationary axisymmetric body P_3 have the form

$$\ddot{\vec{r}}_3 = grad_{\vec{r}_3}\tilde{U} + \tilde{A}_{23}\dot{\vec{r}}_2 + \tilde{B}_{23}\vec{r}_2, \tag{4}$$

$$\begin{aligned}
&\frac{d}{dt}(A_3 p_3) - (A_3 - C_3)\, q_3 r_3 = \left[\frac{\partial \tilde{U}}{\partial \psi_3} - \cos\theta_3 \frac{\partial \tilde{U}}{\partial \varphi_3}\right] \frac{\sin\varphi_3}{\sin\theta_3} + \cos\varphi_3 \frac{\partial \tilde{U}}{\partial \theta_3},\\
&\frac{d}{dt}(A_3 q_3) - (C_3 - A_3)\, r_3 p_3 = \left[\frac{\partial \tilde{U}}{\partial \psi_3} - \cos\theta_3 \frac{\partial \tilde{U}}{\partial \varphi_3}\right] \frac{\cos\varphi_3}{\sin\theta_3} - \sin\varphi_3 \frac{\partial \tilde{U}}{\partial \theta_3},\\
&\frac{d}{dt}(C_3 r_3) = 0,
\end{aligned} \tag{5}$$

$$\begin{gathered}
p_3 = \dot{\psi}_3 \sin\theta_3 \sin\varphi_3 + \dot{\theta}_3 \cos\varphi_3, \qquad q_3 = \dot{\psi}_3 \sin\theta_3 \cos\varphi_3 - \dot{\theta}_3 \sin\varphi_3,\\
r_3 = \dot{\psi}_3 \cos\theta_3 + \dot{\varphi}_3,
\end{gathered} \tag{6}$$

$$\tilde{U} = f\left(\frac{m_1}{r_{31}} + \frac{m_2}{r_{32}}\right) + f\,(C_3 - A_3)\,\frac{1}{2}\left(\frac{1 - 3\gamma_{31}^2}{r_{31}^2} + \frac{1 - 3\gamma_{32}^2}{r_{32}^2}\right) \tag{7}$$

$$\tilde{A}_{23} = -\frac{2\tilde{A}}{\nu_1}, \tilde{B}_{23} = -\frac{\tilde{B}}{\nu_1} + 4\tilde{A}\frac{\dot{\nu}_1}{\nu_1^2},$$

$$\begin{aligned}
\gamma_{31} &= a_{13}\frac{x_1 - x_3}{r_{31}} + a_{23}\frac{y_1 - y_3}{r_{31}} + a_{33}\frac{z_1 - z_3}{r_{31}},\\
\gamma_{32} &= a_{13}\frac{x_2 - x_3}{r_{32}} + a_{23}\frac{y_2 - y_3}{r_{32}} + a_{33}\frac{z_2 - z_3}{r_{32}},
\end{aligned} \tag{8}$$

In equations (1) - (3), (4) - (6), the notation used in [9] is retained.

Solutions for the translational-rotational motion of two primary spherical bodies with variable masses in a barycentric coordinate system are unknown [13], [14], therefore the differential equations of the two primary bodies and the differential equations of the non-stationary small body are investigated jointly. The problem is complex, so the translational-rotational motion of a three-body system is investigated using perturbation theory methods [15], [16], [17] in analogues of Delaunay-Andoyer variables.

3 Exact analytical solutions for the rotational motion of two primary spherical bodies in analogues of Euler variables

Note that the rotational motion of spherical bodies P_1, P_2 has a simple solution, since throughout its evolution, a spherically symmetric body retains its spherical density distribution and spherical external shape.

From equations (2) we obtain

$$A_i p_i = const = A_{i0} p_{i0}, \qquad B_i q_i = const = B_{i0} q_{i0}, \qquad C_i r_i = const = C_{i0} r_{i0}, \tag{9}$$

From this follows the module of the kinetic momentum vector $\vec{K}_{i0}$, of the body P_i a constant value

$$A_i^2 p_i^2 + B_i^2 q_i^2 + C_i^2 r_i^2 = A_{i0}^2 \left(p_{i0}^2 + q_{i0}^2 + r_{i0}^2\right) = const = K_{i0}^2 \tag{10}$$

Let the vector $\vec{K}_{i0}$ be directed along the OZ axis, then the following formulas can be written [18]

$$\begin{array}{l} A_i p_i = A_i p_i = K_{i0} \sin\theta_i \sin\varphi_i = K_{iOx}, \\ B_i q_i = A_i q_i = K_{i0} \sin\theta_i \cos\varphi_i = K_{iOy}, \\ C_i r_i = A_i r_i = K_{i0} \cos\theta_i = K_{iOz}, \end{array} \tag{11}$$

Since $C_i r_i = A_i r_i = A_{i0} r_{i0} = K_{i0} \cos\theta_i = K_{iOz}$,

$$\cos\theta_i = \frac{C_i r_i}{K_{i0}} = \frac{K_{iOz}}{K_{i0}} = \frac{A_{i0} r_{i0}}{A_{i0}\sqrt{p_{i0}^2 + q_{i0}^2 + r_{i0}^2}} = \frac{r_{i0}}{\sqrt{p_{i0}^2 + q_{i0}^2 + r_{i0}^2}} = const \tag{12}$$

Therefore,

$$\dot{\theta}_i = 0 \tag{13}$$

Substituting (13) into equations (3), we obtain

$$p_i = \dot{\psi}_i \sin\theta_i \sin\varphi_i, \qquad q_i = \dot{\psi}_i \sin\theta_i \cos\varphi_i, \qquad r_i = \dot{\psi}_i \cos\theta_i + \dot{\varphi}_i, \tag{14}$$

From the first equation (14) and the first equation (11), the following solution of $\dot{\psi}_i$ is obtained

$$p_i = \dot{\psi}_i \sin\theta_i \sin\varphi_i, \qquad A_i p_i = K_{i0} \sin\theta_i \sin\varphi_i, \tag{15}$$

$$\dot{\psi}_i = \frac{K_{i0}}{A_i}, \tag{16}$$

From the last equation (14), we find the solution $\dot{\varphi}_i$. Let us substitute the solution $\dot{\psi}_i$ and obtain the following

$$\dot{\varphi}_i = r_i - \dot{\psi}_i \cos\theta_i = r_i - \frac{K_{i0}}{A_i}\cos\theta_{i0} = \frac{A_i r_i \cos\theta_{i0}}{A_i} = \frac{A_{i0} r_{i0} \cos\theta_{i0}}{A_i}, \tag{17}$$

As a result, we obtain the following results

$$\cos\theta_i = \cos\theta_{i0} = \frac{K_{i0z}}{K_{i0}} = const, \qquad \theta_i = \theta_{i0} = const, \qquad \dot{\theta}_i = 0, \tag{18}$$

$$\dot{\psi}_i = \frac{K_{i0}}{A_i} \neq const, \tag{19}$$

$$\dot{\varphi}_i = \frac{A_{i0} r_{i0} - K_{i0}\cos\theta_{i0}}{A_i} = \frac{A_{i0} r_{i0} - C_{i0} r_{i0}}{A_i} = \frac{A_{i0} r_{i0} - A_{i0} r_{i0}}{A_i} = 0,$$

$$\dot{\varphi}_i = 0, \qquad \varphi_i = \varphi_i\left(t_0\right) = \varphi_{i0} = const \tag{20}$$

Substituting solutions (18), (19) and (20) into equations (14), we obtain the following equations

$$p_i = \frac{K_{i0}}{A_i}\sin\theta_{i0}\sin\varphi_{i0}, \qquad q_i = \frac{K_{i0}}{A_i}\sin\theta_{i0}\cos\varphi_{i0}, \qquad q_i = \frac{K_{i0}}{A_i}\cos\theta_{i0}, \tag{21}$$

Thus, in a non-rotating coordinate system, we get

$$\begin{aligned} \omega_{iX} &= \dot{\theta}_i \cos\psi_i + \dot{\varphi}_i \sin\theta_i \sin\psi_{i0}, \\ \omega_{iY} &= \dot{\theta}_i \sin\psi_i - \dot{\varphi}_i \sin\theta_i \cos\psi_{i0}, \\ \omega_{iZ} &= \dot{\psi}_i + \dot{\varphi}_i \cos\theta_i, \end{aligned} \tag{22}$$

$$\omega_{iX} = 0, \qquad \omega_{iY} = 0, \qquad \omega_{iZ} = \dot{\psi}_i, \tag{23}$$

$$\omega_i = \sqrt{\omega_{iX}^2 + \omega_{iY}^2 + \omega_{iZ}^2} = \dot{\psi}_i \neq const \tag{24}$$

The solutions found for the differential equations of rotational motion retain their form even in secular perturbations. They will be used in calculations of the total kinetic moment of translational-rotational motion of a gravitational system when analysing dynamic evolution within the framework of a restricted three-body problem in analogues of the Delaunay-Andoyer variables.

4 Exact analytical solutions of the equation of secular perturbations of the translational motion of the centers of mass of two primary spherical bodies with variable masses in the baricentric coordinate system.

4.1 Derivation of the perturbation function

Due to its complexity, so the translational-rotational motion of a restricted three-body system is studied via perturbation theory methods in Delaunay-Andoyer type variables and exact analytical solutions are obtained for secular perturbation equations of the translational-rotational motion of two primary spherical bodies in analogues of Delaunay-Andoyer variables.

Using perturbation theory based on aperiodic motion along a quasi-conical section [12], we derive the perturbing functions from equations (1). We rewrite equations (1) as

$$\ddot{\vec{r}}_1 = -f\tilde{m}_1\frac{\vec{r}_1}{r_1^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\dot{\vec{r}}_1 + \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\frac{\dot{\gamma}_1}{\gamma_1}\right]\vec{r}_1 + \vec{F}_{1pert}, \tag{25}$$

$$\ddot{\vec{r}}_1 + f\tilde{m}_1\frac{\vec{r}_1}{r_1^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\dot{\vec{r}}_1 - \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\frac{\dot{\gamma}_1}{\gamma_1}\right]\vec{r}_1 = \vec{F}_{1pert}, \tag{26}$$

$$\vec{F}_{1pert} = \tilde{B}_1\vec{r}_1 - \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\frac{\dot{\gamma}_1}{\gamma_1}\right]\vec{r}_1, \tag{27}$$

The unknown arbitrary function $\gamma_1 = \gamma_1(t)$ is defined by the following conditions

$$\frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right) = \tilde{A}_1 = 2\frac{\dot{\nu}_2}{\nu_2}, \tag{28}$$

Then we get

$$\gamma_1 = \gamma_1(t) = \frac{\tilde{m}_1(t_0)}{\tilde{m}_1(t)}e^{2\int_{t_0}^{t}\tilde{A}_1 dt} = \frac{m_2}{m_{20}}\frac{m_0^2}{m^2}, \tag{29}$$

Taking into account equations (28) and (29) from (27), we obtain

$$\vec{F}_{1pert} = \tilde{B}_1^*(t)\,\vec{r}_1, \tag{30}$$

Accordingly, using γ_1 we can formulate an explicit form of the function $\dot{B}_1^*$

$$\tilde{B}_1^*(t) = \tilde{B}_1 - \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \tilde{A}_1\frac{\dot{\gamma}_1}{\gamma_1}\right], \tag{31}$$

In the result, equation (26) takes the form

$$\ddot{\vec{r}}_1 + f\tilde{m}_1\frac{\vec{r}_1}{r_1^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\dot{\vec{r}}_1 - \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\frac{\dot{\gamma}_1}{\gamma_1}\right]\vec{r}_1 = grad_{\vec{r}_1}U_1, \tag{32}$$

where, $\vec{F}_{1pert} = \tilde{B}_1^*\vec{r}_1 = grad_{\vec{r}_1}U_1$ - perturbing force, U_1 - perturbing function

$$U_1 = \frac{1}{2}\tilde{B}_1^*(t)r_1^2, \tag{33}$$

Similarly, we write the differential equations (1) in a form convenient for using perturbation theory for the body P_2.

$$\ddot{\vec{r}}_2 + f\tilde{m}_2\frac{\vec{r}_2}{r_2^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_2}{\tilde{m}_2} + \frac{\dot{\gamma}_2}{\gamma_2}\right)\dot{\vec{r}}_2 - \left[\frac{\ddot{\gamma}_2}{\gamma_2} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_2}{\tilde{m}_2} + \frac{\dot{\gamma}_2}{\gamma_2}\right)\frac{\dot{\gamma}_2}{\gamma_2}\right]\vec{r}_2 = grad_{\vec{r}_2}U_2, \tag{34}$$

where, $\vec{F}_{2pert} = \tilde{B}_2^*(t)\vec{r}_2 = grad_{\vec{r}_2}U_2$ - perturbing force, U_2 - perturbing function

$$U_2 = \frac{1}{2}\tilde{B}_2^*(t)r_2^2, \tag{35}$$

$$\tilde{B}_2^*(t) = \tilde{B}_2 - \left[\frac{\ddot{\gamma}_2}{\gamma_2} - \tilde{A}_2\frac{\dot{\gamma}_2}{\gamma_2}\right], \qquad \gamma_2 = \gamma_2(t) = \frac{m_1}{m_{10}}\frac{m_0^2}{m^2}, \tag{36}$$

4.2 Unperturbed motion

In the case where $\vec{F}_{1pert} = 0, \vec{F}_{2pert} = 0$ from equations (32), (34) follows unperturbed motion [12]

$$\ddot{\vec{r}}_i + f\tilde{m}_j\frac{\vec{r}_j}{r_j^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_j}{\tilde{m}_j} + \frac{\dot{\gamma}_j}{\gamma_j}\right)\dot{\vec{r}}_j - \left[\frac{\ddot{\gamma}_j}{\gamma_j} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_j}{\tilde{m}_j} + \frac{\dot{\gamma}_j}{\gamma_j}\right)\frac{\dot{\gamma}_j}{\gamma_j}\right]\vec{r}_j = 0, \tag{37}$$

The solution of the unperturbed motion (37) is well investigated and has the following form

$$\begin{aligned} x_j &= \gamma_j\rho_j\left[\cos u_j \cos\Omega_j - \sin u_j \sin\Omega_j \cos i_j\right], \\ y_j &= \gamma_j\rho_j\left[\cos u_j \sin\Omega_j + \sin u_j \cos\Omega_j \cos i_j\right], \\ z_j &= \gamma_j\rho_j\left[\sin u_j \sin i_j\right], r_j^2 = x_j^2 + y_j^2 + z_j^2 = \gamma_j^2\rho_j^2 \end{aligned} \tag{38}$$

$$\begin{aligned} \dot{x}_j &= \left(\frac{\dot{\gamma}_j}{\gamma_j} + \frac{\dot{\rho}_j}{\rho_j}\right)x_j + \gamma_j\rho_j\dot{u}_j\left[-\sin u_j \cos\Omega_j - \cos u_j \sin\Omega_j \cos i_j\right], \\ \dot{y}_j &= \left(\frac{\dot{\gamma}_j}{\gamma_j} + \frac{\dot{\rho}_j}{\rho_j}\right)y_j + \gamma_j\rho_j\dot{u}_j\left[-\sin u_j \sin\Omega_j - \cos u_j \cos\Omega_j \cos i_j\right], \\ \dot{z}_j &= \left(\frac{\dot{\gamma}_j}{\gamma_j} + \frac{\dot{\rho}_j}{\rho_j}\right)z_j + \gamma_j\rho_j\dot{u}_j\left[\cos u_j \sin i_j\right], \end{aligned} \tag{39}$$

$$\rho_j = \frac{a_j(1-e_j^2)}{1+e_j\cos\theta_j}, j = 1, 2 \tag{40}$$

where θ_j is the true anomaly, and the parameters $a_j, e_j, \omega_j, \Omega_j, i_j$ are analogous to the semi-major axis, eccentricity, inclination, longitude of the perihelion and longitude of the ascending node of the P_1 and P_2 bodies.

$$\dot{\rho}_j = \frac{1}{\gamma_j^2}\sqrt{\frac{\mu_{j0}}{p_j}}e_j\sin\theta_j, \qquad \mu_{j0} = f\tilde{m}_{j0} = const \tag{41}$$

$$\dot{u}_j = \frac{1}{\gamma_j^2}\frac{\sqrt{\mu_{j0}p_j}}{\rho_j^2}, u_j = \theta_j + \omega_j \tag{42}$$

$$\int_0^v \frac{dv}{(1+e\cos\theta)^2} = \frac{\sqrt{\mu_{j0}}}{p^{3/2}}\left[\phi(t) - \phi(\tau)\right], \tag{43}$$

where $\phi(t)$ is the antiderivative of the $(\tilde{m}_i/\tilde{m}_{i0}\gamma_i^3)^{1/2}$ function, and τ_j is the time of passage through pericenter.

4.3 Exact analytical solutions to the equations of secular perturbations of translational motion in osculating analogues of Delaunay variables

The translational motion of the body P_1 in Delaunay variables can be written in the following form:

$$\begin{aligned} \dot{L}_1 &= \frac{\partial U_1^*}{\partial l_1}, \ \dot{G}_1 = \frac{\partial U_1^*}{\partial g_1}, \ \dot{H}_1 = \frac{\partial U_1^*}{\partial h_1}, \\ \dot{l}_1 &= -\frac{\partial U_1^*}{\partial L_1}, \ \dot{g}_1 = -\frac{\partial U_1^*}{\partial G_1}, \ \dot{h}_1 = -\frac{\partial U_1^*}{\partial H_1}, \end{aligned} \tag{44}$$

Accordingly, the Hamiltonian U_1^* expression takes the following form:

$$U_1^* = \left(\frac{\tilde{m}_1}{\tilde{m}_{10}\gamma_1^3(t)}\right)^{1/2}\frac{\mu_{10}^2}{2L_1^2} + \left(\frac{\tilde{m}_{10}}{\tilde{m}_1\gamma_1^3(t)}\right)^{1/2}U(...) \tag{45}$$

where $\mu_{10} = f\tilde{m}_{10}$

The perturbing part of the Hamiltonian:

$$U = \frac{1}{2}\tilde{B}_1^*(t)\vec{r}_1^2 \tag{46}$$

The quantity $\vec{r}_1^{\,2}$ is expanded into a series in terms of the small parameter e the eccentricity

$$r_1^2 = \gamma_1^2 \rho_1^2 = \gamma_1^2 a_1^2 \left(\frac{\rho_1}{a_1}\right)^2 = \gamma_1^2 a_1^2 \left[1 - 2e_1 \cos M + \frac{e_1^2}{2}(3 - \cos 2M) + ...\right] \tag{47}$$

The relation between the analogues of the Keplerian elements and the Delaunay variables is as follows:

$$a = \frac{L^2}{\mu_0}, \qquad e^2 = 1 - \frac{G^2}{L^2}, \qquad M \equiv l, \qquad \omega = g, \qquad \Omega = h, \qquad \cos i = \frac{H}{G} \tag{48}$$

If the above quantities are substituted, the perturbing function takes the following form:

$$U = \frac{1}{2}\tilde{B}_1^*(t)\vec{r}_1^{\,2} = \frac{1}{2}\tilde{B}_1^*(t)\gamma_1^2 a_1^2 \left[1 - 2e_1 \cos M + \frac{e_1^2}{2}(3 - \cos 2M) + ...\right] \tag{49}$$

By averaging over the mean anomaly $M \equiv l$, we obtained the secular part of the perturbation function in the analogues of Delaunay variables

$$U_{sec} = \frac{1}{2}\tilde{B}_1^*(t)\vec{r}_1^{\,2} = \frac{1}{2}\tilde{B}_1^*(t)\gamma_1^2 \frac{L_1^2}{\mu_{10}^2}\left[1 + \frac{3}{2}\left(1 - \frac{G_1^2}{L_1^2}\right)\right] \tag{50}$$

Substituting the secular part of the perturbation function (50) into equations (44)-(45), we get

$$\begin{array}{lll} \dot{L}_1 = 0, & \dot{G}_1 = 0, & \dot{H}_1 = 0, \\ \dot{l}_1 = -\dfrac{\partial U_1^*}{\partial L_1} = -\dfrac{5}{2}\dfrac{\tilde{B}_1^*(t)\gamma_1^2}{\mu_{10}^2} L_1, & \dot{g}_1 = -\dfrac{\partial U_1^*}{\partial G_1} = -\dfrac{3}{2}\dfrac{\tilde{B}_1^*(t)\gamma_1^2}{\mu_{10}^2} G_1, & \dot{h}_1 = 0, \end{array} \tag{51}$$

Hence it follows that

$$\begin{array}{l} L_1 = L_1(t_0) = L_{10} = const, \qquad G_1 = G_1(t_0) = G_{10} = const, \\ H_1 = H_1(t_0) = H_{10} = const, \qquad h_1 = h_1(t_0) = h_{10} = const, \\ l_1 = l_1(t) = l_1(t_0) - \dfrac{5}{2}\dfrac{L_{10}}{\mu_{10}^2}\displaystyle\int_{t_0}^{t} \tilde{B}_1^*(t)\gamma_1^2(t)dt, \ \dot{g}_1 = g_1(t_0) - \dfrac{3}{2}\dfrac{G_{10}}{\mu_{10}^2}\displaystyle\int_{t_0}^{t} \tilde{B}_1^*(t)\gamma_1^2(t)dt, \end{array} \tag{52}$$

where according to (29) -(31) $\gamma_1 = \gamma_1(t) = m_2 m_0^2 / m_{20} m^2$, $\tilde{B}_1^* = \tilde{B}_1 - \left[\ddot{\gamma}_1/\gamma_1 - \tilde{A}_1\dot{\gamma}_1/\gamma_1\right]$.

Thus, formulas (52), (48), (38), (39) completely determine the coordinates and velocities of body P_1 in a secular perturbation.

Similarly, we obtain the coordinates and velocities of the body P_2 using exact analytical solutions of the equation of secular perturbations for the body P_2.

$$\begin{array}{l} L_2 = L_2(t_0) = L_{20} = const, \qquad G_2 = G_2(t_0) = G_{20} = const, \\ H_2 = H_2(t_0) = H_{20} = const, \qquad h_2 = h_2(t_0) = h_{20} = const, \\ l_2 = l_2(t) = l_2(t_0) - \dfrac{5}{2}\dfrac{L_{20}}{\mu_{20}^2}\displaystyle\int_{t_0}^{t} \tilde{B}_2^*(t)\gamma_2^2(t)dt, \ \dot{g}_2 = g_2(t_0) - \dfrac{3}{2}\dfrac{G_{20}}{\mu_{20}^2}\displaystyle\int_{t_0}^{t} \tilde{B}_2^*(t)\gamma_2^2(t)dt, \end{array} \tag{53}$$

Where, according to (36) $\tilde{B}_2^*(t) = \tilde{B}_2 - \left[\ddot{\gamma}_2/\gamma_2 - \tilde{A}_2\dot{\gamma}_2/\gamma_2\right], \gamma_2 = \gamma_2(t) = m_1 m_0^2/m_{10}m^2$.

Formulas (53), (48), (38), (39) completely determine the coordinates and velocities of body P_2 in a secular perturbation.

5 Analytical expressions for the coordinates and velocities of two primary spherical bodies with variable masses based on exact solutions of differential equations of secular perturbations

Consequently, taking into account the formulas of unperturbed motion, which retain their form in perturbed motion, coordinates and velocities in the equations of secular perturbations, in analogues of Kepler's variables, appear as follows:

$$\begin{aligned}
x_{jsec} &= \gamma_j a_j \left[\left[\left(-\frac{3}{2}e\right)\cos\omega_j\right]\cos\Omega_j - \left[\left(-\frac{3}{2}e\right)\sin\omega_j\right]\sin\Omega_j\cos i_j\right],\\
y_{jsec} &= \gamma_j a_j \left[\left[\left(-\frac{3}{2}e\right)\cos\omega_j\right]\sin\Omega_j + \left[\left(-\frac{3}{2}e\right)\sin\omega_j\right]\cos\Omega_j\cos i_j\right],\\
z_{jsec} &= \gamma_j a_j \left[\left[\left(-\frac{3}{2}e\right)\sin\omega_j\right]\sin i_j\right]
\end{aligned} \tag{54}$$

$$\begin{aligned}
\dot{x}_{jsec} &= \frac{e_j}{2a_j\gamma_j}\left(\sqrt{\mu_{j0}p_j}\left(\sin\omega_j\cos\Omega_j + \cos\omega_j\sin\Omega_j\cos i_j\right) - 3a_j\dot{\gamma}_j\left(\sin\omega_j\sin\Omega_j + a_j\gamma_j\cos\omega_j\cos\Omega_j\right)\right),\\
\dot{y}_{jsec} &= \frac{e_j}{2a_j\gamma_j}\left(\sqrt{\mu_{j0}p_j}\left(\sin\omega_j\sin\Omega_j - \cos\omega_j\cos\Omega_j\cos i_j\right) + 3a_j\dot{\gamma}_j\left(\sin\omega_j\cos\Omega_j - a_j\gamma_j\cos\omega_j\sin\Omega_j\right)\right),\\
\dot{z}_{jsec} &= \frac{e_j}{2a_j}\left(-\frac{\sqrt{\mu_{j0}p_j}}{\gamma_j}\cos\omega_j + 3a_j^2\dot{\gamma}_j\sin\omega_j\right)\sin i_j,
\end{aligned} \tag{55}$$

Further, formulas (54) – (55) will be rewritten in analogues of Delaunay variables using known transformation formulas (48), which will be used in the study of translational-rotational motion of a non-stationary small axisymmetric body.

In equations (4) – (6) of translational-rotational motion of a small non-stationary axisymmetric body P_3, the values $\vec{r}_j\left(x_j, y_j, z_j\right), \dot{\vec{r}}_j\left(\dot{x}_j, \dot{y}_j, \dot{z}_j\right), j = 1, 2$, according to the formulas (54) - (55) found above, are already known functions of time.

Thus, the problem of investigating secular perturbations of the translational -rotational motion of a non-stationary small axisymmetric body of variable mass in a Newtonian gravitational field of two primary spherical bodies with variable masses within the framework of a restricted three-body problem with variable masses in a baricentric coordinate system is reduced only to finding the coordinates x_3, y_3, z_3 and velocity $\dot{x}_3, \dot{y}_3, \dot{z}_3$ of a small non-stationary body.

6 Conclusion

In this paper, we investigated the problem of translational-rotational motion of a non-stationary axisymmetric small body in the Newtonian gravitational field of two primary spherically symmetric bodies with variable masses in a restricted formulation. We studied secular perturbations in analogues of the Delaunay-Andoyer variables.

In general, the solution to the problem of translational-rotational motion of two primary spherical bodies with isotropically varying masses is unknown, so the problem is investigated by jointly considering the differential equations of the problem of two primary bodies with variable masses and the differential equations of motion of a small non-stationary axisymmetric body. The problem is complex, so the problem investigated using perturbation theory methods.

As the main new result of this work, we have found exact analytical solutions to the equations of secular perturbations of the translational-rotational motion of two primary spherical bodies with variable masses.

Due to the results obtained in this work, the problem of investigating secular perturbations of the translational-rotational motion of a non-stationary small axisymmetric body of variable mass in the Newtonian gravitational field of two primary spherical bodies with variable masses within the framework of a restricted three-body problem with variable masses in a barycentric coordinate system is summarised as finding only the coordinates $\vec{r}_3\,(x_3, y_3, z_3)$ and velocity $\dot{\vec{r}}_3\,(\dot{x}_3, \dot{y}_3, \dot{z}_3)$ of a small non-stationary body.

Thus, the investigation of secular perturbations of the problem considered is greatly simplified, as in the classical restricted problem, in the following only the motion of a small non-stationary axisymmetric body will be investigated.

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