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M. Konyrkulzhayeva[1*], **G. Auzerkhan**[2]
[1]International Information Technologies University, Almaty, Kazakhstan
[2]Al -Farabi Kazakh National university, Almaty, Kazakhstan
e-mail: *m.konyrkulzhayeva@edu.iitu.kz

AN ELLIPTIC SELF-ADJOINT OPERATOR OF THE SECOND ORDER ON A GRAPH WITH SMALL EDGES

This work is devoted to the study of a second-order elliptic self-adjoint operator on a metric graph with short edges. The underlying structure is constructed by rescaling a given graph by the factor ε^{-1} and attaching it to another fixed graph, where $\varepsilon > 0$ is a small parameter. No substantial restrictions are imposed on the pair of graphs. On this combined structure, we define a general second-order elliptic self-adjoint operator whose differential expression involves derivatives of arbitrary order with variable coefficients and a non-constant potential. The vertex conditions are taken in a general form as well. All coefficients, both in the differential expression and in the vertex conditions, are allowed to depend analytically on the small parameter ε. It was previously established that the components of the resolvent corresponding to the restrictions of the operator to the fixed-length edges and to the short edges are analytic in ε as operators in the corresponding functional spaces, with the restriction on short edges additionally conjugated by dilation operators. Analyticity here means representability of these operator families by Taylor series. The first principal result of the paper is a recursive procedure, reminiscent of the method of matched asymptotic expansions, for determining all coefficients of such Taylor series. The second main result provides a convergent expansion of the resolvent in the form of a Taylor-type series, together with effective estimates of the remainder terms.

Key words: graph, differential operator, resolvent, boundary conditions, Taylor series.

М. Коныркулжаева [1*], Г. Аузерхан [2]
[1]Халықаралық ақпараттық технологиялар университеті, Алматы, Қазақстан
[2]Әл-Фараби атындағы Қазақ Ұлттық Университеті, Алматы, Қазақстан
e-mail: *m.konyrkulzhayeva@edu.iitu.kz

Шағын доғасы бар графтар бойында анықталған өзіне-өзі түйіндес екінші ретті эллиптикалық дифференциалдық оператор

Бұл жұмыс қысқа доғалары бар метрикалық графтағы екінші ретті эллиптикалық өзіне-өзі түйіндес операторды зерттеуге арналған. Бастапқы құрылым берілген графты ε^{-1} коэффициентіне дейін масштабтау және оны басқа бекітілген графқа жалғау арқылы құрылады, мұндағы $\varepsilon > 0$ - кіші параметр. Графтардың жұбына айтарлықтай шектеулер қойылмайды. Осы біріктірілген құрылымда екінші ретті эллиптикалық өзіне-өзі түйіндес оператор анықталады, дифференциалдық өрнегі айнымалы коэффициенттері бар туындылар және тұрақты емес потенциал арқылы анықталған. Графтың төбелеріндегі шарттар да жалпы түрде беріледі. Дифференциалдық өрнекте және төбелердегі шарттардағы барлық коэффициенттер кіші ε параметріне аналитикалық тәуелді болуы мүмкін. Алдыңғы зерттеулерде оператордың тұрақты ұзындықтағы доғалардағы және қысқа доғалардағы шектеулеріне сәйкес резольвентаның компоненттері ε параметріне қатысты тиісті функционалдық кеңістіктердегі операторлар ретінде аналитикалық екендігі дәлелденген. Сонымен қатар, қысқа доғалардағы шектеулер қосымша түрде дилатация операторларымен үйлестіріледі. Мұндағы аналитикалық дегеніміз - осы операторлар тобын Тейлор қатары арқылы өрнектеу мүмкіндігі. Жұмыстың бірінші негізгі нәтижесі - Тейлор қатарларының барлық коэффициенттерін табуға арналған, келісілген асимптотикалық жіктемелер әдісіне ұқсас рекурсивті процедура болып табылады. Екінші негізгі нәтиже резольвентаны тейлорлық типтегі қатар түрінде жинақты жіктеу мен қалдық мүшелердің тиімді бағалауларын ұсынады.

Түйін сөздер: граф, дифференциалдық оператор, резольвента, шекаралық шарттар, Тейлор қатары.

М. Коныркулжаева [1*], Г. Аузерхан [2]
[1]Международный университет информационных технологий, Алматы, Казахстан
[2]Казахский Национальный Университет имени Аль-Фараби, Алматы, Казахстан
e-mail: *m.konyrkulzhayeva@edu.iitu.kz

Эллиптический самосопряженный оператор второго порядка на графе с малыми ребрами

Данная работа посвящена исследованию эллиптического самосопряжённого оператора второго порядка на метрическом графе с малыми рёбрами. Исходная структура строится путём масштабирования данного графа с коэффициентом ε^{-1} и присоединения его к другому фиксированному графу, где $\varepsilon > 0$ - малый параметр. Существенных ограничений на пару графов не накладывается. На этой комбинированной структуре определяется общий эллиптический самосопряжённый оператор второго порядка, дифференциальное выражение которого включает производные произвольного порядка с переменными коэффициентами и непостоянным потенциалом. Условия в вершинах также задаются в общей форме. Все коэффициенты - как в дифференциальном выражении, так и в условиях в вершинах - допускается зависимыми от малого параметра ε аналитическим образом. Ранее было установлено, что компоненты резольвенты, соответствующие ограничениям оператора на рёбра фиксированной длины и на короткие рёбра, аналитичны по ε как операторы в соответствующих функциональных пространствах, при этом ограничение на коротких рёбрах дополнительно сопрягается с операторами дилатации. Под аналитичностью здесь понимается представимость этих семейств операторов в виде рядов Тейлора.Первым основным результатом работы является рекурсивная процедура, напоминающая метод согласованных асимптотических разложений, для нахождения всех коэффициентов таких рядов Тейлора. Второй главный результат даёт сходящийся разложение резольвенты в виде ряда тейлоровского типа вместе с эффективными оценками остаточных членов.

Ключевые слова: граф, дифференциальный оператор, резольвента, граничные условия, ряд Тейлора.

1 Introduction

One of the actively developing areas of modern spectral theory of operators over the past two decades is the theory of quantum graphs [1–6]. In this theory, special attention is paid to perturbations caused by geometric features of the graph, in particular, the presence of small edges. Early studies concerning Taylor series for resolvents of operators on graphs with short edges were focused on the approximation of certain boundary conditions at vertices using graphs of a special structure with small edges. Such an approximation was understood as uniform convergence of resolvents of the corresponding operators on the original and approximating graphs.

Questions of this kind were considered for Schrödinger operators on a number of simple model graphs in [7, 8]. In particular, in [7], a star graph of three edges was studied, one of which was considered small, with δ- or δ'-type conditions imposed at the central vertex. In [8], a graph of arbitrary structure was added to a graph including a loop and two fixed edges, obtained by scaling the given finite graph with a coefficient ε^{-1}, where ε is a small parameter. In these studies, it was found that the resolvents and spectra of such operators depend on ε analytically, which was unexpected, given the singular nature of such perturbations. Usually, for singular perturbations, it is possible to construct only asymptotic expansions for eigenvalues, while the questions of convergence of such series and, especially, their analytical dependence on a small parameter remain open [9,10]. Advances in the analysis of model operators became the motivation for studying more general operators on graphs with arbitrary topology and small edges. An elliptic self-adjoint operator of the second order with variable coefficients and general boundary conditions on a graph was considered, to which another graph with small edges whose lengths were proportional to the parameter ε was glued. Moreover, both the coefficients of the differential expression and the boundary conditions could analytically depend on ε. In this paper, we continue the study begun in [7,8] and focus on the analysis of the resolvent of the general operator. The main result is the construction of a recurrent procedure for the coefficients of the Taylor series of the resolvent components. Based on the results of [9,10], a uniformly convergent expansion of the resolvent in a Taylor series is obtained, and effective estimates of the remainders of this expansion in various operator norms are given.

1.1 Research methodology

The text does not specify the specific time, place, and conditions of the study, since the work is of a theoretical and mathematical nature. The study was conducted within the framework of mathematical modeling and analytical analysis, without involving experimental data or a sample of subjects. Abstract graphs constructed by compressing one graph and then gluing it to another, as well as the corresponding second-order elliptic operators, were considered as the "material". The main research tool was the methods of functional analysis, operator theory, and asymptotic expansions.

2 Research results

2.1 Statement of the Problem

The main object of the study of this article is a self-adjoint elliptic operator of the second order on a singular perturbed graph. The essence of a singular perturbation is the presence of small edges. A graph with small edges is obtained by gluing in a certain way a small graph to a given graph with edges of a fixed length. The latter graph is denoted by the symbol Γ and is a finite metric graph. This means that it contains a finite number of edges and vertices, on each edge a direction and a corresponding variable are introduced. As a measure on each edge, the standard Lebesgue measure is chosen. The graph Γ is allowed to have edges of infinite length. At the same time, we assume that this graph does not contain isolated edges and vertices.

By γ we denote another finite metric graph without isolated vertices and edges, and now we assume that the graph γ contains edges of only finite length. We compress the graph γ by a factor of ε^{-1}, i.e., we replace each of its edges e of length $|e|$ with an edge $\varepsilon|e|$ while preserving the rest of the graph structure. We denote the resulting graph by γ_ε.

In what follows, we will often identify the original graphs Γ and γ_ε with the corresponding subgraphs of Γ_ε, for which the same notations will be used. The directions and variables on the edges of Γ and γ_ε are preserved after the described gluing. Therefore, each function defined on the graphs Γ and γ_ε is simultaneously considered to be defined on the corresponding subgraphs of Γ_ε. And vice versa, a function defined on Γ_ε, is considered to be defined on the graphs Γ and γ_ε.

On the graph Γ_ε we consider the elliptic operator $\mathcal{H}_\varepsilon$ with the differential expression

$$\widehat{\mathcal{H}}(\varepsilon) := -\frac{d}{dx}p_\varepsilon\frac{d}{dx} + i\Big(\frac{d}{dx}q_\varepsilon + q_\varepsilon\frac{d}{dx}\Big) + V_\varepsilon, \tag{1}$$

where i is the imaginary unit, and the coefficients are given by the equalities

$$p_\varepsilon := \begin{cases} p_\Gamma(\cdot,\varepsilon) & \text{on } \Gamma, \\ S_\varepsilon p_\gamma(\cdot,\varepsilon) & \text{on } \gamma_\varepsilon, \end{cases} q_\varepsilon := \begin{cases} q_\Gamma(\cdot,\varepsilon) & \text{on } \Gamma, \\ \varepsilon_\varepsilon^{-1} p_\gamma(\cdot,\varepsilon) & \text{on } \gamma_\varepsilon, \end{cases} V_\varepsilon := \begin{cases} V_\Gamma(\cdot,\varepsilon) & \text{on } \Gamma, \\ \varepsilon^{-2} S_\varepsilon V_\gamma(\cdot,\varepsilon) & \text{on } \gamma_\varepsilon, \end{cases}$$

Here $p_\Gamma = p_\Gamma(\cdot,\varepsilon) \in W_\infty^1(\Gamma), q_\Gamma = q_\Gamma(\cdot,\varepsilon) \in W_\infty^1(\Gamma), V_\Gamma = V_\Gamma(\cdot,\varepsilon) \in L_2(\Gamma)$ and $p_\gamma = p_\gamma(\cdot,\varepsilon) \in W_\infty^1(\gamma), q_\gamma = q_\gamma(\cdot,\varepsilon) \in W_\infty^1(\gamma), V_\gamma = V_\gamma(\cdot,\varepsilon) \in L_2(\gamma)$–are some real functions defined respectively on the graphs Γ and γ and analytic in ε in the norm of the indicated spaces $\mathcal{S}_\varepsilon : L_2(\gamma) \to L_2(\gamma_\varepsilon)$-is a linear operator, defined by the formula

$$(\mathcal{S}_\varepsilon u)(x) := u\Big(\frac{x}{\varepsilon}\Big),\ x \in e_\varepsilon \tag{2}$$

on each edge e_ε of the graph γ_ε.

The differential expression $\widehat{\mathcal{H}}(\varepsilon)$ is considered uniformly elliptic: taking into account the analyticity of the functions p_Γ and p_γ with respect to ε, we assume that the inequalities

$$p_\Gamma(x,0) \ge c_\mathcal{H} \ \text{ on } \Gamma, p_\gamma(\xi,0) \ge c_\mathcal{H} \ \text{ on } \gamma$$

with some fixed constant $c_\mathcal{H} > 0$.

The boundary conditions for the operator $\mathcal{H}_\varepsilon$ are defined as follows. For an arbitrary vertex $M \in \Gamma_\varepsilon$ with degree $d(M) > 0$, we denote by $e_j(M), i = 1, ..., d(M)$, the edges coming out of M. We introduce a pair of $d(M)$-dimensional vectors

$$U_M(u) := \begin{pmatrix} u_1(M) \\ \vdots \\ u_{d(M)}(M) \end{pmatrix}, \quad U'_M(u) := \begin{pmatrix} \frac{du_1}{dx_1}(M) \\ \vdots \\ \frac{du_d}{dx_d}(M) \end{pmatrix}, \tag{3}$$

where x_i-variable on the edge e_i. The boundary conditions at the vertex $M \in \Gamma_\varepsilon$ are specified in the general form:

$$A_M(\varepsilon)U_M(u) + B_M(\varepsilon)U'_M(u) = 0, \tag{4}$$

where $A_M(\varepsilon)$ and $B_M(\varepsilon)$–analytic matrices of size $d(M) \times d(M)$ in ε.

Strictly,$\mathcal{H}_\varepsilon$ is defined as an unbounded operator in $L_2(\Gamma_\varepsilon)$ whose action is described by the differential expression (1) on the domain of definition composed of functions from the space $W_2^2(\Gamma_\varepsilon)$ satisfying the boundary conditions (4); here we use the notation $W_2^j(\cdot) := \oplus W_2^j(e),\ j = 1, 2$. All other operators that are used further in the paper are strictly defined in a similar way based on their differential expressions and boundary conditions.

We restrict our consideration to self-adjoint operators, which means that it is necessary to impose certain conditions on the coefficients of the differential expression (1) and the matrix in the boundary conditions (4). The criterion for the self-adjointness of the operator $\mathcal{H}_\varepsilon$ is the simultaneous fulfillment of the equality

$$\operatorname{rank}\Big(A_M(0)B_M(0)\Big) = d(M) \tag{5}$$

and the presence of self-adjointness of the matrix

$$A_M(\varepsilon)\Pi_M^{-1}(\varepsilon)B_M^*(\varepsilon) + iB_M(\varepsilon)\Pi_M^{-1}(\varepsilon)\Theta_M\Pi_M^{-1}(\varepsilon)B_M^*(\varepsilon),$$

where

$$\begin{aligned}&\prod\nolimits_M(\varepsilon) := \operatorname{diag}\{\vartheta_i(M)p_\varepsilon\big|_{e_i(M)}(M)\}_{i=1,\dots,d(M)},\\ &\Theta_M(\varepsilon) := \operatorname{diag}\{\vartheta_i(M)q_\varepsilon\big|_{e_i(M)}(M)\}_{i=1,\dots,d(M)},\end{aligned} \tag{6}$$

$e_i(M)$ are the edges emanating from the vertex M, and the numbers $\vartheta_i(M)$ are defined as follows: $e_i(M)$, if the direction on the edge $e_i(M)$ inward from the vertex M coincides with the initially chosen direction on this edge, and $\vartheta_i(M) := -1$, if these directions are opposite.

The boundary condition (4), in essence, does not change when multiplying it from the left by non-singular square matrices of size $d(M) \times d(M)$. Taking into account equality (5), we partially limit such freedom in the choice of matrices $A_M(\varepsilon)$ and $B_M(\varepsilon)$ by the following condition. We denote $r(M) := \operatorname{rank} B_M(0)$ and assume that the first $r(M)$ rows of the matrix $B_M(0)$ are linearly independent, and the last $d(M) - r(M)$ rows vanish. We simultaneously assume that the last $d(M) - r(M)$ rows of the matrix $A_M(0)$ are not equal to zero.

The main goal of this paper is to describe in detail the dependence of the resolvent of the operator $\mathcal{H}_\varepsilon$ on the parameter ε. To formulate the main result, we need to introduce auxiliary notations. This will be done in sections 2.2 and 2.3.

2.2 Auxiliary notations and the main condition

For convenience and to simplify a number of technical calculations, we assume throughout the paper that the directions on the edges $e_i, i = 1, ..., d_0$, of the graph Γ, emanating from the vertex M_0, are chosen inside the edges from the vertex M_0. If there is a loop among the edges e_i, then in order to ensure that such a condition is satisfied, we introduce an additional artificial vertex on the loops, on which we set the standard Kirchhoff condition. Such a vertex does not change either the operator $\mathcal{H}_\varepsilon$, or its resolvent, or its spectrum.

We introduce another auxiliary graph γ_∞, which is obtained by attaching edges of infinite length $e_i^\infty, i \in J_j, j = 1, ..., n$, to the vertices $M_j, j = 1, ..., n$, of the graph γ. The variable on the graph γ is denoted by ξ. An auxiliary operator $\mathcal{H}_\infty$ is defined on the graph $\mathcal{H}_\infty$. This is an unbounded operator in $L_2(\gamma_\infty)$ with the differential expression

$$\begin{gathered}\widehat{\mathcal{H}}_\infty := -\tfrac{d}{d\xi}p_\gamma(\cdot,0)\tfrac{d}{d\xi} + i\Big(\tfrac{d}{d\xi}q_\gamma(\cdot,0) + q_\gamma(\cdot,0)\tfrac{d}{d\xi}\Big) + V_\gamma(\cdot,0) \text{ on } \gamma,\\ \widehat{\mathcal{H}}_\infty := -p_i(0)\tfrac{d^2}{d\xi^2} \text{ on } e_i^\infty,\ i \in J_j,\ j = 1, ..., n,\ \ p_i(\varepsilon) := p_\Gamma\big|_{e_i}(M_0,\varepsilon),\end{gathered}$$

and boundary conditions

$$A_M^{(0)}U_M(u) + B_M^{(0)}U_M'(u) = 0 \text{ at the vertices } M \in \gamma_\infty.$$

Here the vectors $U_M(u)$ and $U'_M(u)$ are introduced similarly to (3) with the replacement of derivatives $\frac{du_i}{dx_i}$ on $\frac{du_i}{d\xi_i}$. The matrices $A_M^{(0)}$ and $B_M^{(0)}$ are defined by the formulas

$$A_M^{(0)} := \begin{pmatrix} 0 \\ A_M^-(0) \end{pmatrix}, B_M^{(0)} := \begin{pmatrix} B_M^+(0) \\ \frac{dB_M^-}{d\varepsilon}(0) \end{pmatrix},$$

where $A_M^-(\cdot)$ and $B_M^-(\cdot)$ are the matrices composed of the last $d(M) - r(M)$ rows of the matrices $A_M(\cdot)$ and $B_M(\cdot)$ and the matrix $B_M^+(\cdot)$ is formed by the first $r(M)$ rows of the matrix B_M. The operator $\mathcal{H}_\infty$ is self-adjoint, and its essential spectrum is the semi-axis $[0, \infty)$.

The fundamental condition imposed on the operator $\mathcal{H}_\varepsilon$ is expressed in terms of the operator $\mathcal{H}_\infty$:

(A) *The operator $\mathcal{H}_\infty$ has no embedded eigenvalues at the edge of its essential spectrum.*

Equivalently, condition (A) can be reformulated as follows. Consider the boundary value problem

$$\widehat{\mathcal{H}}_\infty \psi = 0 \text{ on } \gamma_\infty, \;\; A_M^{(0)} U_M(\psi) + B_M^{(0)} U'_M(\psi) = 0 \text{ at the vertices } M \in \gamma_\infty. \tag{7}$$

By virtue of the definition of the differential expression $\widehat{\mathcal{H}}_\infty$ on semi-infinite edges e_i^∞, the solution of this problem on these edges is given by a linear function. Therefore, the absence of an embedded eigenvalue on the edge of the essential spectrum of the operator $\mathcal{H}_\infty$ is equivalent to the absence of nontrivial solutions $\psi \in W_2^2(\gamma) \oplus W_{2,loc}^2(e_i^\infty)$ of problem (7), which identically vanish on all edges e_i^∞. At the same time, the presence of nontrivial solutions that are constant on these edges and do not simultaneously vanish on all edges e_i^∞, is not excluded. In other words, the presence of a virtual level is allowed on the edge of the essential spectrum of the operator $\mathcal{H}_\infty$.

Let $\psi^{(j)}, j = 1, ..., k$, be linearly independent solutions of problem (7), constant on the edges e_i^∞. By condition (A), these functions do not vanish identically simultaneously on all edges e_i^∞. It is clear that $k \le d_0$. If there are no such solutions, then we set $k := 0$.

For an arbitrary function u defined on the edges $e_i^\infty, i \in J_j, j = 1, ..., n$, at least in the neighborhood of the vertex M_j and continuous up to this vertex, we introduce the notation

$$U_\gamma(u) := \left(u\Big|_{e_i^\infty}(M_j) \right)_{i \in J_j, j=1,\dots,n}.$$

We denote $\Psi^{(j)} := U_\gamma(\psi^{(j)}), j = 1, ..., k$ and we choose the functions $\psi^{(j)}$ so that the introduced vectors $\Psi^{(j)}$ are orthonormal in C^{d_0}. If $k < d_0$, then we additionally choose arbitrary vectors $\Psi^{(j)} \in C^{d_0}, j = k+1, ..., d_0$, so that the entire set of vectors $\Psi^{(j)} \in C^{d_0}, j = 1, ..., d_0$, forms an orthonormal basis in C^{d_0}. This means that the matrix $\Psi := \left(\Psi^{(1)} ... \Psi^{(k)} \Psi^{(k+1)} ... \Psi^{(d_0)} \right)$ is unitary.

An important role will be played by another auxiliary operator on yet another graph, denoted by γ_{ex} and obtained by attaching unit edges $e_i^{ex}, i \in J_j, j = 1, ..., n$, to vertices $M_j, j = 1, \dots, n$, of the graph γ. Vertices M_j will be considered the beginning of edges e_i^∞, i.e., the direction on these edges is chosen inward from M_j. Vertices that are the ends of edges

e_i^∞ will be denoted by M_i^{ex}, $i \in J_j$, $j = 1, ..., n$. The mentioned auxiliary operator is denoted by $\mathcal{H}_{ex}(\varepsilon)$ and is determined by the differential expression

$$\begin{gathered}\widehat{\mathcal{H}}_{ex}(\varepsilon) := -\tfrac{d}{d\xi}p_\gamma(\cdot,\varepsilon)\tfrac{d}{d\xi} + i\Big(\tfrac{d}{d\xi}q_\gamma(\cdot,\varepsilon) + q_\gamma(\cdot,\varepsilon)\tfrac{d}{d\xi}\Big) + V_\gamma(\cdot,\varepsilon) \text{ on } \gamma \\ \widehat{\mathcal{H}}_{ex}(\varepsilon) := -p_i(\varepsilon)\tfrac{d^2}{d\xi^2} + 2i\varepsilon q_i(\varepsilon) \text{ on } e_i^{ex},\ i \in J_j,\ j = 1, ...n,\end{gathered} \tag{8}$$

with boundary conditions

$$\varepsilon A_M(\varepsilon)U_M(u) + B_M(\varepsilon)U'_M(u) = 0 \tag{9}$$

at the vertices $M \in \gamma_\infty$ and boundary conditions of the third type

$$\Pi_{\Gamma,M_0}(\varepsilon)U'_{ex}(u) - i\varepsilon\Theta_{\Gamma,M_0}(\varepsilon)U_{ex}(u) = 0$$

at the vertices M_i^{ex}, where it is denoted

$$U'_{ex}(u) := \begin{pmatrix} \frac{du|_{e_1^{ex}}}{d\xi_1}(M_1^{ex}) \\ \vdots \\ \frac{du|_{e_{d_0}^{ex}}}{d\xi_{d_0}}(M_{d_0}^{ex}) \end{pmatrix}, U_{ex}(u) := \begin{pmatrix} u(M_1^{ex}) \\ \vdots \\ u(M_{d_0}^{ex}) \end{pmatrix}, q_i(\varepsilon) := \vartheta_i(M_0)q_\Gamma|_{e_i}(M_0,\varepsilon).$$

2.3 Parts of the resolvent and known results

Let $\mathcal{R}_\Gamma : L_2(\Gamma_\varepsilon) \to L_2(\Gamma)$ and $\mathcal{R}_{\gamma_\varepsilon} : L_2(\Gamma_\varepsilon) \to L_2(\gamma_\varepsilon)$–restriction operators to subgraphs of Γ and γ_ε, namely $\mathcal{R}_\Gamma f := f|_\Gamma$, $\mathcal{R}_{\gamma_\varepsilon} f := f|_{\gamma_\varepsilon}$. It is clear that

$$L_2(\Gamma_\varepsilon) = L_2(\Gamma) \oplus L_2(\gamma_\varepsilon), \mathcal{R}_\Gamma \oplus \mathcal{R}_{\gamma_\varepsilon} = \mathcal{J}_{\Gamma_\varepsilon}, \tag{10}$$

where $\mathcal{J}_{\Gamma_\varepsilon}$- is the identity operator in $L_2(\Gamma_\varepsilon)$.

Since the operator $\mathcal{H}_\varepsilon$ is self-adjoint, its resolvent $(\mathcal{H}_\varepsilon - \lambda)^{-1}$ is well defined for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Therefore, we can introduce a pair of operators

$$R_\Gamma(\varepsilon,\lambda) := \mathcal{R}_\Gamma(\mathcal{H}_\varepsilon - \lambda)^{-1}(\mathcal{J}_\Gamma \oplus \mathcal{S}_\varepsilon), R_\gamma(\varepsilon,\lambda) := \mathcal{S}_\varepsilon^{-1}\mathcal{R}_{\gamma_\varepsilon}(\mathcal{H}_\varepsilon - \lambda)^{-1}(\mathcal{J}_\Gamma \oplus \mathcal{S}_\varepsilon),$$

where $\mathcal{J}_\Gamma$– is the identity operator in $L_2(\Gamma)$. These operators are linear and bounded as acting from $L_2(\Gamma) \oplus L_2(\gamma)$ to $W_2^2(\Gamma)$ and $W_2^2(\gamma)$, respectively.

Let us explain the action of the operators R_Γ and R_γ. For an arbitrary pair of functions $(f_\Gamma, f_\gamma) \in L_2(\Gamma) \oplus L_2(\gamma)$, we construct a function $f \in L_2(\Gamma_\varepsilon)$ by the rule $f := f_\Gamma$ on Γ and $f := \mathcal{S}_\varepsilon f_\gamma$ on γ_ε. Next, the resolvent is applied to f and restrictions of the result to the subgraphs of Γ and γ_ε are considered, i.e., the functions $\mathcal{R}_\Gamma(\mathcal{H}_\varepsilon - \lambda)^{-1}f$ and $\mathcal{R}_{\gamma_\varepsilon}(\mathcal{H}_\varepsilon - \lambda)^{-1}f$. The first of these functions is the action of the operator $\mathcal{R}_\Gamma(\varepsilon,\lambda)$ on (f_Γ, f_γ). To the second restriction we additionally apply the operator $\mathcal{S}_\varepsilon^{-1}$: the resulting function $\mathcal{S}_\varepsilon^{-1}\mathcal{R}_{\gamma_\varepsilon}(\mathcal{H}_\varepsilon - \lambda)^{-1}f$- is the action of the operator $\mathcal{R}_\gamma(\varepsilon,\lambda)$ on (f_Γ, f_γ). We also note the formula

$$(\mathcal{H}_\varepsilon - \lambda)^{-1}f = \big(R_\Gamma(\varepsilon,\lambda) \oplus \mathcal{S}_\varepsilon R_\gamma(\varepsilon,\lambda)\big)\big(\mathcal{R} \oplus \mathcal{S}_\varepsilon^{-1}\mathcal{R}_{\gamma_\varepsilon}\big) \tag{11}$$

Next, we need another auxiliary operator on the graph Γ, considered as a separate graph. This operator is denoted by $\mathcal{H}_0$, and it corresponds to the differential expression $\widehat{\mathcal{H}}(0)$ and the boundary conditions

$$A_M^{(0)} U_M(u) + B_M^{(0)} U_M'(u) = 0 \ \text{ at the vertices } \ M \in \Gamma. \tag{12}$$

For $M \neq M_0$, the matrices $A_M^{(0)}$ and $B_M^{(0)}$ are introduced simply: $A_M^{(0)} := A_M(0), B_M^{(0)} := B_M(0)$. In the case of $M \doteq M_0$ the description of the matrices $A_{M_0}^{(0)}$ and $B_{M_0}^{(0)}$ is much more cumbersome. Namely, these matrices are of size $d_0 \times d_0$ and they have the form

$$A_{M_0}^{(0)} := \begin{pmatrix} Q & 0 \\ 0 & I_{d_0-k} \end{pmatrix} \Psi^* + i \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Psi^* \Theta_{\Gamma,M_0}(0), \ B_{M_0}^{(0)} := - \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Psi^* \Pi_{\Gamma,M_0}(0) \tag{13}$$

where for an arbitrary d the symbol I_d denotes the identity matrix of size $d \times d$, and the symbol 0 denotes the zero matrices of the corresponding sizes. The matrices Π_{Γ,M_0} and Θ_{Γ,M_0} are described by the formulas

$$\Pi_{\Gamma,M_0}(\varepsilon) := diag\{\vartheta_i(M_0) p_\Gamma|_{e_i}(M_0,\varepsilon)\}_{i=1,\dots,d_0},$$
$$\Theta_{\Gamma,M_0}(\varepsilon) := diag\{\vartheta_i(M_0) q_\Gamma|_{e_i}(M_0,\varepsilon)\}_{i=1,\dots,d_0}$$

where, recall, e_i are the edges of the graph Γ emanating from the vertex M_0, the numbers $\vartheta_i(M_0)$ are defined as and in (6). The matrix Q has the form

$$Q := \begin{pmatrix} Q^{(11)} & \dots & Q^{(k1)} \\ \dots & \dots & \dots \\ Q^{(1k)} & \dots & Q^{(kk)} \end{pmatrix}, \ Q^{(ij)} := Q_\gamma^{(ij)} + \sum_{M \in \gamma_\infty} Q_M^{(ij)} \tag{14}$$

$$\begin{gathered} Q^{(ij)} := \left(\tfrac{dp_\gamma}{d\varepsilon}(\cdot,0) \tfrac{d\psi^{(i)}}{d\xi}, \tfrac{d\psi^{(j)}}{d\xi} \right)_{L_2(\gamma)} + \left(\tfrac{d\psi^{(i)}}{d\xi}, i\tfrac{dq_\gamma}{d\varepsilon}(\cdot,0)\psi^{(j)} \right)_{L_2(\gamma)} + \\ + \left(i\tfrac{dq_\gamma}{d\varepsilon}(\cdot,0)\psi^{(j)}, \tfrac{d\psi^{(j)}}{d\xi}, \right)_{L_2(\gamma)} + \left(\tfrac{dV_\gamma}{d\varepsilon}(\cdot,0)\psi^{(i)}, \psi^{(j)} \right)_{L_2(\gamma)}, \end{gathered} \tag{15}$$

$$Q_M^{(ij)} := \left(L_M(\psi^{(i)}), U_M(\psi^{(j)}) \right)_{C^{d(M)}} - \frac{i}{2} \left(\varepsilon_M(\psi^{(i)}), \vartheta_M(\psi^{(j)}), \right)_{C^{d(M)}}, \tag{16}$$

$$\mathcal{L}_M(\psi^{(i)}) := \frac{d\Pi_{\gamma,M}}{d\varepsilon}(0) U_M'(\psi^{(i)}) - i \frac{d\Theta_{\gamma,M}}{d\varepsilon}(0) U_M(\psi^{(i)}) + \big(Y_M + I_{d(M)}\big)^{-1} P_M^{\perp} \varepsilon_M(\psi^{(i)}), \tag{17}$$

$$\varepsilon_M(\cdot) := 2iC_M\big(A_M^{(1)} U_M(\cdot) + B_M^{(1)} U_M'(\cdot)\big), \tag{18}$$

$$U_M := -C_M\big(\tilde{A}_M + i\tilde{B}_M\big), \ \vartheta_M(\cdot) := \Pi_{\gamma,M}(0) U_M'(\cdot) - \Theta_{\gamma,M}(0) U_M(\cdot), \tag{19}$$

$$\tilde{A}_M := A_M^{(0)} + iB_M^{(0)} \Pi_{\gamma,M}^{-1}(0) \Theta_{\gamma,M}(0), \ \tilde{B}_M := B_M^{(0)} \Pi_{\gamma,M}^{-1}(0), C_M := \big(\tilde{A}_M - i\tilde{B}_M\big)^{-1},$$

$$A_M^{(1)} := \begin{pmatrix} A_M^+(0) \\ \frac{dA_M^-}{d\varepsilon}(0) \end{pmatrix}, B_M^{(1)} := \begin{pmatrix} \frac{dB_M^+}{d\varepsilon}(0) \\ \frac{1}{2}\frac{d^2B_M^-}{d\varepsilon^2}(0), \end{pmatrix},$$

$$\Pi_{\gamma,M_0}(\varepsilon) := diag\{\vartheta_i(M_0)p_\gamma|_{e_i}(M,\varepsilon)\}_{i=1,\dots,d_M},$$
$$\Theta_{\gamma,M_0}(\varepsilon) := diag\{\vartheta_i(M_0)q_\gamma|_{e_i}(M,\varepsilon)\}_{i=1,\dots,d_M}$$

where the matrix $A_M^+(\cdot)$ is formed by the first $r(M)$ rows of the matrix $A_M(\cdot), e_i(M)$ are the edges emanating from the vertex M, the numbers $\vartheta_i(M)$ are defined in (6), and the functions p_γ and q_γ are extended to the edges e_i^∞, $i \in J_j$, $j = 1, ..., n$, by the formulas $p_\gamma(\cdot,\varepsilon) \equiv p_i(\varepsilon)$, $q_\gamma(\cdot,\varepsilon) \equiv \varepsilon q_i(\varepsilon)$.

By P_M we denote the projection in $\mathbb{C}^{d(M)}$ onto the eigenspace of the matrix U_M corresponding to the eigenvalue -1, and we also set $P_M^\perp := I_{d(M)} - P_M$.

Let's define more spaces:

$$\tau(\cdot) := \oplus_{e\in\cdot} C(\overline{e}) \cap L_\infty(e), \|u\|_{\tau(\overline{e})} := \sum_{e\in\cdot} \|u\|_{L_{\infty(e)}}$$

$$\tau^1(\cdot) := \oplus_{e\in\cdot} C^1(\overline{e}) \cap W^1_\infty(e), \|u\|_{\tau^1(\overline{e})} := \sum_{e\in\cdot} \|u\|_{W^1_{\infty(e)}}.$$

Theorem 1 *Let the matrices $A_M(\varepsilon), B_M(\varepsilon)$ satisfy the above conditions. Then the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ are self-adjoint. Let condition (A) also be satisfied. Then the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ are linear and bounded as acting from $L_2(\Gamma) \oplus L_2(\gamma)$ to $W_2^2(\Gamma)$ and $W_2^2(\gamma)$ and to $\tau^1(\Gamma)$ and $\tau^1(\gamma)$. For each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists $\varepsilon_0(\lambda) > 0$, such that for $\varepsilon_0(\lambda) > 0$ the operators $\mathcal{R}_\Gamma(\varepsilon,\lambda)$ and $\mathcal{R}_\gamma(\varepsilon,\lambda)$ are analytic in ε as are the operators from $L_2 \oplus L_2(\gamma)$ in $W_2^2(\Gamma)$ and $W_2^2(\gamma)$ and in $\tau^1(\Gamma)$ and $\tau^1(\gamma)$. In both cases the first terms of the Taylor series of these operators are of the form*

$$\begin{gathered}\mathcal{R}_\Gamma(\varepsilon,\lambda) = \big(\mathcal{H}_0 - \lambda\big)^{-1}\mathcal{P}_\Gamma + O(\varepsilon), \mathcal{R}_\Gamma(\varepsilon,\lambda) = \mathcal{R}_\gamma^0\mathcal{P}_\Gamma + O(\varepsilon), \\ \mathcal{R}_\gamma^0(\lambda)f := \sum_{i=1}^k c_i(f)\psi^{(i)},\ \big(c_1(f),...,c_k(f)\big)^t := \Big(\Psi^{(1)},...,\Psi^{(k)}\Big)^* U_{M_0}\Big(\big(\mathcal{H}_0 - \lambda\big)^{-1}f\Big).\end{gathered} \tag{20}$$

2.4 Main result

The main result of this paper describes all the coefficients of the Taylor series for the operators $\mathcal{R}_\Gamma$ and $\mathcal{R}_\gamma$, as well as the analogue of the Taylor series for the resolvent of the operator $\mathcal{H}_\varepsilon$.

Let's define a family of auxiliary functions - solutions to problems

$$\begin{gathered}\Big(\mathcal{H}(0) - \lambda\Big)\vartheta_{i,\Gamma} = 0, \text{ on } \Gamma, \\ A_M^{(0)}U_M(\vartheta_{i,\Gamma}) + B_M^{(0)}U'_M(\vartheta_{i,\Gamma}) = 0, \text{ in } M \neq M_0,\ U_{M_0}(\vartheta_{i,\Gamma}) = \Psi^{(i)}.\end{gathered} \tag{21}$$

The application of Lemma 2 from Section 3 guarantees the unique solvability of these problems. Let us consider two systems of boundary value problems. The first is introduced on the graph Γ, considered as an independent graph ($p \geq 0$):

$$\left(\widehat{\mathcal{H}}(0)-\lambda\right)u_p^\Gamma = -\sum_{q=1}^{p}\frac{1}{q!}\frac{d^q\widehat{\mathcal{H}}}{d\varepsilon^q}(\cdot,0)u_{p-q}^\Gamma \text{ in } \Gamma, \tag{22}$$

$$\begin{gathered} A_M^{(0)}U_M(\vartheta_{i,\Gamma})+B_M^{(0)}U_M'(\vartheta_{i,\Gamma})= \\ -\sum_{q=1}^{p}\tfrac{1}{q!}\left(\tfrac{d^qA_M}{d\varepsilon^q}(0)U_M\left(u_{p-q}^\Gamma\right)+\tfrac{d^qB_M}{d\varepsilon^q}(0)U_M'\left(u_{p-q}^\Gamma\right)\right)\ M\in\Gamma,\ M\neq M_0. \end{gathered} \tag{23}$$

$$\widehat{\mathcal{H}}_{ex}(0)u_p^\gamma = \delta_{2p}\chi_\gamma f_\gamma - \sum_{q=1}^{p}\frac{1}{q!}\frac{d^q\widehat{\mathcal{H}}_{ex}}{d\varepsilon^q}(0)\chi_\gamma u_{p-q}^\gamma + \lambda\chi_\gamma u_{p-2}^\gamma \text{ on } \gamma_{ex}, p\geq 0, \tag{24}$$

$$P_MU_M(u_p^\gamma) = -P_MC_Mg_p^\gamma,\ P_M^\perp\vartheta_M(u_p^\gamma)+K_M^\perp U_M(u_p^\gamma) = 2iC_M\left(U_M+E_{d(M)}\right)^{-1}C_Mg_p^\gamma, \tag{25}$$

$$g_p^\gamma := \sum_{i=1}^{p}\left(A_M^{(i)}U_M(u_{p-i}^\gamma)+B_M^{(i)}U_M'(u_{p-i}^\gamma)\right),$$

$$A_M^i := \begin{pmatrix}\frac{1}{(i-1)!}\frac{d^{i-1}A_M^+}{d\varepsilon^{i-1}}(0)\\ \frac{1}{i!}\frac{d^iA_M^-}{d\varepsilon^i}(0)\end{pmatrix}, B_M^i := \begin{pmatrix}\frac{1}{i!}\frac{d^iB_M^+}{d\varepsilon^i}(0)\\ \frac{1}{(i+1)!}\frac{d^{i+1}B_M^-}{d\varepsilon^{i+1}}(0)\end{pmatrix}, M\in\gamma_\infty$$

where we set $u_p^\gamma := 0$ for $p \leq -1$, χ_γ denotes the characteristic function of the graph γ, δ_{qp} is the Kronecker-Capelli symbol and

$$K_M^\perp := i\left(U_M+E_{d(M)}\right)^{-1}P_M^\perp\left(U_M-E_{d(M)}\right).$$

The main result about Taylor series for the operators $\mathcal{R}_\Gamma$ and $\mathcal{R}_\gamma$ is as follows.

Theorem 2 *Let all the above conditions on the coefficients of the differential expression $\widehat{\mathcal{H}}(\varepsilon)$ and the matrices $A_M(\varepsilon), B_M(\varepsilon)$ be satisfied, and let condition (A) be satisfied. For each pair $(f_\Gamma, f_\gamma) \in L_2(\Gamma)\oplus L_2(\gamma)$ the Taylor series of the functions $\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)$ and $\mathcal{R}_\gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)$ have the form*

$$\begin{aligned} \mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma) &= \sum_{p=0}^{\infty}\varepsilon^pu_p^\Gamma,\ \ u_0^\Gamma := \left(\mathcal{H}_0-\lambda\right)^{-1}f_\Gamma \\ \mathcal{R}_\gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma) &= \sum_{p=0}^{\infty}\varepsilon^pu_p^\gamma,\ \ u_0^\gamma := \sum_{i=1}^{k}c_i(f_\Gamma)\psi^{(i)}, \end{aligned} \tag{26}$$

and converge uniformly in ε in the spaces $W_2^2(\Gamma)$ and $W_2^2(\gamma)$, as well as in the norms of the spaces $\tau^2(\Gamma)$ and $\tau^2(\gamma)$.

The coefficients of these series are given by the formulas

$$u_p^\Gamma = u_{p,*}^\Gamma + \sum_{i=1}^{k} c_{i,p}\vartheta_{i,\Gamma},\ u_p^\gamma = u_{p,*}^\Gamma + \sum_{i=1}^{k} c_{i,p}\psi^{(i)},\ p \geq 1, \tag{27}$$

where $u_{p,*}^\Gamma$ is the unique solution of problem (22), (23) with boundary conditions

$$U_{M_0}\big(u_{p,*}^\Gamma\big) = U_\gamma\big(u_{p,*}^\gamma\big), \tag{28}$$

and $u_{p,*}^\gamma$ is a particular solution of problem (24), (25) with the boundary condition

$$U'_{\gamma_{ex}}\big(u_{p,*}^\gamma\big) = U'_{M_0}\big(u_{p-1}^\Gamma\big), \tag{29}$$

determined by the orthogonality conditions

$$\Big(U_\gamma\big(u_{p,*}^\gamma\big), \Psi^{(j)}\Big)_{\mathbb{C}^{d_0}} = 0,\ j = 1, ..., k. \tag{30}$$

The constants $c_{i,p}$, $p \geq 1$, are given by the formulas

$$c_p = \big(Q + L\big)^{-1} h_p, \tag{31}$$

$$c_p := \begin{pmatrix} c_{1,p} \\ \vdots \\ c_{k,p} \end{pmatrix}, h_p := \begin{pmatrix} h_{1,p} \\ \vdots \\ h_{k,p} \end{pmatrix}, L := \begin{pmatrix} \big(\vartheta_0(\vartheta_{1,\Gamma}), \Psi^{(1)}\big)_{\mathbb{C}^{d_0}} & \dots & \big(\vartheta_0(\vartheta_{k,\Gamma}), \Psi^{(1)}\big)_{\mathbb{C}^{d_0}} \\ \dots & \dots & \dots \\ \big(\vartheta_0(\vartheta_{1,\Gamma}), \Psi^{(k)}\big)_{\mathbb{C}^{d_0}} & \dots & \big(\vartheta_0(\vartheta_{k,\Gamma}), \Psi^{(k)}\big)_{\mathbb{C}^{d_0}} \end{pmatrix} \tag{32}$$

$$\begin{gathered} h_{j,p} := -\sum_{q=2}^{p} \tfrac{1}{q!}\Big(\tfrac{d^q\widehat{\mathcal{H}}_{ex}}{d\varepsilon^q}(0)u_{p-q}^\gamma, \psi^{(i)}\Big)_{L_2(\gamma)} - \Big(\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)u_{p-1,*}^\gamma, \psi^{(j)}\Big)_{L_2(\gamma)} + \\ +\delta_{1p}(f_\gamma, \psi^{(j)})_{L_2(\gamma)} + \lambda\big(u_{p-2}^\gamma, \psi^{(j)}\big)_{L_2(\gamma)} - \Big(\Pi_{\Gamma,M_0}(0)U'_{M_0}\big(u_{p-1,*}^\Gamma, \Psi^{(j)}\big)\Big)_{\mathbb{C}^{d_0}} - \\ -\sum_{M\in\gamma}\Bigg(\Big(P_M g_{M,j}, P_M U'_M(\psi^{(j)})\Big)_{\mathbb{C}^{d(M)}} + 2i\Big(\big(U_M(0) + E_{d(M)}\big)^{-1} P_M^\perp g_{M,j}, P_M^\perp U_M(\psi^{(j)})\Big)_{\mathbb{C}^{d(M)}}\Bigg), \end{gathered}$$

$$\begin{gathered} g_{M,j} := P_M C_M\Bigg(\textstyle\sum_{i=2}^{p}\Big(A_M^{(i)}U_M(u_{p-i}^\gamma) + B_M^{(i)}U'_M(u_{p-1,*}^\gamma) + A_M^{(1)}U_M(u_{p-1,*}^\gamma) + B_M^{(1)}U'_M(u_{p-1,*}^\gamma)\Big)\Bigg), \\ \vartheta_0(\cdot) := \Pi_{\Gamma,M_0}(0)U'_{M_0}(\cdot) - i\Theta\Gamma, M_0(0)U_{M_0}(\cdot) \end{gathered} \tag{33}$$

The second main result describes an analogue of the Taylor series for the resolvent of the operator $\mathcal{H}_\varepsilon$.

Theorem 3 *Let all the above conditions on the coefficients of the differential expression $\widehat{\mathcal{H}}(\varepsilon)$ and the matrices $A_M(\varepsilon), B_M(\varepsilon)$, be satisfied, and let condition (A) be satisfied. For each* *function $f \in L_2(\Gamma_\varepsilon)$ the function $(\mathcal{H}_\varepsilon-\lambda)^{-1}f$ can be represented by a series converging in $W_2^2(\Gamma_\varepsilon)$ and $\tau^2(\Gamma_\varepsilon)$*

$$(\mathcal{H}_\varepsilon-\lambda)^{-1}f=\sum_{p=0}^{\infty}\varepsilon^p u_p^\Gamma \oplus \mathcal{S}_\varepsilon^{-1}u_p^\gamma, \tag{34}$$

where the functions u_p^Γ and u_p^γ are the coefficients of the series (26) *with* $f_\Gamma := \mathcal{P}_\Gamma f,\ f_\gamma := \mathcal{S}_\varepsilon \mathcal{P}_{\gamma_\varepsilon} f$.
For an arbitrary $N \in \mathbb{Z}_+$ the following estimates are valid:

$$\|\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|_{W_2^2(\Gamma)} \leq C^{N+1}\varepsilon^{N+1/2}\|f\|_{L_2(\Gamma_\varepsilon)} \tag{35}$$

$$\|\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|_{\tau^2(\Gamma)} \leq C^{N+1}\varepsilon^{N+1/2}\|f\|_{L_2(\Gamma_\varepsilon)} \tag{36}$$

$$\|\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^p \mathcal{S}_\varepsilon u_p^\Gamma\|_{L_2(\gamma_\varepsilon)} \leq C^{N+1}\varepsilon^{N+1/2}\|f\|_{L_2(\Gamma_\varepsilon)} \tag{37}$$

$$\|\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^p \mathcal{S}_\varepsilon u_p^\Gamma\|_{W_2^i(\gamma_\varepsilon)} \leq C^{N+1}\varepsilon^{N+1/2}\|f\|_{L_2(\Gamma_\varepsilon)}\ i=1,2, \tag{38}$$

$$\|\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^p \mathcal{S}_\varepsilon u_p^\Gamma\|_{\tau(\gamma_\varepsilon)} \leq C^{N+1}\varepsilon^{N+1/2}\|f\|_{L_2(\Gamma_\varepsilon)} \tag{39}$$

$$\|\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^p \mathcal{S}_\varepsilon u_p^\Gamma\|_{\tau^1(\gamma_\varepsilon)} \leq C^{N+1}\varepsilon^{N-1/2}\|f\|_{L_2(\Gamma_\varepsilon)} \tag{40}$$

where C is some fixed constant independent of ε, N and f.

3 Auxiliary lemmas

To prove the main result, we will need a series of auxiliary lemmas and facts, which are presented in this section.

Lemma 1 *Suppose that condition (A) is satisfied. For an arbitrary family of vectors $g_M \in P_M\mathbb{C}^{d(M)}, g_{M,\perp} \in P_{M,\perp}\mathbb{C}^{d(M)}$, $M \in \gamma_\infty$, an arbitrary vector $g_{ex} \in \mathbb{C}^{d_0}$ and an arbitrary function $g \in L_2(\gamma_{ex})$ the boundary value problem*

$$\begin{gathered}\widehat{\mathcal{H}}_{ex}(0)u = g, \text{ in } \gamma_{ex}, \ \Pi_{\Gamma,M_0}(0)U'_{ex}(u) = \mathfrak{g}_{ex}, \\ P_M U_M(u) = \mathfrak{g}_M, \ P_M^\perp \vartheta_M(u) + K_M^\perp U_M(u) = \mathfrak{g}_{M,\perp} \text{ in } M \in \gamma_\infty\end{gathered} \tag{41}$$

is solvable in $W_2^2(\gamma_{ex})$, if and only if for all $j = 1, \ldots, k$ the equality holds

$$\begin{gathered}\left(g, \psi^{(j)}\right)_{L_2(\gamma_{ex})} = -\Big(\mathfrak{g}_{ex}, U_\gamma\big(\psi^{(j)}\big)\Big)_{\mathbb{C}^{d_0}} + \\ \sum_{M\in\gamma_\infty} \Big(\mathfrak{g}_{M,\perp}, U_\gamma\big(\psi^{(j)}\big)\Big)_{\mathbb{C}^{d(M)}} - \sum_{M\in\gamma_\infty} \Big(\mathfrak{g}_M, \vartheta_M\big(\psi^{(j)}\big)\Big)_{\mathbb{C}^{d(M)}}.\end{gathered} \tag{42}$$

Lemma 2 *For an arbitrary family of vectors $\mathfrak{g}_M \in \mathbb{C}^{d(M)}, M \in \Gamma$, an arbitrary function $g \in L_2(\Gamma)$ and each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the boundary value problem*

$$\begin{gathered}\Big(\widehat{\mathcal{H}}(0) - \lambda\Big)u = g \text{ on } \Gamma, \\ U_{M_0}(u) = \mathfrak{g}_{M_0} \text{ in } M \in \Gamma, \ M \neq M_0\end{gathered}$$

is uniquely solvable in $W_2^2(\Gamma)$.

The following auxiliary lemma guarantees the invertibility of the matrix $Q + L$ from the formulation of Theorem 2 (see (31)).

Lemma 3 *The matrix $Q + L$ is non-singular.*

We multiply the equation in (21) by $\vartheta_{j,\Gamma}$ scalarly in $L_2(\Gamma)$ and integrate by parts twice, taking into account the boundary conditions from (21). Then we obtain that the matrix $L - iIm\lambda G_\Gamma$ is self-adjoint, where G_Γ is the positive definite self-adjoint Gram matrix of the functions $\vartheta_{i,\Gamma}$. Since the matrix Q is also self-adjoint, it follows from here that for all $c \in \mathbb{C}^{d_0}$ we have

$$Im\Big(\big(Q + L\big)c, c\Big)_{\mathbb{C}^k} = -Im\lambda\big(G_\Gamma c, c\big)_{\mathbb{C}^k} \neq 0$$

which immediately implies the non-degeneracy of the matrix Q and completes the proof of the lemma.

4 Taylor series for parts of the resolvent

In this section we prove Theorem 2. According to Theorem 1, the operators $\mathcal{R}_\Gamma$ and $\mathcal{R}_\gamma$ are analytic in ε as operators from $L_2(\Gamma) \oplus L_2(\gamma)$ in $W_2^2(\Gamma)$ and $W_2^2(\gamma)$ and in $\tau^1(\Gamma)$ and $\tau^1(\gamma)$. This means that for an arbitrary pair $(f_\Gamma, f_\gamma) \in L_2(\Gamma) \oplus L_2(\gamma)$ the functions $u_\varepsilon^\Gamma := \mathcal{R}_\Gamma(\varepsilon, \gamma)(f_\Gamma, f_\gamma)$ and $u_\gamma^\varepsilon := \mathcal{R}_\gamma(\varepsilon, \gamma)(f_\Gamma, f_\gamma)$ are represented by series (26) converging uniformly in ε in the norms $\tau^1(\Gamma)$ and $\tau^1(\gamma)$. It also follows from formula (11) that in the sense of decompositions (10) the equality

$$\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1} f =: u_\varepsilon = u_\varepsilon^\Gamma \oplus \mathcal{S}_\varepsilon u_\varepsilon^\gamma, \ f := f_\Gamma \oplus \mathcal{S}_\varepsilon f_\gamma \tag{43}$$

From this and from the equation for the resolvent $\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1}$ it follows that the function u_ε is a solution of the differential equation $\big(\widehat{\mathcal{H}}_\varepsilon - \lambda\big)^{-1} u_\varepsilon^\Gamma = f_\Gamma$ on the graph Γ and satisfies the boundary conditions (4) at the vertices $M \in \Gamma$, $M \neq M_0$. We substitute the series for $\mathcal{R}_\Gamma$ from (26) into this equation and the boundary conditions, expand all the coefficients in Taylor series in ε and collect the coefficients at the same powers of ε. Then we obtain a recurrent system of boundary value problems (22), (23) for the functions u_p^Γ.

Our next step is to obtain similar boundary value problems for the functions u_p^γ. To do this, we first extend these functions from the graph γ to the graph γ_{ex} according to the following rule:

$$u_p^\gamma(\xi_i) := \frac{du_{p-1}^\Gamma|_{e_i}}{dx_i}(M_0)\vartheta_i(M_0)\xi_i + u_p^\Gamma|_{e_i}(M_0), \ i \in J_j, \ j = 1, ..., n. \tag{44}$$

Due to such continuation, equality (43) and continuous differentiability of the function u_ε, we immediately conclude that the following equalities must be satisfied:

$$U_{M_0}\big(u_p^\Gamma\big) = U_\gamma\big(u_p^\gamma\big), \ p \geq 1, \tag{45}$$

$$U'_{\gamma_{ex}}\big(u_0^\gamma\big) = 0, \ U'_{\gamma_{ex}}\big(u_p^\gamma\big) = U'_{M_0}\big(u_{p-1}^\Gamma\big) \ \ p \geq 1, \tag{46}$$

These relations are continuity conditions connecting the restrictions of the function u_ε to the subgraphs Γ and γ on the edges e_i; here we should keep in mind the replacement $\xi = x_i\varepsilon^{-1}$, connecting the variables on the edges e_i^{ex} and e_i. Further, we also consider these continuity conditions as boundary conditions for the functions u_p^Γ and u_p^γ.

From formulas (43), the equation for the resolvent $\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1}$, the definition of the operator $\mathcal{H}_{ex}(\varepsilon)$ in Section 3 and the continuation formulas (43) it follows that the function u_ε^γ is a solution of the differential equation $\big(\mathcal{H}_{ex}(\varepsilon) - \varepsilon^2\lambda\big)u_\gamma^\varepsilon = \varepsilon^2 f$ on γ with boundary conditions (9). We substitute into this problem the series for $\mathcal{R}$ from (26), expand all the coefficients into Taylor series in ε and collect the coefficients at the same powers of ε. Then, taking into account the continuation formulas (44) and the definition (8) of the differential expression $\widehat{\mathcal{H}}_{ex}(0)$ on the edges e_i^{ex}, we obtain a recurrent system of boundary equations (24) for the functions u_p^γ with the boundary conditions

$$\begin{gathered} A_M^{(0)} U_M\big(u_p^\gamma\big) + B_M^{(0)} U'_M\big(u_p^\gamma\big) = -g_p^\gamma, \\ g_p^\gamma := \sum_{i=1}^{p} \Big(A_M^{(i)} U_M\big(u_{p-i}^\gamma\big) + B_M^{(0)} U'_M\big(u_{p-i}^\gamma\big)\Big), \ M \in \gamma_\infty. \end{gathered} \tag{47}$$

We now investigate the solvability of problems (22), (23), (45) and (24), (25), (46). The function u_0^Γ is already defined in (26). Since problem (24), (25), (46) for u_0^γ is homogeneous, its solution is a linear combination of functions $\psi^{(i)} : u_0^\gamma = \sum_{i=1}^k c_{i,0}\psi^{(i)}$. Due to the boundary conditions for u_0^Γ at the vertex M_0 and the definition of the vectors $\Psi^{(i)}$, $i \leq k+1$ we obviously have $\Big(U_{M_0}\big(u_0^\Gamma\big), \Psi^{(i)}\Big)_{\mathbb{C}^{d_0}} = 0$ for $j \geq k+1$. Consequently,

$$c_{0,i} = c_i(f_\Gamma), \ U_{M_0}\big(u_0^\Gamma\big) = \sum_{i=1}^{k} c_i(f_\Gamma)\Psi^{(i)},$$

where the functionals $c_i(f)$ were defined in (20). This leads to the formula for u_0^γ from (26).

Let us now consider problem (24), (25), (46) for u_1^γ. The right-hand side $U'_{M_0}\big(u_0^\Gamma\big)$ in the boundary condition (46) with $p=1$ is a known quantity, since the function u_0^Γ is already completely defined. This problem is a special case of problem (41) with

$$f=-\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)u_0^\gamma,\ \mathfrak{g}_{M^{ex}}=\Pi_{\Gamma,M_0}(0)U'_{M_0}\big(u_0^\Gamma\big)$$
$$\mathfrak{g}_M=\tfrac{i}{2}P_M\varepsilon_M\big(u_0^\gamma\big),\ \mathfrak{g}_{M,\perp}=\Big(U_M+E_{d(M)}\Big)^{-1}P_M^\perp\varepsilon_M\big(u_0^\gamma\big),$$

where the operator EM is defined in (18). The solvability condition for this problem is given by equality (42), which in this case takes the form

$$0=-\bigg(\Pi_{\Gamma,M_0}(0)U'_{M_0}\big(u_0^\Gamma\big),U_\gamma\big(\psi^{(j)}\big)\bigg)_{\mathbb{C}^{d_0}}+\bigg(\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)u_0^\gamma,\big(\psi^{(j)}\big)\bigg)_{L_2(\gamma)}-$$
$$-\sum_{M\in\gamma_\infty}\tfrac{i}{2}\bigg(P_M\varepsilon_M\big(u_0^\gamma\big),\vartheta_M\big(\psi^{(j)}\big)\bigg)_{\mathbb{C}^{d(M)}}+\sum_{M\in\gamma_\infty}\bigg(\Big(U_M+E_{d(M)}\Big)^{-1}P_M^\perp\varepsilon_M\big(u_0^\gamma\big),U_M\big(\psi^{(j)}\big)\bigg)_{\mathbb{C}^{d(M)}} \tag{48}$$

Using the obvious equalities

$$\frac{d\psi^{(i)}}{d\xi}=0,\ \text{on }\gamma\setminus\gamma_{ex},\ \frac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)\psi^{(j)}=\begin{cases}\frac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)\chi_\gamma\big(\psi^{(j)}\big) & \text{on, }\gamma\\ 0 & \text{on }\gamma\setminus\gamma_{ex}\end{cases}$$

and definition (15) quantities $Q_\gamma^{(ij)}$, by integrating by parts we verify that

$$\begin{aligned}\bigg(\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)\chi_\gamma\big(\psi^{(j)}\big)\bigg)_{L_2(\gamma)}&=\bigg(\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)\chi_\gamma\big(\psi^{(j)}\big)\bigg)_{L_2(\gamma)_{ex}}=\\ =Q_\gamma^{(ij)}+\sum_{M\in\gamma_\infty}&\bigg(\tfrac{d\Pi_{\gamma,M}}{d\varepsilon}(0)U'_M\big(\psi^{(i)}\big)-i\tfrac{d\Theta_{\gamma,M}}{d\varepsilon}(0)U_M\big(\psi^{(i)}\big),U_M\big(\psi^{(i)}\big)\bigg)_{\mathbb{C}^{d(M)}}.\end{aligned} \tag{49}$$

Taking into account the last relation and the definition of the function u_0^γ in (26), it is easy to see that the solvability condition (48) of the problem for u_1^γ is equivalent to the boundary condition (12) at the vertex M_0 with matrices (13). Since this condition is satisfied, the problem for u_1^γ is solvable and its general solution has the form

$$u_1^\gamma=u_{1,*}^\gamma+\sum_{i=1}^k c_{i,1}\psi^{(i)}, \tag{50}$$

where $u_{1,*}^\gamma$ is a particular solution of problem (24), (25), (46) satisfying the orthogonality condition (30), and $c_{i,1}$ are some constants that will be found later.

Having found the function u_1^γ, we can already determine the function u_1^Γ. It is found as a solution to problem (22), (23), (45). According to Lemma 2, problem (24), (47), (44) for u_1^Γ is uniquely solvable and its solution has the form

$$u_1^\gamma=u_{1,*}^\gamma+\sum_{i=1}^k c_{i,1}\vartheta_{i,\Gamma}, \tag{51}$$

where $u^{\Gamma}_{1,*}$ is the solution of problem (22), (23) with the boundary condition $U_{M_0}\big(u^{\Gamma}_{1,*}\big) = U_{\gamma}\big(u^{\gamma}_{1,*}\big)$.

Let us now study problem (24), (25), (46). We substitute formula (50) into the right-hand sides of equalities (24), (25), and we substitute equality (51) into the right-hand side of (46). The resulting problem is a special case of problem (41), the solvability of which is determined by condition (42). Writing out this condition for this problem and taking into account relations (49) and (14)-(17), we obtain

$$\sum_{i=1}^{k} Q^{(ij)} c_{i,1} + \sum_{i=1}^{k} c_{i,1}\Big(\vartheta_0\big(\vartheta_{i,\Gamma}\big), \Psi^{(j)}\Big)_{\mathbb{C}^{d_0}} = h_{j,1}, \tag{52}$$

where the numbers $h_{j,1}$ are defined by the equalities

$$\begin{aligned} h_{j,1} := \left(f_{\gamma} - \tfrac{1}{2}\tfrac{d^2\widehat{\mathcal{H}}_{ex}}{d\varepsilon^2}(0)\chi_{\gamma}u^{\gamma}_0 + \lambda u^{\gamma}_0 - \tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)u^{\gamma}_{1,*}, \psi^{(i)} \right)_{L_2(\gamma)} - \\ \Big(\Pi_{\Gamma,M_0}(0)u'_{M_0}\big(u^{\Gamma}_{1,*}\big), \Psi^{(j)}\Big)_{\mathbb{C}^{d_0}} - \\ -2i\sum_{M\in\gamma_\infty}\bigg(\Big(U_M(0)+E_{d(M)}\Big)^{-1}P^{\perp}_M C_M\Big(A^{(1)}_M U_M\big(u^{\gamma}_{1,*}\big) + B^{(1)}_M U'_M\big(u^{\gamma}_{1,*}\big) + \\ + A^{(2)}_M U_M\big(u^{\gamma}_0\big) + B^{(2)}_M U'_M\big(u^{\gamma}_0\big), P^{\perp}_M U_M\big(\psi^{(j)}\big)\Big)\bigg)_{\mathbb{C}^{d(M)}} - \\ - \textstyle\sum_{M\in\gamma_\infty}\bigg(P_M C_M\Big(A^{(1)}_M U_M\big(u^{\gamma}_{1,*}\big) + B^{(1)}_M U'_M\big(u^{\gamma}_{1,*}\big) + \\ + A^{(2)}_M U_M\big(u^{\gamma}_0\big) + B^{(2)}_M U'_M\big(u^{\gamma}_0\big), P_M\vartheta_M\big(\psi^{(j)}\big)\Big)\bigg)_{\mathbb{C}^{d(M)}}. \end{aligned}$$

In matrix form, equalities (52) are rewritten to the equation $(Q+L)c_1 = h_1$, and Lemma 3 allows us to uniquely solve this equation by finding the coefficients $c_{i,1}$. This leads to formulas (31), (32) with $p = 1$.

The remaining functions, u^{Γ}_p and u^{γ}_p, are defined in a similar way. Namely, the function u^{γ}_p is defined up to a linear combination of the functions $\psi^{(j)}$ with some coefficients $c_{i,p}$ by the formula from (27). Then problem (22), (23) for the function u^{Γ}_p is uniquely solvable and its solution has the form (27). Now we can solve problem (24), (46), (47) for u^{γ}_{p+1},since all the right-hand sides in this problem are expressed through the functions already found. The solvability condition for the last problem is given by equality (42) and leads to a system of linear equations $(Q+L)c_p = h_p$, where the vector h_p is from (32) with coefficients from (33). This system is uniquely solved thanks to Lemma 3 and the solution is given by formula (31). The described procedure allows us to determine all the coefficients of the Taylor series (26).

Theorem 2 is completely proved.

5 Analogue of the Taylor series for the resolvent

This section is devoted to the proof of Theorem 3. Equality (34) follows immediately from (26) and (11); it suffices to substitute the series (26) into formula (11).

Since, by Theorem 1, the operators $\mathcal{R}_\Gamma(\varepsilon,\lambda)$ and $\mathcal{R}_\Gamma(\varepsilon,\lambda)$ are analytic in ε and, according to Theorem 2, their Taylor series are given by equalities (26), the following inequalities hold:

$$\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|^2_{W_2^2(\Gamma)}\le C^{2N+2}\varepsilon^{2N+2}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma)}\Big),\tag{53}$$

$$\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|^2_{W_2^2(\gamma)}\le C^{2N+2}\varepsilon^{2N+2}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma)}\Big),\tag{54}$$

$$\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|^2_{\tau^2(\Gamma)}\le C^{N+1}\varepsilon^{N+1}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma)}\Big)^{\frac{1}{2}},\tag{55}$$

$$\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|^2_{\tau^2(\gamma)}\le C^{N+1}\varepsilon^{N+1}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma)}\Big)^{\frac{1}{2}},\tag{56}$$

where C is a fixed constant independent of ε, N, f_Γ and f_γ. From definition (2) of the operator $\mathcal{S}_\varepsilon$ it follows that

$$\Big\|\frac{d^i\mathcal{S}_\varepsilon u}{dx^i}\Big\|^2_{L_2(\gamma_\varepsilon)}=\varepsilon^{1-2i}\Big\|\frac{d^iu}{dx^i}\Big\|^2_{L_2(\gamma)},\Big\|\frac{d^i\mathcal{S}_\varepsilon u}{dx^i}\Big\|^2_{\tau(\gamma_\varepsilon)}=\varepsilon^{-i}\Big\|\frac{d^iu}{dx^i}\Big\|_{\tau(\gamma)},\tag{57}$$

and, in particular,

$$\Big\|f_\gamma\Big\|^2_{L_2(\gamma)}=\varepsilon^{-1}\Big\|f_\gamma\Big\|^2_{L_2(\gamma_\varepsilon)}\tag{58}$$

We also note that from formula (11) follows the equality

$$\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^pu_p^\Gamma\oplus\mathcal{S}_\varepsilon u_p^\gamma=\Big(\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^pu_p^\Gamma\Big)\oplus\mathcal{S}_\varepsilon\Big(\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^pu_p^\gamma\Big)\Big)\tag{59}$$

Then from (53), (56) we have

$$\Big\|\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^pu_p^\Gamma\Big\|^2_{W_2^2(\Gamma)}=\Big\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^pu_p^\Gamma\Big\|^2_{W_2^2(\Gamma)}\le$$
$$\le C^{2N+2}\varepsilon^{2N+2}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma_\varepsilon)}\Big)$$

and from here it follows (35). Similarly, using (55) instead of (53), it is easy to prove estimate (36).

As above, applying inequality (54), the first formula in (57) and equalities (59), (**??**) we obtain estimates (37), (38):

$$\Big\|(\mathcal{H}_\varepsilon-\lambda)^{-1}f-\sum_{p=0}^{N}\varepsilon^p u_p^{\Gamma}\Big\|^2_{W_2^i(\gamma_\varepsilon)}=\Big\|\mathcal{S}_\varepsilon\Big(\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^{\gamma}\Big)\Big\|^2_{W_2^i(\gamma_\varepsilon)}\leq$$
$$\leq\varepsilon^{1-2i}\Big\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^{\gamma}\Big\|^2_{W_2^i(\gamma_\varepsilon)}\leq C^{2N+2}\varepsilon^{2N+2}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\varepsilon^{-1}\|f_\Gamma\|^2_{L_2(\gamma_\varepsilon)}\Big),$$

where $i=0,1,2$, and for $i=0$ for convenience we set $W_2^i:=L_2$. Reasoning as above and using inequality (56) instead of (54) and the second equality in (57) instead of the first, we easily prove inequalities (39), (40). Theorem 3 is completely proven.

6 Discussion

The results obtained show that even in the case of graphs with complex structure and arbitrary boundary conditions, the behavior of the resolvent of an elliptic operator can be accurately described using analytical methods. This is important for further study of the spectral properties of operators on quantum graphs, as well as for applications in physics, where such structures model real systems with small scales. The work develops and refines previously known approaches, offering a more general and flexible analysis scheme without strict restrictions on the graph structure or the type of coefficients. The results are consistent with previous studies in the field of asymptotic analysis, but take a step forward due to the rigorous description of the remainder terms. Prospects include extending the methods to nonlinear operators, systems with variable scales, and numerical implementations for specific applications.

7 Conclusion

In this paper, we study the behavior of an elliptic self-adjoint operator of the second order on a graph with small edges, depending on a small parameter ε. It is established that the resolvent of such an operator is analytic in ε and can be represented as a convergent series with the possibility of exact calculation of all coefficients. An effective method, similar to the matching of asymptotic expansions, is developed for constructing this series and estimates of the remainder terms are obtained. The results confirm the possibility of a rigorous description of the spectral properties of the operator as $\varepsilon\to 0$ and open the way to further studies of operators on graphs of complex structure.

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Авторлар туралы мәлімет:

Коныркулжаева Марал Нурлановна (корреспондент автор) – Халықаралық ақпараттық технологиялар университетінің ассистент профессоры (Алматы, Қазақстан, электрондық пошта: m.konyrkulzhayeva@edu.iitu.kz);

Аузерхан Гаухар Сұлтанбекқызы – әл-Фараби атындағы Қазақ ұлттық университетінің математика кафедрасының аға оқытушысы (Алматы, Қазақстан, электрондық пошта: Auzerkhan@math.kz).

Сведения об авторах:

Коныркулжаева Марал Нурлановна (корреспондент автор) – ассистент профессор Международного университета информационных технологий (Алматы, Казахстан, электронная почта: m.konyrkulzhayeva@edu.iitu.kz);

Аузерхан Гаухар Султанбеккызы – старший преподаватель кафедры математики Казахского национального университета имени аль-Фараби(Алматы, Казахстан, электронная почта: Auzerkhan@math.kz).

Information about authors:

Konyrkulzhaeva Maral Nurlanovna (corresponding author) – assistant professor of the International University of Information Technologies (Almaty, Kazakhstan, email: m.konyrkulzhayeva@edu.iitu.kz);

Auzerkhan Gauhar Sultanbekkyzy – Senior Lecturer at the Department of Mathematics, Al Farabi Kazakh National University (Almaty, Kazakhstan, email: Auzerkhan@math.kz).

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