1-бөлім	Раздел 1	Section 1
Математика	**Математика**	**Mathematics**

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CONVEXITY AND CONCAVITY IN SUBMAJORIZATION INEQUALITIES FOR τ-MEASURABLE OPERATORS

We proved the following result. Consider a semi-finite von Neumann algebra equipped with a trace, and let there be several τ-measurable operators together with a nonnegative function defined on the nonnegative real line. Suppose also that we are given positive weights whose total sum equals one. If the function obtained by applying f to the square root of its argument is convex and if f vanishes at zero, then the weighted sum of the values of f applied to the absolute values of the operators is at least as large as a certain expression involving f evaluated at both the average of all the operators and their pairwise differences. If the same function of the square root is concave, the inequality reverses: the mentioned expression becomes no smaller than the weighted sum of the transformed absolute values. This theorem yields a significant generalization of Clarkson-type inequalities in the noncommutative setting and extends the result previously established by Alrimawi, Hirzallah, and Kittaneh.
Keywords: Clarkson inequality, $\tau-$measurable operator, von Neumann algebra, generalized singular value function, submajorization inequality, convex function, concave function.

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$\tau-$өлшемді операторлар үшін субмажорланған теңсіздіктердегі ойыс және дөңес функциялар

Біз келесі нәтижені дәлелдедік. Жартылай ақырлы фон Нейман алгебрасы қарастырылсын, сонымен қатар, мұнда бірнеше τ-өлшемді операторлар және нақты санның теріс емес мәндерінде анықталған функция бар болсын. Сондай-ақ қосындысы бірге тең болатын оң салмақтар берілсін. Егер нөлдегі мәні нөлге тең f функциясы өзінің аргументінің квадрат түбіріне қолданылған кезде ойыс болса, онда операторлардың модульдеріне осы f функцияны қолдану арқылы алынған мәндердің салмақталған қосындысы барлық операторлардың орташа мәніне және олардың айырмашылықтарына қолданылған f функциядан құралған белгілі бір өрнектен кем болмайды. Ал егер аталған функция квадрат түбірге қолданылғанда дөңес болса, онда бұл теңсіздік қарама-қарсы бағытта орындалады: көрсетілген өрнек салмақталған қосындыдан кем емес болады. Бұл нәтиже бейкоммутативтік ортадағы Кларксон типті теңсіздіктерді едәуір кеңейтеді және Alrimawi, Hirzallah және Kittaneh еңбегінде алынған нәтижені толықтырады.
Түйін сөздер: Кларксон теңсіздігі, τ-өлшемді оператор, фон Нейман алгебрасы, сингулярлы жалпылама функция, субмажоризация теңсіздігі, дөңес функция, ойыс функция.

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Выпуклость и вогнутость в субмажоризационных неравенствах для $\tau-$измеримых операторов

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Мы доказали следующий результат. Рассматривается полу-конечная алгебра фон Неймана, несколько τ-измеримых операторов и неотрицательная функция, определённая на неотрицательной полуоси. Также заданы положительные веса, сумма которых равна единице. Если функция, которая получается при применении f к квадратному корню своего аргумента, является выпуклой и при этом значение f в нуле равно нулю, то взвешенная сумма значений функции, применённой к модулям операторов, не меньше определённого выражения, построенного из значений f, взятых на среднем операторе и на их попарных различиях. Если же указанная функция, применённая к квадратному корню аргумента, является вогнутой, то справедливо обратное неравенство: это выражение не меньше взвешенной суммы соответствующих значений. Полученный результат существенно расширяет неравенства типа Кларксона в некоммутативной среде и дополняет форму, ранее установленную Alrimawi, Hirzallah и Kittaneh.
Ключевые слова: неравенство Кларксона, τ-измеримый оператор, алгебра фон Неймана, обобщённая сингулярная функция, неравенство субмажоризации, выпуклая функция, вогнутая функция.

1 Introduction

Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$, where τ is a faithful normal semifinite trace on $\mathcal{M}$. The $*$-algebra $S(\tau)$ of all τ-measurable operators affiliated with $\mathcal{M}$ provides a natural framework for extending classical inequalities to the noncommutative setting.

An operator $x \in S(\tau)$ is said to be positive (denoted $x \geq 0$) if $\langle x\xi, \xi\rangle \geq 0$ for all $\xi \in \mathrm{dom}(x)$. For two self-adjoint operators $x, y \in S(\tau)$, we write $x \geq y$ if $x - y \geq 0$.

In this framework, unitarily invariant norms play a central role. We say $|||\cdot|||$ is unitarily invariant if

$$|||uxv||| = |||x|||$$

for all $x \in S(\tau)$ and for all unitary operators $u, v \in \mathcal{M}$. Typical examples include the generalized Schatten p-norms given by

$$\|x\|_p = \tau\left(|x|^p\right)^{1/p}, \quad 0 < p < \infty.$$

A fundamental result in the theory of unitarily invariant norms is given by the classical *Clarkson inequalities* in Schatten classes, which provide bounds for operator combinations under p-norms. These inequalities have been studied and generalized in various contexts, particularly in the setting of bounded operators on Hilbert spaces.

We now consider $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. This algebra is a von Neumann algebra under the weak operator topology and forms one of the most important examples in functional analysis. Within $\mathcal{B}(\mathcal{H})$, the positive operators are those x for which

$$\langle x\xi, \xi\rangle \geq 0 \quad \text{for all } \xi \in \mathcal{H}.$$

A notable extension of Clarkson-type inequalities to this noncommutative setting was established by Alrimawi, Hirzallah, and Kittaneh in [1]. They proved the following result:

Let $|||\cdot|||$ be a unitarily invariant norm, $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ be positive operators, and let $\alpha_1, \ldots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^{n} \alpha_j = 1$

Define the index set $S_\ell = \{1, \ldots, n\} \setminus \{\ell\}$. Then:

1. If f is a nonnegative function on $[0,\infty)$ with $f(0)=0$ and $g(t)=f\left(\sqrt{t}\right)$ is convex on $[0,\infty)$, then for every unitarily invariant norm,

$$\left|\left|\left| f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|A_\ell-\sum_{j=1}^{n}\alpha_jA_j\right|\right)+\sum_{j,k\in S_\ell}f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|A_j-A_k|\right)\right.\right.\right.$$
$$\left.\left.\left.+f\left(\left|\sum_{j=1}^{n}\alpha_jA_j\right|\right)\right|\right|\right| \le \left|\left|\left|\sum_{j=1}^{n}\alpha_j f(|A_j|)\right|\right|\right|,$$

   for each $\ell=1,\ldots,n$.

2. If f is a nonnegative function on $[0,\infty)$ such that $g(t)=f\left(\sqrt{t}\right)$ is concave on $[0,\infty)$, then for every unitarily invariant norm,

$$\left|\left|\left|\sum_{j=1}^{n}\alpha_j f(|A_j|)\right|\right|\right| \le \left|\left|\left| f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|A_\ell-\sum_{j=1}^{n}\alpha_jA_j\right|\right)\right.\right.\right.$$
$$\left.\left.\left.+\sum_{j,k\in S_\ell}f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|A_j-A_k|\right)+f\left(\left|\sum_{j=1}^{n}\alpha_jA_j\right|\right)\right|\right|\right|,$$

   for each $\ell=1,\ldots,n$.

This paper develops new norm inequalities for operators that generalize and refine the classical Clarkson inequalities. Prior results on non-commutative Clarkson-type inequalities for the symmetric norm of τ-measurable operators can be found in [7], while related submajorization inequalities were established in [4, 5].

The structure of the paper is as follows. Section 2 develops several key operator identities on Hilbert spaces that form the foundation for our results. In Section 3, we establish the main norm inequalities for convex and concave functions of τ-measurable operators.

2 Preliminaries

Throughout this paper, we denote by $\mathcal{M}$ a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, equipped with a faithful normal semifinite trace τ. A densely defined, closed linear operator x on $\mathcal{H}$ with domain $D(x)$ is said to be *affiliated* with $\mathcal{M}$ if

$$u^*xu=x \quad \text{for all unitary } u\in\mathcal{M}',$$

where $\mathcal{M}'$ is the commutant of $\mathcal{M}$.

An operator x affiliated with $\mathcal{M}$ is said to be *τ-measurable* if, for every $\varepsilon>0$, there exists a projection $e\in\mathcal{M}$ such that $e(\mathcal{H})\subseteq D(x)$ and $\tau\left(e^\perp\right)<\varepsilon$, where $e^\perp:=1-e$.

The set of all τ-measurable operators is denoted by $S(\tau)$. This space forms a $*$-algebra under strong closures of algebraic operations.

Let $\mathcal{P}(\mathcal{M})$ denote the lattice of projections in $\mathcal{M}$. For $\varepsilon,\delta>0$, define the sets

$$\mathcal{N}(\varepsilon,\delta):=\left\{x\in S(\tau):\exists\, e\in\mathcal{P}(\mathcal{M}) \text{ such that } \|xe\|<\varepsilon \text{ and } \tau\left(e^\perp\right)<\delta\right\}.$$

These sets form a neighborhood base at 0 for a metrizable Hausdorff topology on $S(\tau)$, called the *measure topology.* With this topology, $S(\tau)$ becomes a complete topological $*$ algebra (see [11]).

For $x \in S(\tau)$, *generalized singular value function* $\mu(t;x)$ is defined by

$$\mu(t;x) := \inf\left\{\|xe\| : e \in \mathcal{P}(\mathcal{M}),\ \tau\left(e^{\perp}\right) \leq t\right\}, \quad t \geq 0,$$

(see [9] for more details about the generalized singular value function).

If $x, y \in S(\tau)$, then we say that x is *submajorized* by y and write $x \preceq y$ if and only if

$$\int_0^t \mu(s;x)ds \leq \int_0^t \mu(s;y)ds, \quad t \in [0,\infty).$$

See [2, 3] and the references therein.

3 Main results

The next result is a restatement of [6, Lemma 2].

Lemma 1 *Let $x_0, \cdots, x_{n-1}$ be τ-measurable operators and let $\alpha_0, \cdots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j x_j = 0$ and $\sum_{j=0}^{n-1} \alpha_j = 1$. For each $\ell \in \{0, \ldots, n-1\}$, we have*

$$\frac{\alpha_\ell}{(1-\alpha_\ell)}|x_\ell|^2 + \sum_{j,k \in S_\ell} \frac{\alpha_j \alpha_k}{2(1-\alpha_\ell)}|x_j - x_k|^2 = \sum_{j=0}^{n-1} \alpha_j |x_j|^2,$$

where $S_\ell = \{0, \ldots, n-1\} \setminus \{\ell\}$.

Lemma 2 *Let $\sum_{j=0}^{n-1} \alpha_j = 1$. Then for each $\ell = 0, \ldots, n-1$,*

$$\frac{\alpha_\ell}{(1-\alpha_\ell)}\left|x_\ell - \sum_{j=0}^{n-1} \alpha_j x_j\right|^2 + \sum_{j,k \in S_\ell} \frac{\alpha_j \alpha_k}{2(1-\alpha_\ell)}|x_j - x_k|^2 = \sum_{j=0}^{n-1} \alpha_j |x_j|^2 - \left|\sum_{j=0}^{n-1} \alpha_j x_j\right|^2.$$

In particular,

$$\frac{\alpha_\ell}{(1-\alpha_\ell)}\left|x_\ell - \sum_{j=0}^{n-1} \alpha_j x_j\right|^2 \leq \sum_{j=0}^{n-1} \alpha_j |x_j|^2 - \left|\sum_{j=0}^{n-1} \alpha_j x_j\right|^2,$$

with equality if and only if $x_j = x_k$ for all $j, k \in S_\ell$.

Proof. Let

$$\bar{x} := \sum_{j=0}^{n-1} \alpha_j x_j.$$

Define new variables:

$$y_j := x_j - \bar{x}, \quad \text{so that} \quad \sum_{j=0}^{n-1} \alpha_j y_j = 0.$$

Applying Lemma 1 to the y_j (since they satisfy the same assumptions), we get:

$$\frac{\alpha_\ell}{1-\alpha_\ell}\left|y_\ell\right|^2+\sum_{j,k\in S_\ell}\frac{\alpha_j\alpha_k}{2\left(1-\alpha_\ell\right)}\left|y_j-y_k\right|^2=\sum_{j=0}^{n-1}\alpha_j\left|y_j\right|^2. \tag{1}$$

Note that:

$$y_\ell=x_\ell-\bar{x},\quad \left|y_j-y_k\right|^2=\left|(x_j-\bar{x})-(x_k-\bar{x})\right|^2=\left|x_j-x_k\right|^2,$$

and

$$\begin{aligned}
\sum_{j=0}^{n-1}\alpha_j\left|y_j\right|^2&=\sum_{j=0}^{n-1}\alpha_j\left|x_j-\bar{x}\right|^2\\
&=\sum_{j=0}^{n-1}\alpha_j\left(\left|x_j\right|^2-x_j^*\bar{x}-\bar{x}^*x_j+\left|\bar{x}\right|^2\right)\\
&=\sum_{j=0}^{n-1}\alpha_j\left|x_j\right|^2-\left(\sum_{j=0}^{n-1}\alpha_jx_j^*\right)\bar{x}-\bar{x}^*\sum_{j=0}^{n-1}\alpha_jx_j+\left|\bar{x}\right|^2\sum_{j=0}^{n-1}\alpha_j\\
&=\sum_{j=0}^{n-1}\alpha_j\left|x_j\right|^2-\bar{x}^*\bar{x}-\bar{x}^*\bar{x}+\left|\bar{x}\right|^2\\
&=\sum_{j=0}^{n-1}\alpha_j\left|x_j\right|^2-\left|\bar{x}\right|^2.
\end{aligned}$$

Substituting these into equation (1), we obtain:

$$\frac{\alpha_\ell}{1-\alpha_\ell}\left|x_\ell-\bar{x}\right|^2+\sum_{j,k\in S_\ell}\frac{\alpha_j\alpha_k}{2\left(1-\alpha_\ell\right)}\left|x_j-x_k\right|^2=\sum_{j=0}^{n-1}\alpha_j\left|x_j\right|^2-\left|\bar{x}\right|^2,$$

which is exactly the statement of the lemma.

The second part of the lemma follows from the nonnegativity of the second term on the left-hand side:

$$\sum_{j,k\in S_\ell}\frac{\alpha_j\alpha_k}{2\left(1-\alpha_\ell\right)}\left|x_j-x_k\right|^2\geq 0,$$

with equality if and only if $x_j=x_k$ for all $j,k\in S_\ell$.

We obtain the following result, similar to [9, Proposition 4.6] (see also [6, Lemma 3]).

Lemma 3 *Let $x_0,\ldots,x_{n-1}$ be positive τ-measurable operators and let $\alpha_0,\ldots,\alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1}\alpha_j=1$.*

(i) If $g:[0,\infty)\to[0,\infty)$ is a convex function with $g(0)=0$, then

$$\mu\left(t;g\left(\sum_{j=0}^{n-1}\alpha_jx_j\right)\right)\leq\mu\left(t;\sum_{j=0}^{n-1}\alpha_jg\left(x_j\right)\right).$$

(ii) If $h:[0,\infty)\to[0,\infty)$ is a concave function, then

$$\mu\left(t;\sum_{j=0}^{n-1}\alpha_j h\left(x_j\right)\right)\leq\mu\left(t;h\left(\sum_{j=0}^{n-1}\alpha_j x_j\right)\right).$$

We recall the following well-known result (see [8, Theorem 5.3]).

Lemma 4 *Let for all $x_0,\cdots,x_{n-1}$ be positive $\tau-$measurable operators.*

(i) If $g:[0,\infty)\longrightarrow[0,\infty)$ is convex function with $g(0)=0$, then

$$\sum_{j=0}^{n-1}g\left(x_j\right)\preceq g\left(\sum_{j=0}^{n-1}x_j\right)$$

(ii) If $h:[0,\infty)\longrightarrow[0,\infty)$ is concave function, then

$$h\left(\sum_{j=0}^{n-1}x_j\right)\preceq\sum_{j=0}^{n-1}h\left(x_j\right)$$

The following theorem offers a natural extension of [1, Theorem 3.8].

Theorem 1 *Let $x_0,\ldots,x_{n-1}$ be τ-measurable operators and $\alpha_0,\ldots,\alpha_{n-1}$ be positive real numbers such that*

$$\sum_{j=0}^{n-1}\alpha_j=1.$$

Then for each $\ell\in\{0,\ldots,n-1\}$:

(i) If $f:[0,\infty)\to[0,\infty)$ is a non-negative function with $f(0)=0$ and $g(t)=f(\sqrt{t})$ is convex, then

$$f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell-\sum_{j=0}^{n-1}\alpha_j x_j\right|\right)+\sum_{j,k\in S_\ell}f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j-x_k|\right)$$
$$+f\left(\left|\sum_{j=0}^{n-1}\alpha_j x_j\right|\right)\preceq\sum_{j=0}^{n-1}\alpha_j f(|x_j|).$$

(ii) If $h(t)=f(\sqrt{t})$ is concave, then the reverse inequality holds:

$$\sum_{j=0}^{n-1}\alpha_j f(|x_j|)\preceq f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell-\sum_{j=0}^{n-1}\alpha_j x_j\right|\right)$$
$$+\sum_{j,k\in S_\ell}f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j-x_k|\right)+f\left(\left|\sum_{j=0}^{n-1}\alpha_j x_j\right|\right).$$

Proof. Let $f : [0,\infty) \to [0,\infty)$ be a non-negative function. Define $g(t) = f(\sqrt{t})$ and $h(t) = f(\sqrt{t})$, depending on convexity or concavity of $f(\sqrt{t})$.

We prove each case separately.

(i) Assume that $g(t) = f(\sqrt{t})$ is convex and $f(0) = 0$.

Let

$$A_\ell = \sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell - \sum_{j=0}^{n-1}\alpha_j x_j\right|, \quad B_{\ell,j,k} = \sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j - x_k|, \quad C = \left|\sum_{j=0}^{n-1}\alpha_j x_j\right|.$$

Then,

$$f(A_\ell) + \sum_{j,k\in S_\ell} f\left(B_{\ell,j,k}\right) + f(C) = g(A_\ell^2) + \sum_{j,k\in S_\ell} g\left(B_{\ell,j,k}^2\right) + g(C^2).$$

By convexity of g and Lemma 4(i), we have

$$g(A_\ell^2) + \sum_{j,k\in S_\ell} g\left(B_{\ell,j,k}^2\right) + g(C^2) \preceq g\left(A_\ell^2 + \sum_{j,k\in S_\ell} B_{\ell,j,k}^2 + C^2\right).$$

Note that by Lemma 2, we can write:

$$A_\ell^2 + \sum_{j,k\in S_\ell} B_{\ell,j,k}^2 + C^2 = \sum_{j=0}^{n-1}\alpha_j|x_j|^2. \tag{2}$$

Hence,

$$f(A_\ell) + \sum_{j,k\in S_\ell} f\left(B_{\ell,j,k}\right) + f(C) \preceq g\left(\sum_{j=0}^{n-1}\alpha_j|x_j|^2\right).$$

Applying Lemma 3(i) (since g is convex and $g(0) = 0$), we conclude:

$$g\left(\sum_{j=0}^{n-1}\alpha_j|x_j|^2\right) \preceq \sum_{j=0}^{n-1}\alpha_j g(|x_j|^2) = \sum_{j=0}^{n-1}\alpha_j f(|x_j|).$$

Therefore,

$$\begin{aligned} f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell - \sum_{j=0}^{n-1}\alpha_j x_j\right|\right) + \sum_{j,k\in S_\ell} f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j - x_k|\right) \\ + f\left(\left|\sum_{j=0}^{n-1}\alpha_j x_j\right|\right) \preceq \sum_{j=0}^{n-1}\alpha_j f(|x_j|). \end{aligned}$$

(ii) Now assume $h(t) = f(\sqrt{t})$ is concave.

Then using Lemma 3(ii) and (2),

$$\sum_{j=0}^{n-1}\alpha_j f(|x_j|) = \sum_{j=0}^{n-1}\alpha_j h(|x_j|^2) \preceq h\left(\sum_{j=0}^{n-1}\alpha_j|x_j|^2\right) = h\left(A_\ell^2 + \sum_{j,k\in S_\ell} B_{\ell,j,k}^2 + C^2\right).$$

Then by Lemma 4(ii) and concavity of h, we get:

$$\begin{aligned} h\left(A_\ell^2 + \sum_{j,k\in S_\ell} B_{\ell,j,k}^2 + C^2\right) &\preceq h(A_\ell^2) + \sum_{j,k\in S_\ell} h\left(B_{\ell,j,k}^2\right) + h(C^2) \\ &= f(A_\ell) + \sum_{j,k\in S_\ell} f\left(B_{\ell,j,k}\right) + f(C). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{j=0}^{n-1} \alpha_j f(|x_j|) \preceq f&\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell - \sum_{j=0}^{n-1}\alpha_j x_j\right|\right) \\ &+ \sum_{j,k\in S_\ell} f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j - x_k|\right) + f\left(\left|\sum_{j=0}^{n-1}\alpha_j x_j\right|\right). \end{aligned}$$

This completes the proof.

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