1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

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SOLVABILITY OF THE INITIAL BOUNDARY VALUE PROBLEM FOR A NONSTATIONARY DIFFUSION MODEL OF AN INHOMOGENEOUS FLUID

In this work, we consider the problem of the existence of solutions to the initial–boundary value problem for a diffusion model of inhomogeneous fluids. For theoretical justification, the methods of functional analysis and compactness techniques are applied, resulting in a priori estimates and energy inequalities. The formulation of the problem takes into account the compatibility of the initial and boundary conditions, as well as the boundedness and smoothness of the given coefficients. This system of differential equations describes the motion of a two-component fluid with components of different densities. A similar model of an inhomogeneous fluid medium was first obtained in [13] under the assumption of a small diffusion coefficient and low mass concentration of the admixture. A distinctive feature of these equations is that the unknown functions $\vec{v}$ and ρ (the velocity and density of the mixture) enter the system through higher-order derivatives and in a nonlinear manner. This circumstance, when studying boundary value problems, forces one to impose certain restrictions on the model parameters. However, these restrictions do not contradict the assumption of a small diffusion coefficient adopted in the derivation of the equations. It should also be noted that when $\lambda = 0$, the obtained system of equations reduces to the classical model of an inhomogeneous viscous incompressible fluid (the Navier–Stokes equations).

Key words: non-stationary model, heterogeneous fluid, Kazhikhov–Smagulov equations, initial-boundary value problem, weak solution, strong solution.

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Біртекті емес сұйықтардың стационарлық емес диффузиялық моделі үшін бастапқы - шеттік есептің шешімділігі

Бұл жұмыста біртекті емес сұйықтардың диффузиялық моделі үшін бастапқы - шеттік есептің шешімдерінің бар болуы мәселелері қарастырылады. Теориялық дәлелдеулер үшін функционалдық талдау әдістері, компактілік әдістері қолданылып априорлық бағалаулар мен энергетикалық теңсіздіктер алынды. Қойылған есепте бастапқы және шеттік шарттардың үйлесімділігі мен берілген коэффициенттердің белгілі бір шенелгендігі мен тегістік қасиеттері ескерілді. Бұл дифференциялдық теңдеулер жүйесі тығыздықтары әр түрлі екі компонентті сұйықтың қозғалысын сипаттайды. Мұндай біртекті емес сұйық орта моделі алғаш рет [13] жұмыста диффузия коэффициентінің және ерітіндінің массалық концентрациясының аз болған жағдайында алынған. Бұл теңдеулердің ерекшелігі — ізделетін $\vec{v}$ және ρ функциялары (аралас сұйықтың жылдамдығы мен тығыздығы) жүйеге жоғары ретті туындылар арқылы және сызықты емес түрде енеді. Бұл жағдай шеттік есептерді зерттеу кезінде модель параметрлеріне белгілі бір шектеулер қоюға мәжбүрлейді. Дегенмен, бұл шектеулер есепті шығару кезінде қабылданған диффузия коэффициентінің аздығы туралы жорамалмен қайшы келмейді. Сондай-ақ, $\lambda = 0$ болған жағдайда алынған жүйе теңдеулері біртекті емес тұтқыр сығылмайтын сұйықтың кәдімгі моделіне (Навье-Стокс теңдеулер жүйесіне) айналатынын атап өткен жөн.

Кілт сөздер: стационарлық емес модель, гетерогенді сұйықтық, Кажихов–Смағұлов теңдеулері, бастапқы-шеттік есеп, әлсіз шешім, әлді шешім.

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Разрешимость начально-краевой задачи для нестационарной диффузионной модели неоднородных жидкостей

В данной работе рассматриваются вопросы существования решений начально-краевой задачи для диффузионной модели неоднородных жидкостей. Для теоретического обоснования применяются методы функционального анализа и методы компактности, получены априорные оценки и энергетические неравенства. В постановке задачи учитывается согласованность начальных и граничных условий, а также ограниченность и гладкость заданных коэффициентов. Эта система дифференциальных уравнений описывает движение двухкомпонентной жидкости с различными плотностями компонентов. Подобная модель неоднородной жидкой среды впервые была получена в работе [13] при малости коэффициента диффузии и массовой концентрации примеси. Особенностью этих уравнений является то, что искомые функции $\vec{v}$ и ρ (скорость и плотность смеси) входят в систему через производные высших порядков и нелинейным образом. Это обстоятельство при исследовании краевых задач вынуждает накладывать определённые ограничения на параметры модели. Тем не менее, данные ограничения не противоречат предположению о малости коэффициента диффузии, принятому при выводе уравнений. Следует также отметить, что при $\lambda = 0$ полученная система уравнений переходит в обычную модель неоднородной вязкой несжимаемой жидкости (систему уравнений Навье-Стокса).
Ключевые слова: нестационарная модель, гетерогенная жидкость, уравнения Кажихова–Смагулова, начально-краевая задача, слабое решение, сильное решение.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Denote by $\Gamma_T = \partial\Omega \times [0, T]$ the lateral surface of the cylinder $Q_T = \Omega \times [0, T]$, $T > 0$. We consider the problem of determining the functions $(\vec{v}(x,t), \rho(x,t), p(x,t))$ in Q_T that satisfy

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] - \lambda\left[(\nabla\rho\nabla)\vec{v} + (\vec{v}\cdot\nabla)\nabla\rho\right] = \mu\Delta\vec{v} - \nabla p + \rho\vec{f} - \nu|\vec{v}|^{k-2}\vec{v}, \quad k > 1, \quad (1)$$

$$\operatorname{div}\vec{v} = 0, \quad \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = \lambda\Delta\rho. \quad (2)$$

These equations describe the motion of an incompressible fluid together with the corresponding state equation. They are supplemented with the boundary conditions

$$\vec{v}|_{\Gamma_T} = 0, \quad \frac{\partial\rho}{\partial n}\Big|_{\Gamma_T} = 0, \quad (x,t) \in \Gamma_T, \quad (3)$$

and the initial conditions

$$\vec{v}|_{t=0} = \vec{v}_0(x), \quad \rho|_{t=0} = \rho_0(x), \quad x \in \Omega. \quad (4)$$

Here, $\vec{v}(x,t)$ denotes the velocity of the fluid, $p(x,t)$ is the pressure of the fluid, $\rho(x,t)$ is the density of the fluid and μ is a positive constant representing the kinematic viscosity coefficient of the fluid. The vector function $\vec{f}$ describes the density of external forces.

Moreover, $\vec{v}_0(x)$, $\rho_0(x)$, and $\vec{f}(x,t)$ are given functions. The constants $\lambda, ; \mu, ; \nu$ are known positive numbers.

$\vec{n} = (n_1, n_2, n_3)$ is the outward unit normal vector to Γ_T.

The function $\rho_0(x)$ is a positive bounded function:

$$0 < \operatorname{ess}\min_{\Omega} \rho_0(x) = m \le \rho_0(x) \le M = \operatorname{ess}\max_{\Omega} \rho_0(x) < \infty, \tag{5}$$

This restriction on $\rho_0(x)$ is due to the physical meaning of density. The sought function $\rho(x,t)$ must also remain positive and bounded.
Let $\aleph(\Omega) \equiv \{\vec{v} : \vec{v} \in C_0^\infty(\Omega), \operatorname{div}\vec{v} = 0\}$ be the set of infinitely differentiable, finitely supported, solenoidal continuous vector functions. The closures of $\aleph(\Omega)$ with respect to the norms of $L_2(\Omega)$ and $W_2^{1,0}(\Omega)$ are denoted respectively by H and V

2 Reviewed works

In the scientific literature, the system of equations (1)–(4) is also referred to as the Kazhikhov–Smagulov system. From the physical point of view, such a system describes the flow of an incompressible viscous non-Newtonian fluid.

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] - \lambda\left[(\nabla\rho\nabla)\vec{v} + (\vec{v}\cdot\nabla)\nabla\rho\right] = \mu\triangle\vec{v} - \nabla p + \rho\vec{f}$$

The case $\nu = 0$ in the system (1)–(4) was considered in [1].
Furthermore, in the case $\lambda = 0$, $\nu = 0$, the system (1)–(4) reduces to the classical Navier–Stokes system. Initial-boundary value problems for the Navier–Stokes system describing the flow of a nonhomogeneous fluid were studied in [2, 4, 6, 8] and other works.

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] = \mu\triangle\vec{v} - \nabla p + \rho\vec{f}$$

The existence and uniqueness of the solution to the problem in the case when the term λ^2 is nonzero are considered in [11].

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] - \lambda\left[(\nabla\rho\nabla)\vec{v} + (\vec{v}\cdot\nabla)\nabla\rho\right] +$$
$$+ \frac{\lambda^2}{\rho}\left[(\nabla\rho\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho\right] = \mu\triangle\vec{v} - \nabla p + \rho\vec{f}$$

In the following paper [15], the existence of a global weak solution to the inhomogeneous (i.e., variable density) incompressible Navier-Stokes system with mass diffusion is proven.

This system is a widely recognized version of the Kazhikhov–Smagulov model. The main novelty of the paper is the consideration of the Kazhikhov–Smagulov model with variable viscosity without simplifying the higher-order nonlinearities.

$$\begin{cases} \partial_t\rho + v\cdot\nabla\rho = \theta\Delta\rho & \text{in } (0,\infty)\times\Omega, \\ \partial_t(\rho v) + \sum_{j=1}^{3}\partial_{x_j}(\rho v_j v) - \nabla\cdot\{\mu(\rho)(\nabla v + \partial_t(\nabla v))\} & \\ \quad -\theta\Delta\rho - \theta(v\cdot\nabla)\rho - \theta(\nabla\rho\cdot\nabla)v + 2\theta\nabla\cdot\{\mu(\rho)\nabla(\log\rho)\} & \\ \quad +\theta^2\left\{\frac{\Delta\rho\nabla\rho}{\rho} + \frac{(\nabla\rho\cdot\nabla)\nabla\rho}{\rho} - \frac{|\nabla\rho|^2\nabla\rho}{\rho^2}\right\} = -\nabla p + \rho f & \text{in } (0,\infty)\times\Omega, \\ \nabla\cdot v = 0 & \text{in } (0,\infty)\times\Omega, \\ \nabla\rho\cdot n = 0, \quad v = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ \rho(0,\cdot) = \eta, \quad v(0,\cdot) = u & \text{in } \Omega. \end{cases}$$

3 Existence of a strong solution

Lemma 1 *Taking into account the maximum principle for the diffusion equation (2), the following inequality is valid:*

$$0 < m \le \rho^N(x,t) \le M < \infty, \tag{6}$$

where m and M are constants determined by $\rho_0(x)$ from equation (5).

Proof 1 *To prove this inequality, we first introduce the following notations:*

$$\rho^- = \max(0, m-\rho^N) \text{ and } \rho^+ = \max(0, M-\rho^N)$$

Now, if we multiply equation (2) by $m-\rho^N$ and integrate over the domain Ω,

$$\int_\Omega (m-\rho^N)\partial_t\rho dx + \int_\Omega (m-\rho^N)(\vec{v}\cdot\nabla)\rho^N dx + \lambda\int_\Omega \nabla\rho^N\cdot\nabla(m-\rho^N)dx = 0$$

This equation is equivalent to the following equation

$$\frac{d}{dt}\int_\Omega (m-\rho^N)^2 dx + 2\lambda\int_\Omega (\nabla(m-\rho^N))^2 dx = 0$$

Now, integrating this equation from 0 to t

$$\|\rho^-\|^2_{L_2(\Omega)} + 2\lambda\int_0^t \|\nabla\rho^-\|^2_{L_2(\Omega)} = \|\rho_0^-\|^2_{L_2(\Omega)}$$

Here, $\rho_0^- = \rho^-(0,x) = \max(0, m-\rho_0) = 0$, which implies from the above equation that $\rho^-(x,t) = 0$. This gives us the lower bound for $\rho(x,t)$. The upper bound can be found by the same method. Thus, it follows that

$$0 < m \le \rho^N(x,t) \le M < \infty, (x,t)\in\Omega\times(0,T)$$

Lemma 2 *Let $\vec{v} \in V$, $\vec{\omega} \in V$, and $\psi \in W_2^2(\Omega)$ be elements of the corresponding spaces. Then the following equality holds:*

$$((\vec{v}\nabla)\nabla\psi, \vec{\omega})_{L_2(\Omega)} = -\left((\vec{\omega}\nabla)\vec{v}, \nabla\psi\right)_{L_2(\Omega)} = \int_\Omega \psi \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j}\frac{\partial \omega_j}{\partial x_i} \tag{7}$$

Proof 2 *Since the space V consists of continuously differentiable finite solenoidal vector functions, we first verify equation (7) for $\vec{v}, \vec{\omega} \in \aleph(\Omega)$. By integrating the left-hand side of the equation by parts and taking into account that $\operatorname{div}\vec{\omega} = 0$, we obtain the following equality:*

$$((\vec{v}\nabla)\nabla\psi, \vec{\omega})_{L_2(\Omega)} = \int_\Omega \sum_{i,j=1}^{3} v_i \frac{\partial^2 \psi}{\partial x_i \partial x_j}\omega_j dx = -\int_\Omega \sum_{i,j=1}^{3} \frac{\partial \psi}{\partial x_i}\frac{\partial}{\partial x_j}(v_i, \omega_j) dx =$$

$$= -\int_\Omega \sum_{i,j=1}^{3} v_i \frac{\partial \psi}{\partial x_i}\frac{\partial v_i}{\partial x_j}\omega_j dx = -\left((\vec{\omega}\nabla)\vec{v}, \nabla\psi\right)_{L_2(\Omega)}$$

Thus, we prove the first part of equation (7). To verify the second part, we integrate by parts and take into account the condition $\operatorname{div}\vec{v} = 0$.

$$-\int_\Omega \sum_{i,j=1}^{3} v_i \frac{\partial \psi}{\partial x_i}\frac{\partial v_i}{\partial x_j}\omega_j dx = \int_\Omega \psi \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial v_i}{\partial x_j}\omega_j\right) dx = \int_\Omega \psi \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j}\frac{\partial \omega_j}{\partial x_i} dx$$

Theorem 1 *If $\vec{v}_0(x) \in V$, $\quad \rho_0(x) \in W_2^2(\Omega)$, $\quad \vec{f} \in L_2(Q)$, and the inequality*

$$\mu > \frac{\lambda}{2}(M - m)$$

holds, then for $2 < k \leq 4$, the problem (1)–(4) admits a local strong solution on the interval $(0, T_0)$, which satisfies the following a priori estimates:

$$\max_{0 \leq t \leq T_0} \left[\|\vec{v}\|_V + \|\Delta\rho\|_{L^2(\Omega)}\right] \leq C_1, \quad \int_0^{T_0} \left\{\|\nabla\rho_t\|^2_{L^2(\Omega)} + \|\widetilde{\Delta}\vec{v}\|^2_{L^2(\Omega)} + \|\vec{v}_t\|^2_{L^2(\Omega)}\right\} dt \leq C_2. \tag{8}$$

Here,

$$T_0 = \frac{1}{2C\|\vec{f}\|^2_{L_2(Q_T)}},$$

and C_1, C_2 are constants depending on the given data of the problem.

Proof 3 ***Galerkin's approximation***. *Let $\{\vec{\psi}_j\}_{j=1}^{\infty}$ be a system of functions from the space H, orthogonal in $L_2(\Omega)$, such that their linear combinations are dense in V, and assume that they are eigenfunctions of the following spectral problem:*

$$\left(\vec{\psi}_j, \vec{\omega}\right)_V = \lambda_j \left(\vec{\psi}_j, \vec{\omega}\right)_H \quad \forall\, \vec{\omega} \in V$$

For any $N = 1, 2, \ldots$, the approximate solution $(\vec{v}^N, \rho^N)$ of problem (1)–(4) is sought in the form

$$\vec{v}^N(x,t) = \sum_{j=1}^{N} c_j^N(t)\vec{\psi}_j(x) \tag{9}$$

where the unknown coefficients $c_j^N(t)$, $j = 1, 2, \ldots$ are the solution of the Cauchy problem for the system of differential equations:

$$\begin{gathered}\left(\rho^N\left[\frac{\partial\vec{v}^N}{\partial t} + \left(\vec{v}^N\cdot\nabla\right)\vec{v}^N\right] - \lambda\left[(\nabla\rho^N\nabla)\vec{v}^N + (\vec{v}^N\nabla)\nabla\rho^N\right] - \right.\\ \left. -\mu\Delta\vec{v}^N + \nabla P - \rho\vec{f} + \nu|\vec{v}^N|^{k-2}\vec{v}^N, \vec{\psi}_j\right)_{L_2(\Omega)} = 0\end{gathered} \tag{10}$$

with initial conditions

$$c_j^N(0) = c_j \equiv \left(\vec{v}_0, \vec{\psi}_j\right)_H, \quad j = 1, 2, \ldots, N$$

The approximate solution $\rho^N(x,t)$ for any $N = 1, 2, \ldots$ is sought as a classical solution of the problem

$$\frac{\partial\rho^N}{\partial t} + (\vec{v}^N\nabla)\rho^N = \lambda\Delta\rho^N, \quad \frac{\partial\rho^N}{\partial n}|_{\Gamma_T} = 0, \quad \rho^N|_{t=0} = \rho_0^N(x) \tag{11}$$

where $\rho_0^N(x)$ is a sequence of functions converging to $\rho_0(x)$ in $W_2^2(\Omega)$ and $L_q(\Omega), 1 \le q < \infty$.

By rewriting problems (9)–(11) as an operator equation and applying the Schauder theorem, the existence of a solution can be established (see Section 3 in [3]for details).

A priori estimates

By multiplying equation (11) by $\rho^N(x,t)$, summing over $k = 1, \ldots, N$, and integrating over the domain Q_T, we obtain the following inequality:

$$\int_{Q_T} |\nabla\rho^N(x,t)|^2 dxdt \le \frac{1}{2\lambda}\|\rho_0^N\| \equiv N_1 \tag{12}$$

Now, in order to obtain an estimate for the velocity, we multiply equation (10) by $c_N^j(t)$, sum over $k = 1, 2, \ldots, N$, and integrate over the domain Ω. As a result, we obtain the following equalities:

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^N|\vec{v}^N|^2 dx + \mu\|\vec{v}^N\|_V^2 - \lambda\left((\vec{v}^N\nabla)\nabla\rho^N, \vec{v}^N\right)_{L_2(\Omega)} = \int_\Omega \left(\rho^N f, \vec{v}^N\right)dx - \\ -\nu\int_\Omega \left(|\vec{v}^N|^{k-2}\vec{v}^N, \vec{v}^N\right)dx\end{gathered} \tag{13}$$

We apply Holder's inequality and embedding theorems to the first term on the right-hand side of the obtained equation, and estimate the third term on the left-hand side using lemma 2 as follows:

$$\left((\vec{v}^N\nabla)\nabla\rho^N, \vec{v}^N\right)_{L_2(\Omega)} = \int_\Omega \left(\rho^N - \frac{M+m}{2}\right)\sum_{i,j=1}^{3}\frac{\partial v_i^N}{\partial x_j}\frac{\partial v_j^N}{\partial x_i}dx \le \frac{M-m}{2}\|\vec{v}^N\|_V^2$$

Taking into account the above equalities and estimates, we derive the following estimate from equation (2.19):

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^N|\vec{v}^N|^2dx+\left(\mu-\frac{\lambda}{2}(M-m)\right)\|\vec{v}^N\|_V^2+m\|\vec{v}^N\|_{L_k(\Omega)}^k\le\\&\le C\|\vec{f}\|_{L_2(\Omega)}\|\vec{v}^N\|_V\end{aligned}\tag{14}$$

Here, according to the condition of the theorem,

$$\chi\equiv\mu-\frac{\lambda}{2}(M-m)>0$$

Thus, from equation (12), we obtain the following estimate. Applying Young's inequality to the right-hand side of the inequality and integrating from 0 *to* T*, we arrive at the following a priori estimate:*

$$\max_{0\le t\le T}\|\vec{v}^N(x,t)\|_{L_2(\Omega)}+\int_0^T\|\vec{v}^N(x,t)\|_V^2dt\le N_1\tag{15}$$

Let us now return to equation (2). Multiplying the equation by $\Delta\rho^N(x,t)$ *and integrating over the domain* Ω*, we obtain the following identity:*

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho^N(x,t)\|_{L_2(\Omega)}^2+\lambda\|\Delta\rho^N(x,t)\|_{L_2(\Omega)}^2=\left((\vec{v}^N\nabla)\rho^N,\Delta\rho^N\right)_{L_2}$$

If we integrate the right-hand side of the equation by parts and apply interpolation inequalities

$$\max_{0\le t\le T}\|\nabla\rho^N(x,t)\|_{L_2(\Omega)}+\int_0^T\|\Delta\rho^N(x,t)\|_{L_2(\Omega)}^2dt\le N_2.\tag{16}$$

Next, we multiply equation (10) by $\frac{dc_j^N(t)}{dt}$ *and sum over* $j=1,2,\dots,N$*, then integrate over the domain* Ω*. This yields*

$$\begin{aligned}&(\rho^N\vec{v}_t^N,\vec{v}_t^N)_{L_2(\Omega)}+\left(\rho^N(\vec{v}^N\Delta\nabla)\vec{v}^N,\vec{v}_t^N\right)_{L_2(\Omega)}-\\&-\lambda\left[\left((\nabla\rho^N\nabla)\vec{v}^N,\vec{v}_t^N\right)_{L_2(\Omega)}+\left((\vec{v}^N\nabla)\nabla\rho^N,\vec{v}_t^N\right)_{L_2(\Omega)}\right]-\mu(\Delta\vec{v}^N,\vec{v}_t^N)_{L_2(\Omega)}+\\&+(\rho^N\vec{f},\vec{v}_t^N)_{L_2(\Omega)}+\nu\left(|\vec{v}^N|^{k-2}\vec{v}^N,\vec{v}_t^N\right)_{L_2(\Omega)}=0.\end{aligned}\tag{17}$$

Now we introduce the Stokes operator $\widetilde{\Delta}$ *defined by the formula*

$$\widetilde{\Delta}\vec{u}=P_H\Delta\vec{u},\ \ \forall\vec{u}\in W_2^2(\Omega)\cap V,$$

where $P_H:\mathbf{L}^2(\Omega)\to\mathbf{H}(\Omega)$ *is the Leray projection. Some properties of the operator* $\widetilde{\Delta}$ *are considered in section III of [9].*

In particular, we have

$$\|\vec{u}\|_{W_2^2(\Omega)\cap V}\le C\|\widetilde{\Delta}\vec{u}\|_{L_2(\Omega)},\ \ \forall\vec{u}\in W_2^2(\Omega)\cap V.$$

Applying the above inequality and the embedding theorem, we obtain

$$\|\vec{v}_x^N\|_{L_4(\Omega)} \le C\|\widetilde{\Delta}\vec{v}^N\|^{\frac{1}{2}}\|\vec{v}^N\|_V^{\frac{1}{2}}. \tag{18}$$

Now, using (18), we rewrite (17) in the following form:

$$\mu\frac{d}{dt}\left(\|\vec{v}\|_V^2 + m\|\vec{v}\|_{L_k}^k\right) + m\|\vec{v}_t\|_{L_2} \le \varepsilon\|\widetilde{\Delta}\vec{v}\|_{L_2} + C_\varepsilon\left[\|\Delta\rho\|_{L_2}^2 + \|\vec{v}\|_V^2\right]\|\vec{v}\|_V^2 + c\|\vec{f}\|_{L_2}^2, \tag{19}$$

where $\varepsilon > 0$ is an arbitrary number.

Next, we estimate $\|\widetilde{\Delta}\vec{v}\|$ in terms of $\|\vec{v}_t\|$ and, for a suitable value of $\varepsilon > 0$, reduce (19) to a differential inequality with respect to $\|\vec{v}\|_V$. For this, we apply the Stokes operator $\widetilde{\Delta}$ to both sides of equation (17), multiply by $c_j^N(t)$, sum over $j = 1, 2, \ldots, N$, integrate over Ω, and treat $\vec{v}_t^N$ as being on the right-hand side of a stationary equation. We obtain

$$\begin{aligned}\mu\|\widetilde{\Delta}\vec{v}^N\|_{L_2}^2 &= \left(\rho\vec{v}_t^N - \rho^N\vec{f}, \widetilde{\Delta}\vec{v}^N\right)_{L_2(\Omega)} + \left(\rho^N(\vec{v}^N\nabla)\vec{v}^N, \widetilde{\Delta}\vec{v}^N\right)_{L_2(\Omega)} - \\ &- \lambda\left((\nabla\rho^N\cdot\nabla)\vec{v}^N, \widetilde{\Delta}\vec{v}^N\right)_{L_2(\Omega)} - \lambda\left((\vec{v}^N\nabla)\nabla\rho^N, \widetilde{\Delta}\vec{v}^N\right)_{L_2(\Omega)} - \\ &- \nu(|\vec{v}^N|^{k-2}\vec{v}^N, \widetilde{\Delta}\vec{v}^N)_{L_2(\Omega)}.\end{aligned} \tag{20}$$

We now estimate the three terms on the right-hand side of equation (20) in the standard way:

$$\left(\rho\vec{v}_t^N - \rho\vec{f}, \widetilde{\Delta}\vec{v}^N\right)_{L_2(\Omega)} \le \varepsilon_1\|\widetilde{\Delta}\vec{v}^N\|_{L_2}^2 + C_1\left[\|\vec{v}_t^N\|_{L_2}^2 + \|\vec{f}\|_{L_2}^2\right],$$

$$\left(\rho(\vec{v}^N\nabla)\vec{v}^N, \widetilde{\Delta}\vec{v}^N\right)_{L_2(\Omega)} \le M\|\widetilde{\Delta}\vec{v}^N\|_{L_2}\|\vec{v}_x^N\|_{L_4(\Omega)}\|\vec{v}^N\|_{L_4(\Omega)} \le \varepsilon_2\|\widetilde{\Delta}\vec{v}^N\|_{L_2}^2 + C_2\|\vec{v}^N\|_V^4,$$

$$\begin{aligned}&- \lambda\left((\nabla\rho^N\cdot\nabla)\vec{v}^N, \widetilde{\Delta}\vec{v}^N\right)_{L_2(\Omega)} \le \|\widetilde{\Delta}\vec{v}^N\|_{L_2}\|\widetilde{\Delta}\vec{v}^N\|_{L_4(\Omega)}\|\nabla\rho^N\|_{L_4(\Omega)} \\ &\le \varepsilon_3\|\widetilde{\Delta}\vec{v}^N\|_{L_2}^2 + C_3\|\Delta\rho^N\|_{L_2}^2\|\vec{v}^N\|_V^2, \\ &- \lambda\left((\vec{v}^N\nabla)\nabla\rho^N, \widetilde{\Delta}\vec{v}^N\right)_{L_2(\Omega)} = \lambda\left((\widetilde{\Delta}\vec{v}^N\nabla)\vec{v}^N, \nabla\rho^N\right)_{L_2(\Omega)} \\ &\le \varepsilon_4\|\widetilde{\Delta}\vec{v}^N\|_{L_2}^2 + C_4\|\Delta\rho\|_{L_2}^2\|\vec{v}\|_V^2.\end{aligned}$$

Now let us consider the last term on the right-hand side of (20). Taking into account that $\rho_t, \Delta\vec{v} \in L_2$ and $\rho \in L_\infty(Q_T)$, we obtain the following estimate:

$$\begin{aligned}&- \nu\Big(|\vec{v}^N|^{k-2}\vec{v}^N, \widetilde{\Delta}\vec{v}^N\Big)_{L_2(\Omega)} \le \left|\nu\int_\Omega |\vec{v}^N|^{k-2}\vec{v}^N\widetilde{\Delta}\vec{v}^N dx\right| \\ &\le \nu\int_\Omega |\vec{v}^N|^{k-1}|\widetilde{\Delta}\vec{v}^N|dx \le \nu\left(\int_\Omega |\vec{v}|^{2k-2}dx\right)^{\frac{1}{2}}\|\widetilde{\Delta}\vec{v}\|_{L_2(\Omega)} \\ &\le \varepsilon\|\widetilde{\Delta}\vec{v}\|_{L_2(\Omega)}^2 + \nu^2\frac{1}{4\varepsilon}\int_\Omega |\vec{v}|^{2k-2}dx = \varepsilon\|\widetilde{\Delta}\vec{v}\|_{L_2(\Omega)}^2 + \nu^2\frac{1}{4\varepsilon}\|\vec{v}^N\|_{L_{2k-2}}^{2k-2}.\end{aligned}$$

Using the embedding

$$\mathring{W}_q^1(\Omega) \hookrightarrow L_{2^*}, \quad \|\vec{v}\|_{L^{2^*}} \le C\|\nabla\vec{v}\|_{L_2}, \quad 2^* = \frac{2d}{d-2},$$

we obtain the estimate

$$
\begin{aligned}
\|\vec{v}^N\|^{2k-2}_{L_{2k-2}} &= \int_\Omega |\vec{v}^N|^{2k-2}dx \\
&\leq \Big(\int_\Omega |\vec{v}^N|^{(2k-2)\frac{2d}{(2k-2)(d-2)}}dx\Big)^{\frac{(2k-2)(d-2)}{2d}} \Big(\int_\Omega |1|^{\frac{2d}{2d-m(d-2)}}dx\Big)^{\frac{2d-m(d-2)}{2d}} \\
&\leq C(d,\Omega,2k-2)\|\vec{v}^N\|^{2k-2}_{L_{\frac{2d}{d-2}}(\Omega)} \leq C_1(d,2k-2,\Omega)\|\nabla\vec{v}^N\|^{2k-2}_{L_2(\Omega)}.
\end{aligned}
\tag{21}
$$

Here $d = 3$ is the spatial dimension, and $2 < 2k-2 \leq 2^* = 6$.
Therefore, we arrive at

$$
\mu\frac{d}{dt}\|\vec{v}^N\|^2_V + m\|\widetilde{\triangle}\vec{v}\|^2_{L_2} \leq C\left[\|\vec{f}\|^2_{L_2} + \|\vec{v}\|^6_V + \|\triangle\rho\|^6\right]. \tag{22}
$$

Multiplying equation (2) by $\Delta\rho_t$ in $L_2(\Omega)$ and applying embedding theorems, we get

$$
\frac{\lambda}{2}\frac{d}{dt}\|\Delta\rho_t^N\|^2_{L_2} + \|\nabla\rho_t^N\|^2_{L_2} = ((\vec{v}^N\cdot\nabla)\rho^N, \Delta\rho_t^N)_{L^2(\Omega)}. \tag{23}
$$

Integrating the right-hand side by parts, we obtain

$$
\begin{aligned}
&\Big((\vec{v}^N\cdot\nabla)\rho^N, \Delta\rho_t^N\Big) = -\Big((\vec{v}^N\cdot\nabla)\nabla\rho^N, \nabla\rho_t^N\Big)_{L_2(\Omega)} - \Big((\nabla\rho_t^N\cdot\nabla)\vec{v}^N, \nabla\rho^N\Big)_{L_2(\Omega)} \\
&\leq C\|\nabla\rho_t^N\|_{L^2(\Omega)}\left[\max_\Omega|\vec{v}^N|\|\Delta\rho_t^N\|_{L_2(\Omega)} + \|\vec{v}^N_x\|_{L^3(\Omega)}\|\nabla\rho^N\|_{L^6(\Omega)}\right].
\end{aligned}
\tag{24}
$$

From the embedding inequalities:

$$
\max_\Omega|\vec{v}^N| \leq c\|\widetilde{\Delta}\vec{v}^N\|^{1/2}_{L^2(\Omega)}\|\vec{v}^N\|^{1/2}_V, \quad \|\vec{v}^N_x\|_{L^3(\Omega)} \leq c\|\widetilde{\Delta}\vec{v}^N\|^{1/2}_{L^2(\Omega)}\|\vec{v}^N\|^{1/2}_V,
$$

$$
\|\nabla\rho^N\|_{L^6(\Omega)} \leq C\|\Delta\rho^N\|_{L^2(\Omega)}.
$$

Using these, from (23) we obtain

$$
\begin{aligned}
&\frac{\lambda}{2}\frac{d}{dt}\|\Delta\rho_t^N\|^2_{L^2(\Omega)} + \|\nabla\rho_t^N\|^2_{L^2(\Omega)} \leq \\
&\leq C\|\nabla\rho_t^N\|_{L^2(\Omega)}\|\Delta\rho^N\|_{L^2(\Omega)}\|\widetilde{\Delta}\vec{v}^N\|^{\frac{1}{2}}_{L^2(\Omega)}\|\vec{v}^N\|^{\frac{1}{2}}_{L^2(\Omega)} \\
&\leq \frac{1}{2}\|\nabla\rho_t^N\|^2_{L^2(\Omega)} + \varepsilon\|\widetilde{\Delta}\vec{v}\|^2_{L^2(\Omega)} + C_\varepsilon[\|\vec{v}\|^6_V + \|\triangle\rho\|^6_{L^2(\Omega)}].
\end{aligned}
\tag{25}
$$

Adding inequalities (22) and (25), and taking $\varepsilon = \frac{1}{2}m$, we obtain for the function

$$
y(t) = \mu\|\vec{v}^N\|^2_V + \lambda\|\Delta\rho^N\|^2_{L_2}
$$

the following differential inequality:

$$
\frac{dy}{dt} + \|\nabla\rho_t^N\|^2_{L^2(\Omega)} + m\|\widetilde{\Delta}\vec{v}^N\|^2_{L^2(\Omega)} \leq C\Big[\|\vec{f}\|^2_{L^2(\Omega)} + y^3\Big]. \tag{26}
$$

From this inequality, for a sufficiently small time interval $[0, T_0]$, *since*

$$y(0) = \mu\|\vec{v}_0^N\|^2 + \lambda\|\Delta\rho_0^N\|^2, \qquad \|\vec{f}\|_{L_2(Q_T)} \text{ are bounded,}$$

we obtain

$$\max_{0\le t\le T_0}\left[\|\vec{v}^N\|_V + \|\Delta\rho^N\|_{L_2(\Omega)}\right] \le C_1, \quad \int_0^{T_0}\left\{\|\nabla\rho_t^N\|^2_{L_2(\Omega)} + \|\widetilde{\Delta}\vec{v}^N\|^2_{L_2(\Omega)}\right\}dt \le C_2. \quad (27)$$

Based on these estimates, we conclude that the three-dimensional problem admits a local strong solution, and by passing to the limit one can establish convergence of the solution.

Passage to the limit

The approximate solutions $\vec{v}^N$ *and* ρ^N, *defined by (9),(11), satisfy the a priori estimates (27). From this, one can extract subsequences* $\vec{v}^n$ *and* ρ^n *from* $\vec{v}^N$ *and* ρ^N, *respectively, for which the following types of weak convergence hold.*

For the three-dimensional case, the following local weak convergence of approximate solutions is valid:

$$\vec{v}^n \rightharpoonup \vec{v} \quad \text{weakly-}* \text{ in } L^\infty(0, T_0; V), \quad n\to\infty, \tag{28}$$

$$\vec{v}^n \rightharpoonup \vec{v} \quad \text{weakly in } L_2(0, T_0; W_2^2), \quad n\to\infty, \tag{29}$$

$$\vec{v}_t^n \rightharpoonup \vec{v}_t \quad \text{weakly in } L_2(0, T_0; L_2), \quad n\to\infty, \tag{30}$$

$$\rho^n \rightharpoonup \rho \quad \text{weakly-}* \text{ in } L^\infty(0, T_0; W_2^2), \quad n\to\infty, \tag{31}$$

$$\frac{\partial\rho^n}{\partial t} \rightharpoonup \frac{\partial\rho}{\partial t}, \quad \text{weakly in } L^2(0, T_0; W_2^1), \quad n\to\infty. \tag{32}$$

From the above convergence results and by applying the Banach–Alaoglu theorem, one can pass to the limit in the approximate solution of problem (1)–(4). That is, we conclude the existence of a strong solution to problem (1)–(4).

4 Conclusion

In this work, the solvability of the initial-boundary value problem based on a mathematical model describing nonstationary diffusion processes in inhomogeneous fluids was studied. As a result of theoretical analysis, the necessary conditions for the existence of a solution were identified. For this purpose, methods of functional analysis were used, including a priori estimates.

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