IRSTI 27.31.44 DOI: https://doi.org/10.26577/JMMCS129120262

A.A. Aniyarov
Astana International University, Astana, Kazakhstan
Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
e-mail: aniyarov@math.kz

SYMMETRY EQUIVALENCE OF NON-UNIFORM BEAMS WITH INTERMEDIATE MASSES

This paper investigates the spectral properties of non-uniform Euler–Bernoulli beams resting on a Winkler–type elastic foundation and subjected to an axial load, in the presence of two concentrated masses placed symmetrically with respect to the beam midpoint. The governing equation includes variable bending stiffness, foundation modulus, and distributed mass, while the point masses are modeled through classical continuity and jump conditions. Under an agreed symmetry of the variable coefficients, we prove that the spectral problem admits a symmetry-based decomposition: the eigenvalues and eigenfunctions of the full beam split into symmetric and antisymmetric families. This is achieved through two auxiliary problems on the half-interval, equipped with sliding-hinged and hinged-hinged (or sliding–clamped and hinged–clamped) midpoint conditions. The resulting factorization reduces the complexity of eigenvalue computation and extends known symmetry equivalence results to beams with intermediate masses. Numerical examples confirm the theoretical findings and illustrate the loss of symmetry when coefficient symmetry is violated. The proposed framework significantly reduces the computational complexity of spectral analysis for non-uniform beams with discrete masses and extends existing symmetry-based results to a broader class of structurally heterogeneous systems.
Key words: Euler–Bernoulli beam, non-uniform beam, eigenvalue, symmetry, equivalence, intermediate elements.

А.А. Аниyаров
Астана Халықаралық университеті, Астана, Қазақстан
Математика және математикалық моделдеу институты, Алматы, Қазақстан
e-mail: aniyarov@math.kz

Аралық массалары бар біркелкі емес бөренелердің симметриялық эквиваленттілігі

Бұл мақалада бөрененің ортаңғы нүктесіне қатысты симметриялы түрде орналастырылған екі аралық массасы болған кезде Винклер типті серпімді іргетасқа сүйеніп, осьтік жүктемесі бар біркелкі емес Эйлер-Бернулли бөренелерінің спектрлік қасиеттері зерттеледі. Басқарушы теңдеуде айнымалы иілу қаттылығы, іргетас модулі және қосарланған масса қарастырылады, ал нүктелік массалар классикалық үздіксіздік және секіру шарттары арқылы модельденеді. Коэффициенттердің келісілген симметриясы жағдайында спектрлік есептің симметрияға негізделген жіктелуі орын алатыны дәлелденеді: толық бөрененің меншікті мәндері мен меншікті функциялары симметриялы және антисимметриялы класстарға бөлінеді.Бұған сырғымалы-топсалы және топсалы-топсалы (немесе сырғамалы-қатты және топсалы-қатты) ортаңғы нүкте шарттарымен жабдықталған жартылай аралықтағы екі көмекші есеп арқылы қол жеткізіледі. Нәтижесінде мұндай факторизация меншікті мәндерді есептеудің күрделілігін төмендетеді және симметрия эквиваленттілігі жөніндегі белгілі нәтижелерді аралық массалары бар бөренелерге дейін кеңейтеді. Сандық мысалдар теориялық тұжырымдарды растайды және коэффициенттердің симметриясы бұзылған кезде симметрияның жойылуын көрсетеді. Ұсынылған құрылым дискретті массалары бар біркелкі емес бөренелер үшін спектрлік талдаудың есептеу күрделілігін айтарлықтай төмендетеді және симметрияға негізделген бар нәтижелерді құрылымдық жағынан әртекті жүйелердің кеңірек класына кеңейтеді.
Түйін сөздер: Эйлер–Бернулли бөренесі, біркелкі емес бөрене, меншікті мән, симметрия, эквиваленттік, аралық элементтер.

А.А. Анияров
Международный университет Астана, Астана, Казахстан
Институт математики и математического моделирования, Алматы, Казахстан
e-mail: aniyarov@math.kz

Симметричная эквивалентность неравномерных балок с промежуточными массами

В данной работе исследуются спектральные свойства неоднородных балок Эйлера-Бернулли, опирающихся на упругий фундамент типа Винклера и подверженных осевой нагрузке, при наличии двух сосредоточенных масс, расположенных симметрично относительно середины балки. Управляющее уравнение включает переменную жесткость на изгиб, модуль упругости фундамента и распределенную массу, в то время как точечные массы моделируются с помощью классических условий непрерывности и скачка. При согласованной симметрии переменных коэффициентов мы доказываем, что спектральная задача допускает разложение на основе симметрии: собственные значения и собственные функции полной балки разделяются на симметричные и антисимметричные семейства. Это достигается с помощью двух вспомогательных задач на полуинтервале, снабженных условиями скользящего шарнира и шарнира-шарнира (или скользящего зажима и шарнира-зажима) в средней точке. Полученная факторизация снижает сложность вычисления собственных значений и распространяет известные результаты симметричной эквивалентности на балки с промежуточными массами. Численные примеры подтверждают теоретические выводы и иллюстрируют потерю симметрии при нарушении симметрии коэффициентов. Предлагаемая структура значительно снижает вычислительную сложность спектрального анализа для неравномерных балок с дискретными массами и распространяет существующие результаты, основанные на симметрии, на более широкий класс структурно неоднородных систем.
Ключевые слова: балка Эйлера–Бернулли, неоднородная балка, собственное значение, симметрия, эквивалентность, промежуточные элементы.

1 Introduction

Mechanical structures involving beam elements frequently exhibit spatially varying physical and geometric properties due to manufacturing processes, graded materials, temperature fields, or intentional design features. Such non-uniformity substantially increases the complexity of vibration analysis, leading to fourth-order differential equations with variable coefficients. A broad variety of analytical and numerical approaches have therefore been developed to study free vibrations of beams with variable stiffness or heterogeneous mass distributions, including asymptotic formulations, special-function techniques, regular variation methods, collocation schemes and Gaussian expansions [1–5].

Recent developments in analytical modelling of non-uniform Euler–Bernoulli beams include exact and approximate formulations proposed by Lee and Ke [6], Datta and Sil [7], and [8], as well as the comprehensive monograph by Wang [9] providing closed-form solutions for a wide class of variable-coefficient beam models. Alongside these analytical approaches, significant progress has been achieved in numerical methods, including Bernstein spectral collocation [10], Chebyshev–based Petrov–Galerkin schemes [11], differential quadrature formulations [12], and advanced matrix-collocation techniques for tapered and functionally graded beams [13]. These modern analytical and computational tools constitute an essential framework for studying vibration properties of non-uniform and functionally graded beams, and they form the methodological background for the present work.

In the modeling of mechanical systems, it is essential to have a closed analytical formula for natural frequencies [14–17]. A review of the literature can be found in the work [20]. Several results pertaining to the closed-form expression for the natural frequencies of uniform beams were modified in [17]. The symmetric equivalence of boundary value problems for the uniform and non-uniform beams, both with and without axial loads on a Winkler-type foundation, was studied in [18] and [19], while [20] further established the role of coefficient symmetry in producing spectral decomposition for non-uniform beams.

At the same time, structural symmetry plays an equally significant role in systems containing discrete internal elements such as point masses. Analytical investigations have shown that intermediate masses fundamentally modify the spectral characteristics of a beam through jump conditions and inertial discontinuities. For instance, exact solutions for beams carrying two concentrated masses were constructed in [21], where it was demonstrated that symmetric mass placement naturally separates vibration modes into symmetric and antisymmetric families. A methodological comparison of classical approaches to deriving frequency equations for multi-mass beams was presented in [22], confirming that symmetric configurations lead to noticeable simplifications of characteristic equations. Moreover, the orthogonality properties of beams with multiple masses and piecewise-varying stiffness were generalised in [23], emphasising the need for appropriately weighted modal inner products. A related line of research on beams with internal supports [24] shows that even discrete structural constraints without mass can significantly reshape the spectrum and alter modal interactions. Akhmediyev et al. [25] showed that point masses and elastic?pliable supports substantially modify the vibration response of a multi-element beam, and demonstrated that the presence of concentrated masses leads to a more complex modal structure, including the appearance of nodes at one of the mass locations. Together, these studies indicate that discrete internal elements strongly affect the underlying spectral structure of beam models, but they do not resolve how geometric symmetry interacts with variable coefficients and symmetric mass placement in the non-uniform case.

Recent studies also demonstrate that the geometric symmetry of concentrated masses plays a decisive role in the uniqueness of inverse problems; in particular, symmetric placement of the first and third masses leads to non-uniqueness of reconstruction from the first three natural frequencies [26]. Despite this progress, the symmetric equivalence of boundary value problems for non-uniform beams with concentrated masses remains insufficiently explored.

The goal of the present research is to identify conditions on the variable bending stiffness, foundation coefficient, mass distribution, and types of end fixity under which it is possible to establish the equivalence of eigenvalues and eigenfunctions for non-uniform beams carrying intermediate point masses. The main novelty lies in the introduction of an agreed symmetry of the variable coefficients (see Theorem 1), which allows the reduction of the full problem to two auxiliary problems on the half-interval. The results extend several previously known results [17–20] to a significantly broader class of beam systems containing discrete masses.

2 Mathematical problem formulation

Consider a non-uniform Euler-Bernoulli beam of unit length $(0, 1)$, resting on a Winkler-type elastic foundation and subjected to a constant axial load T. Let $EJ(x)$ is the bending stiffness, $\rho\, A(x)$ is mass of the beam per unit length, $k(x)$ is the variable coefficient of foundation. The equation of transverse free vibrations has the form

$$(EJ(x)v''(\lambda,x))'' + Tv''(\lambda,x) + k(x)v(\lambda,x) = \lambda\rho A(x)v(\lambda,x) + \\ + \lambda m v(a)\delta(x-a) + \lambda m v(1-a)\delta(x-(1-a)), \quad x \in I_p, p = 1,2, \quad (1)$$

where $\lambda = \rho\omega^2$ are the eigenvalues; ω is the circular frequency; ρ is the material density; $v(\lambda, x)$ are the eigenfunctions of the transverse static deflection of the beam; T is corresponding to a constant compressive force if $T > 0$ or a constant tensile force if $T < 0$; $I_1 = (0, 1)$, $I_2 = \left(\frac{1}{2}, 1\right)$; $m > 0$ are two concentrated masses located symmetrically at points $x = a$ and $x = 1 - a$, $0 < a < \frac{1}{2}$; $\delta(x)$ denotes the Dirac delta function. Notice that $J(x)$ and $A(x)$ are assumed twice continuously differentiable and strictly positive, $k(x)$ is the real-valued summable function.

Depending on the physical configuration, the beam is assumed to be fixed at both ends either as follows. The first is the hinged-hinged beam on the interval I_1 with the boundary conditions (see Figure 1)

$$v(\lambda,0) = 0,\ v''(\lambda,0) = 0, v(\lambda,1) = 0,\ v''(\lambda,1) = 0, \quad (2)$$

and the second is the clamped-clamped beam on the interval I_1 with the boundary conditions (see Figure 2)

$$v(\lambda,0) = 0,\ v'(\lambda,0) = 0, v(\lambda,1) = 0,\ v'(\lambda,1) = 0. \quad (3)$$

Let the beam contain two concentrated masses of equal magnitude $m > 0$, located at the

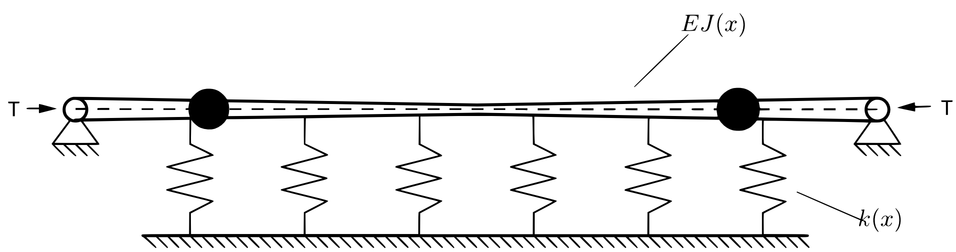

Figure 1: Hinged-hinged Euler–Bernoulli non-uniform beam.

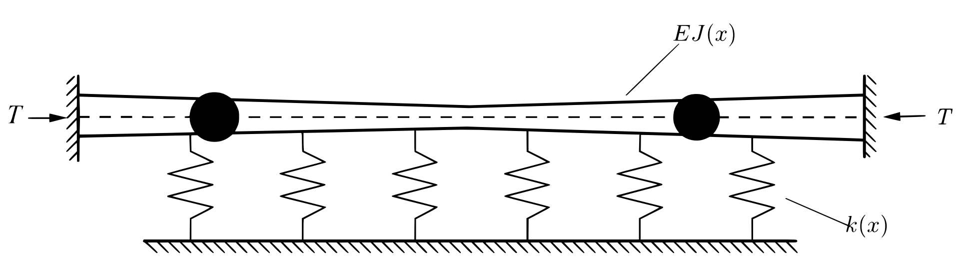

Figure 2: Clamped-clamped Euler–Bernoulli non-uniform beam.

symmetric points $x = a$ and $x = 1 - a$. At each of these points the eigenfunctions of the

spectral problem satisfy the standard interface conditions associated with a concentrated mass in the Euler-Bernoulli model. Let $x_j \in \{a, 1-a\}$. Then the following relations hold:
1. Continuity of deflection and rotation

$$v(x_j^-) = v(x_j^+), \qquad v'(x_j^-) = v'(x_j^+) \tag{4}$$

2. Continuity of bending moment

$$[EJ(x)v''(x)]_{x=x_j} = 0 \tag{5}$$

3. Jump condition for the shear force

$$[EJ(x)v'''(x)]_{x=x_j} = \lambda m v(x_j) \tag{6}$$

In order to formulate auxiliary problems on the half-interval I_2, we introduce two additional types of boundary conditions at the midpoint $x = \frac{1}{2}$. Introduce the sliding-hinged boundary conditions

$$v'\left(\lambda, \frac{1}{2}\right) = 0,\ v'''\left(\lambda, \frac{1}{2}\right) = 0,\ v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0, \tag{7}$$

and hinged-hinged boundary conditions

$$v\left(\lambda, \frac{1}{2}\right) = 0,\ v''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0 \tag{8}$$

which are connected with hinged-hinged fixing on the interval I_2. Furthermore, we introduce the sliding-clamped boundary conditions

$$v'\left(\lambda, \frac{1}{2}\right) = 0,\ v'''\left(\lambda, \frac{1}{2}\right) = 0,\ v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0, \tag{9}$$

and hinged-clamped boundary conditions

$$v\left(\lambda, \frac{1}{2}\right) = 0,\ v''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0 \tag{10}$$

which are connected with clamped-clamped fixing on the interval I_2.

These midpoint conditions define the auxiliary problems B and C on the interval $\left(\frac{1}{2}, 1\right)$, which correspond to symmetric and antisymmetric eigenmodes, respectively. Their introduction enables the decomposition of the spectrum of the full problem into two independent parts [18], [20].

3 Main results

Let $\sigma(A_1), \sigma(B_1), \sigma(C_1)$ be a set of eigenvalues of problems $A_1 - \lambda I,\ B_1 - \lambda I,\ C_1 - \lambda I$ generated by Equation (1) on finite intervals by boundary conditions (2), (7), (8) and interface conditions, respectively.

Theorem 1 *Let $J(x)$, $k(x)$ and $A(x)$ be the symmetric functions with respect to the point $x = \frac{1}{2}$*

$$J(x) = J(1-x), \quad k(x) = k(1-x), \quad A(x) = A(1-x), \quad x \in \left[0; \frac{1}{2}\right] \tag{11}$$

and $T < T_{cr}$ be a subcritical axial load. The following statements are true:
1. The spectrum of the full problem is the union of the spectra of the two half-interval problems:

$$\sigma(A_1) \equiv \sigma(B_1) \cup \sigma(C_1)$$

2. If $\lambda \in \sigma(B_1)$ or $\lambda \in \sigma(C_1)$, then the eigenfunctions of problems $A_1 - \lambda I$ corresponding to the eigenvalues λ are symmetric or asymmetric with respect to the middle of the beam at the point $x = \frac{1}{2}$ on the interval $(0, 1)$, respectively.

This theorem expresses that, under the assumed symmetry of the coefficients and symmetric placement of the two point masses, the spectrum of the hinged-hinged beam with two concentrated masses can be factorized into two parts associated with symmetric and antisymmetric vibration modes, described by the sliding-hinged and hinged-hinged problems on the half-interval.

Let $\sigma(A_2)$, $\sigma(B_2)$, $\sigma(C_2)$ be a set of eigenvalues of problems $A_2 - \lambda I$, $B_2 - \lambda I$, $C_2 - \lambda I$ generated by Equation (1) on finite intervals by boundary conditions (3), (9), (10) and interface conditions, respectively.

Theorem 2 *Let $J(x)$, $k(x)$ and $A(x)$ be the symmetric functions with respect to the point $x = \frac{1}{2}$, i.e. the condition in Equation (11) holds and $T < T_{cr}$. The following statements are true:*
1. $\sigma(A_2) \equiv \sigma(B_2) \cup \sigma(C_2)$
2. If $\lambda \in \sigma(B_2)$ or $\lambda \in \sigma(C_2)$, then the eigenfunctions of problems $A_2 - \lambda I$ corresponding to the eigenvalues λ are symmetric or asymmetric with respect to the middle of the beam at the point $x = \frac{1}{2}$ on the interval $(0, 1)$, respectively.

The proof of Theorems 1 and 2 is ideologically similar to that presented in works [18], [20]. Nevertheless, there is a single discrepancy, which require is calculating of the critical value T_{cr}. Further will be described the scheme for proving Theorems 1 and 2.

First step. The following functions $J(x)$, $k(x)$ and $A(x)$ will be selected to satisfy condition (11).

Second step. The critical value of T_{cr} will be calculated that corresponding to the first step and the value of T will be selected such that $T < T_{cr}$. The calculation of T_{cr} will be conducted using well-known numerical method (see, [28]).

Third step. The final step will employ the same technique used to prove the result presented in [18].

Upon completion of the aforementioned three steps, the proofs of Theorems 1 and 2 will be obtained. In the third step, the analytical or numerical method may be employed. It should be noted that if the functions $J(x)$, $k(x)$ and $A(x)$ satisfy condition (11) and the additional conditions from [1], then the non-uniform beam can be transformed into a uniform one.

4 Examples and discussion

In this section, we calculate approximately the four or five eigenvalues of boundary value problems $A_n-\lambda I$, $B_n-\lambda I$, $C_n-\lambda I$ $(n=1,2,3)$ generated by the Euler–Bernoulli equation for the various coefficients $J(x)$, $k(x)$, $A(x)$ and $p(x)$. The results of calculation of the eigenvalues are shown in the corresponding columns of Tables 1–4.

Example 1 *In this analysis, we examine three steps.*

First step. Let $J(x)=1+x(1-x)$, $k(x)=4x(1-x)$, $A(x)=x(1-x)$ *and* $E=1$.

Second step. In this example $T_{cr}\approx 12.09$ *and take* $T=5$, $a=0.2$ *and* $m=0.25$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 1 are shown in Table 1.*

Table 1: Numerical calculations of the first five eigenvalues from the Example 1

Hinged-hinged at the points $x=0$, $x=1$	Sliding at the point $x=\frac{1}{2}$, hinged at the point $x=1$	Hinged at the point $x=\frac{1}{2}$, hinged at the point $x=1$
(2)	(7)	(8)
11.07	11.07	38.36
38.36	121.5	319.78
121.52	627.37	964.55
319.85	1166.92	1376.94
627.41	1842.48	2510.70

The calculations presented in Example 1 provide corroboration for the validity of Statement 1 of Theorem 1 pertaining to the factorization of the set of eigenvalues.

Example 2 *In this analysis, we consider three steps.*

First step. Let $J(x)=x(1-x)$, $k(x)=5(1+x)^3$, $A(x)=x(1-x)$ *and* $E=1$.

Second step. In this example $T_{cr}\approx 3.73$ *and take* $T=1$, $a=0.2$ *and* $m=0.25$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 2 are shown in Table 2.*

The violation of the regularity of factorization of eigenvalues in Example 2 is due to the failure to satisfy the symmetry condition for the function $k(x)$. The aforementioned calculations in Example 2 confirm the validity of Statement 1 of Theorem 1.

Example 3 *We consider three steps.*

First step. Let $J(x)=1+x(1-x)$, $k(x)=4$, $A(x)=x^2(1-x)^2$ *and* $E=1$.

Second step. In this example $T_{cr}\approx 45.71$ *and take* $T=30$, $a=0.2$ *and* $m=0.25$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 3 are shown in Table 3.*

The calculations which represent in Example 3 confirm the validity of Statement 1 of Theorem 2 on the factorization of the set of eigenvalues.

Table 2: Numerical calculations of the first five eigenvalues from the example 2.

Hinged-hinged at the points $x=0$, $x=1$	Sliding at the point $x=\frac{1}{2}$, hinged at the point $x=1$	Hinged at the point $x=\frac{1}{2}$, hinged at the point $x=1$
(2)	(7)	(8)
6.72	7.39	14.33
14.06	46.29	126.27
46.14	239.09	317.57
126.09	373.45	511.67
239.4	737.47	978.79

Table 3: Numerical calculations of the first five eigenvalues from the Example 3

Clamped-clamped at the points $x=0$, $x=1$	Sliding at the point $x=\frac{1}{2}$, clamped at the point $x=1$	Hinged at the point $x=\frac{1}{2}$, clamped at the point $x=1$
(3)	(9)	(10)
28.28	28.28	59.6
59.61	204.8	635.7
204.82	1344.86	2302.18
635.8	3446.32	4498.7
1345.06	5119.74	5879.77

Example 4 *We consider three steps.*

First step. Let $J(x)=1+x(1+x)$, $k(x)=4$, $A(x)=x^2(1-x)^2$ *and* $E=1$.

Second step. In this example $T_{cr}\approx 67.4$ *and take* $T=30$, $a=0.2$ *and* $m=0.25$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 4 are shown in Table 4.*

Table 4: Numerical calculations of the first four eigenvalues from the Example 4

Clamped-clamped at the points $x=0$, $x=1$	Sliding at the point $x=\frac{1}{2}$, clamped at the point $x=1$	Hinged at the point $x=\frac{1}{2}$, clamped at the point $x=1$
(3)	(9)	(10)
41.97	58.95	98.86
83.57	303.243	880.6
263.85	1805.62	3071.78
784.56	4620.54	6310.1
1634.5	7558.9	8419.15

The violation of the regularity of factorization of eigenvalues in Example 4 is is due to the failure to satisfy the symmetry condition for the function $J(x)$. The aforementioned calculations in Example 4 confirm the validity of Statement 1 of Theorem 2.

All numerical examples were computed using a finite element formulation of the Euler–Bernoulli beam with variable coefficients and concentrated masses [29], [30]. The computations confirm the theoretical symmetry-based factorization of the spectrum.

The results obtained in this work make it possible to investigate the qualitative spectral properties of non-uniform beams with intermediate concentrated masses. The symmetry equivalence principle allows the natural frequencies of a full-length beam to be determined by solving auxiliary spectral problems posed on shorter beam segments with different boundary conditions and fixing types. The paper presents numerical examples illustrating a partial factorization of the eigenvalue spectrum of a full beam in the presence of asymmetric system parameters, including variable foundation coefficients and discrete inertial elements. Moreover, the lengths of the auxiliary beam segments depend on the adopted symmetry assumptions: in particular, when the coefficients and the distribution of intermediate masses are symmetric with respect to $x = 1/2$, each auxiliary problem is defined on an interval equal to half of the original beam length. These properties play an important role in numerical computations and in the modeling of mechanical systems with complex structural configurations. It is expected that future research will extend the proposed framework to more general mechanical systems, including graph-based structures, with star graphs serving as a representative example [31].

5 Conclusions

This work has established a symmetry-based framework for analysing the spectral properties of non-uniform Euler–Bernoulli beams carrying two concentrated masses placed symmetrically with respect to the midpoint. By assuming an agreed symmetry of the bending stiffness, foundation modulus, and distributed mass, we proved that the full eigenvalue problem on $(0, 1)$ decomposes into two independent auxiliary problems on the half-interval. These reduced problems correspond to symmetric and antisymmetric vibration modes and employ sliding-hinged and hinged-hinged (or sliding-clamped and hinged-clamped) conditions at the midpoint. The resulting factorization theorem provides explicit conditions under which the eigenvalues and eigenfunctions of the original system can be recovered from the half-length formulations.

The proposed approach significantly simplifies the spectral analysis of beams with intermediate masses, reduces computational complexity, and extends previously known symmetry equivalence results to systems that include discrete inertial elements. Numerical experiments confirm the theoretical predictions and demonstrate that violation of coefficient symmetry destroys the factorization property.

6 Acknowledgments

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP19579114).

References

[1] Abrate, S. 1995. "Vibration of Non-Uniform Rods and Beams." *Journal of Sound and Vibration* 185 (4): 703–716.

[2] Firouz-Abadi, R. D., H. Haddadpour, and A. B. Novinzadeh. 2007. "An Asymptotic Solution to Transverse Free Vibrations of Variable-Section Beams." *Journal of Sound and Vibration* 304: 530–540.

[3] Huang, Y., J. Chen, and Q.-Z. Luo. 2013. "A Simple Approach for Determining the Eigenvalues of the Fourth-Order Sturm-Liouville Problem with Variable Coefficients." *Applied Mathematics Letters* 26: 729–734.

[4] Saurin, V. 2019. "Analysis of Dynamic Behavior of Beams with Variable Cross-Section." *Lobachevskii Journal of Mathematics* 40 (3): 364–374.

[5] Wang, T., Y. Tang, and Q. Ding. 2023. "Gaussian Expansion Element Method of the New Dynamic Modeling Technique in Non-Uniform and Variable Cross-Section Structures." *Applied Mathematical Modelling* 116: 122–146.

[6] Lee, S. Y., and H. Y. Ke. 1990. "Free Vibrations of a Non-Uniform Beam with General Elastically Restrained Boundary Conditions." *Journal of Sound and Vibration* 136: 425–437.

[7] Datta, A. K., and S. N. Sil. 1996. "An Analysis of Free Undamped Vibration of Beams of Varying Cross-Section." *Computers and Structures* 59: 479–483.

[8] Caruntu, D. I. 2007. "Classical Jacobi Polynomials, Closed-Form Solutions for Transverse Vibrations." *Journal of Sound and Vibration* 306: 467–494.

[9] Wang, C. Y., and C. M. Wang. 2019. *Structural Vibration: Exact Solutions for Strings, Membranes, Beams, and Plates*. Boca Raton: CRC Press.

[10] Garijo, D. 2015. "Free Vibration Analysis of Non-Uniform Euler-Bernoulli Beams by Means of Bernstein Pseudospectral Collocation." *Engineering Computations* 31: 813–823.

[11] Youssri, Y. H., W. M. Abd-Elhameed, A. A. Elmasry, and A. G. Atta. 2025. "An Efficient Petrov–Galerkin Scheme for the Euler–Bernoulli Beam Equation via Second-Kind Chebyshev Polynomials." *Fractal and Fractional* 9 (2): 1–20.

[12] Deng, T., L. Zhao, and F. Jin. 2024. "Flexural Wave Rainbow Trapping Effect in the Periodic Non-Uniform Euler–Bernoulli Beams and Its Application in Energy Harvesting." *Mechanics of Materials* 190: 104892.

[13] Cayan, S., B. B. Ozhan, and M. Sezer. 2025. "A Non-Classical Algorithm for Vibration Analysis of Non-Uniform Beams." *International Journal for Numerical Methods in Engineering* 126 (22): 1–20.

[14] Auciello, N. M. 1996. "Transverse Vibrations of a Linearly Tapered Cantilever Beam with Tip Mass of Rotatory Inertia and Eccentricity." *Journal of Sound and Vibration* 194: 25–34.

[15] Cao, D., Y. Gao, J. Wang, M. Yao, and W. Zhang. 2019. "Analytical Analysis of Free Vibration of Non-Uniform and Non-Homogenous Beams: Asymptotic Perturbation Approach." *Applied Mathematical Modelling* 65: 526–534.

[16] Ghannadiasl, A., A. Zamiri, and A. Borhanifar. 2020. "Free Vibrations of Non-Uniform Beams on a Non-Uniform Winkler Foundation Using the Laguerre Collocation Method." *Journal of the Brazilian Society of Mechanical Sciences and Engineering* 42 (242): 1–12.

[17] Valle, J., D. Fernandez, and J. Madrenas. 2019. "Closed-Form Equation for Natural Frequencies of Beams under Full Range of Axial Loads Modeled with a Spring-Mass System." *International Journal of Mechanical Sciences* 153–154: 380–390.

[18] Nurakhmetov, D., S. Jumabayev, A. Aniyarov, and R. Kussainov. 2020. "Symmetric Properties of Eigenvalues and Eigenfunctions of Uniform Beams." *Symmetry* 12 (2097): 1–13.

[19] Nurakhmetov, D., S. Jumabayev, and A. Aniyarov. 2024. "Symmetric Properties of Eigenvalues and Eigenfunctions of Uniform Beams with Axial Loads." *Trends in Mathematics* 4: 163–164.

[20] Jumabayev, S. A., D. B. Nurakhmatov, and A. A. Aniyarov. 2025. "Symmetry Equivalences of Boundary Value Problems for the Non-Uniform Beams." *Journal of Mathematics, Mechanics and Computer Science* 126 (2): 69–79.

[21] Maiz, S., D. V. Bambill, C. A. Rossit, and P. A. A. Laura. 2007. "Transverse Vibration of Bernoulli–Euler Beams Carrying Point Masses and Taking into Account Their Rotatory Inertia: Exact Solution." *Journal of Sound and Vibration* 303: 895–908.

[22] Low, K. H. 2001. "On the Methods to Derive Frequency Equations of Beams Carrying Multiple Masses." *International Journal of Mechanical Sciences* 43: 871–881.

[23] Branski, A. 2012. "Modes Orthogonality of the Mechanical System Simple Supported Beam Actuators Concentrated Masses." *Acoustic and Biomedical Engineering* 121: 126–131.

[24] Ozkaya, E., S. M. Bagdatli, and H. R. Oz. 2008. "Nonlinear Transverse Vibrations and 3:1 Internal Resonances of a Beam with Multiple Supports." *Journal of Vibration and Acoustics* 130: 1–11.

[25] Akhmediyev, S. K., O. Khabidolda, N. I. Vatin, R. Muratkhan, and N. K. Medeubaev. 2024. "Studying Forced Oscillations of a Single-Span Cantilever Beam Using the Analytical Method." *Journal of Mathematics, Mechanics and Computer Science* 124 (4): 38–47.

[26] Aniyarov, A. A., S. A. Jumabayev, D. B. Nurakhmetov, and R. K. Kussainov. 2021. "A Hybrid Algorithm for Solving Inverse Boundary Problems with Respect to Intermediate Masses on a Beam." *Bulletin of the Karaganda University. Mathematics Series* 4 (104): 4–13.

[27] Bokaian, A. 1988. "Natural Frequencies of Beams under Compressive Axial Loads." *Journal of Sound and Vibration* 126 (1): 49–65.

[28] Bokaian, A. 1990. "Natural Frequencies of Beams under Tensile Axial Loads." *Journal of Sound and Vibration* 142 (3): 481–498.

[29] Reddy, J. N. 2006. *An Introduction to the Finite Element Method.* New York: McGraw-Hill.

[30] Kythe, P., and D. Wei. 2003. *An Introduction to Linear and Nonlinear Finite Element Analysis: A Computational Approach.* Boston: Birkhдuser.

[31] Kanguzhin, B., G. H. Aimal Rasa, and Z. Kaiyrbek. 2021. "Identification of the Domain of the Sturm-Liouville Operator on a Star Graph."*Symmetry* 13 (1210): 1–15.

Автор туралы мәлімет:

Анияров Альмир Аскарович – физика-математика ғылымдарының кандадаты, Математика және математикалық моделдеу институтының аға ғылыми қызметкері (Алматы, Қазақстан, электрондық пошта: aniyarov@math.kz), Астана халықаралық университетінің қауымдастырылған профессоры (Астана, Қазақстан).

Информация об авторе:

Анияров Альмир Аскарович – кандидат физико-математических наук, главный научный сотрудник Института математики и математического моделирования (Алматы, Казахстан, электронная почта: aniyarov@math.kz), ассоциированный профессор Международного университета Астана (Астана, Казахстан).

Information about author:

Aniyarov Almir Askarovich (corresponding author) – Candidate of Physical and Mathematical Sciences, Chief researcher at the Institute of Mathematics and Mathematical Modeling (Almaty, Kazakhstan, email: aniyarov@math.kz), Associate professor at the Astana International University, (Astana, Kazakhstan).

Received: December 5, 2025
Accepted: December 16, 2025