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ON THE STABILITY OF THE “INFECTION-FREE” ALMOST PERIODIC SOLUTION IN AN IMPULSIVE SIR MODEL

The paper investigates the stability of an *infection-free* almost periodic solution of an impulsive SIR model describing the spread of infection in biological systems or vulnerability to malicious software in computer networks. The model accounts for impulsive perturbations interpreted as vaccination or antivirus updates occurring at nonperiodic moments in time. The existence and asymptotic stability of an almost periodic solution are proved under certain conditions imposed on the sequence of impulses and the time intervals between them. The analysis is based on the theory of impulsive differential equations and almost periodic functions, employing Lyapunov stability criteria for the corresponding linearized system. The obtained results generalize known periodic cases and provide a theoretical foundation for constructing robust control strategies in nonlinear impulsive systems arising in epidemiological and cybernetic models.

Key words: impulsive systems, almost periodic solutions, stability, nonlinear dynamics, SIR model, control, stable dynamics.

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Импульстік SIR моделіндегі ”инфекциясыз” периодты дерлік шешімнің тұрақтылығы туралы

Бұл мақалада биологиялық жүйелерде инфекцияның таралуын немесе компьютерлік желілердегі зиянды бағдарламалық жасақтамаға осалдығын сипаттайтын импульстік SIR моделінің *инфекциясыз* периодты дерлік шешімінің тұрақтылығы зерттеледі. Модель уақыттың периодты емес сәттерінде болатын вакцинация немесе антивирустардың жаңартулар ретінде қарастырылатын импульстік ауытқуларды ескереді. Импульстар тізбегіне және олардың арасындағы уақыт аралықтарына белгілі бір шарттар қойылғанда периодты дерлік шешімнің бар болуы және асимптотикалық тұрақтылығы дәлелденген. Талдау сәйкес сызықтандырылған жүйеге Ляпунов тұрақтылық критерийлерін қолдану арқылы импульстік дифференциалдық теңдеулер және периодты дерлік функциялар теорияларына негізделген. Алынған нәтижелер белгілі периодтық жағдайларды жалпылайды және эпидемиология мен кибернетика модельдерінде кездесетін бейсызық импульстік жүйелер үшін робастты басқару стратегияларын құруға теориялық негіз береді.

Түйін сөздер: импульстік жүйелер, периодты шешімдер, тұрақтылық, сызықты емес динамика, SIR моделі, басқару, тұрақты динамика.

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Об устойчивости почти периодического решения ”без инфекции” в импульсной SIR модели

В статье исследуется устойчивость почти периодического решения импульсной SIR модели *без инфекции*, описывающей распространение инфекции в биологических системах или уязвимость к вредоносному программному обеспечению в компьютерных сетях. Модель учитывает импульсные возмущения, интерпретируемые как вакцинация или обновления антивирусов, возникающие в непериодические моменты времени. Доказано существование и асимптотическая устойчивость почти периодического решения при определенных условиях, накладываемых на последовательность импульсов и временные интервалы между ними. Анализ основан на теории импульсных дифференциальных уравнений и почти периодических функций с использованием критериев устойчивости Ляпунова для соответствующей линеаризованной системы. Полученные результаты обобщают известные периодические случаи и дают теоретическую основу для построения робастных стратегий управления в нелинейных импульсных системах, возникающих в моделях эпидемиологии и кибернетики.

Ключевые слова: импульсные системы, почти периодические решения, устойчивость, нелинейная динамика, SIR модель, управление, устойчивая динамика.

1 Introduction

The study of the stability of nonlinear dynamical systems, particularly in the context of modeling biological, ecological, and technological processes, has attracted significant attention in recent years. Among the classical approaches to such analysis, compartmental epidemic models - most notably the SIR model (Susceptible - Infectious - Removed) - play an important role in describing the spread of infectious agents in biological or computer networks. In real-world settings, such systems are frequently subjected to abrupt changes, such as mass vaccination, periodic antivirus updates, or other instantaneous impacts, which are naturally modeled by impulsive differential equations.

In many existing studies, the duration of impulsive influences is neglected, or such influences are assumed to occur with a fixed periodicity. However, in practical scenarios this idealization may not adequately capture the nuanced behaviors introduced by irregular impulsive effects. Accounting for these features is essential for the effectiveness of the model. In practice, it is crucial to ensure *functional stability* [1,2] and *controllability* [3–6] of complex systems that undergo external or internal instantaneous perturbations.

Consequently, impulsive and almost periodic systems constitute an important research direction in modern control theory and stability theory, providing a means of describing processes with discrete interventions within the framework of continuous dynamics. In this study, we examine an impulsive SIR model with almost periodic impulses and investigate the existence and asymptotic stability of the *infection-free* almost periodic solution. This solution may be interpreted as a stable or functionally stable system state achieved under irregular but bounded impulsive control actions.

The results obtained generalize well-known models with periodic impulses to the case of almost periodic perturbations, which more accurately reflect real-world scenarios. The study contributes to the development of stability and control theory for nonlinear impulsive systems and may be used to design adaptive decision-making strategies in epidemiological and cybernetic applications.

2 Motivation and Problem Statement

It is known [7–10] that infection-spread scenarios in biological populations, as well as certain DDoS attack scenarios in computer networks, can be described by the following nonlinear system:

$$\begin{aligned} \dot{S} &= -\lambda S I + \mu - \mu S, \\ \dot{I} &= \quad \lambda S I - \gamma I - \mu I, \end{aligned} \tag{1}$$

which is considered in the region invariant with respect to (1)

$$D = \{(S, I) \mid S \geq 0,\ I \geq 0,\ S + I \leq 1\}.$$

Here, $I(t)$ denotes the infected population, $S(t)$ the susceptible (those who may become infected), and λ, γ, μ are given positive parameters. Of particular importance is the parameter

$$\sigma = \frac{\lambda}{\gamma + \mu}$$

which characterizes the basic reproduction number of the infection. In the phase space of system SOI, there exist two equilibrium points:
(i) $S_0^* = 1$, $I_0^* = 0$ (“infection-free”),
(ii) $S_1^* = \dfrac{1}{\sigma}$, $I_1^* = \dfrac{\mu(\sigma - 1)}{\lambda}$ (“endemic”).

Theorem 1 *[7] For $\sigma \leq 1$, the equilibrium (S_0^*, I_0^*) is globally (in D) asymptotically stable (in this case, $(S_0^*, I_0^*) \neq D$).*

For $\sigma > 1$, the equilibrium (S_0^*, I_0^*) is unstable, while the equilibrium (S_0^*, I_0^*) is globally (in $D \backslash \{I = 0\}$) asymptotically stable.

It turns out that when $\sigma > 1$, the qualitative behavior of solutions of (1) can be fundamentally altered by introducing impulsive perturbations [5].

From the modeling perspective, an *instantaneous* reduction of $S(t)$ at times

$$0 < t_1 < t_2 < \ldots$$

may be interpreted as vaccination in biological models [12], or as antivirus updates for network nodes susceptible to DDoS attacks in computer network models [13].

From the mathematical point of view, this leads to the following impulsive system, where perturbations occur at fixed time instants [11]

$$\begin{aligned} \dot{S} &= \ -\lambda S I + \mu - \mu S, && t \neq t_n, \\ \dot{I} &= \ \lambda S I - \gamma I - \mu I, && t \neq t_n, \\ \Delta S|_{t=t_n} &\equiv \ S(t_n + 0) - S(t_n) = -p_n S(t_n), && n \geq 1, \end{aligned} \tag{2}$$

where $0 < t_1 < t_2 < \ldots, t_n \to \infty$ are fixed impulsive moments, and $\{p_n\} \subset (0, 1)$ are fixed parameters characterizing the magnitude of the impulses.

In work [12], the periodic case was considered, i.e., when

$$t_n = n \cdot T, \quad p_n \equiv p, \quad n \geq 1. \tag{3}$$

Theorem 2 *[12] Under conditions* (3)*, system* (2) *admits a periodic impulsive ("infection-free") solution*

$$\bar{S}(t) = \begin{cases} 1 + \dfrac{p \cdot e^{\mu T}}{1 - p - e^{\mu T}} \cdot e^{-\mu(t-nT)}, & t \in [nT, (n+1)T), \\ \dfrac{(1-p)\left(e^{\mu T} - 1\right)}{p - 1 + e^{\mu T}}, & t = (n+1)T, \end{cases}$$

which is asymptotically stable under the condition

$$\frac{1}{T}\int_0^T \bar{S}(t)dt < \frac{1}{\sigma}. \tag{4}$$

Remark 1 *Condition* (4) *implies that there exists a constant* $T^* > 0$ *such that, for all* $T < T^*$*, the solutions of system* (2) *satisfy*

$$S(t) \to \bar{S}, \qquad I(t) \to 0, \qquad t \to \infty. \tag{5}$$

In the present paper, we extend this result to the case where the parameters of the impulsive perturbation are "almost periodic" in the sense of [11].

Theorem 3 *Assume that in system* (2) *the impulsive moments are given by*

$$t_n = nT + \alpha_n, \qquad |\alpha_n| < \frac{T}{2}, \qquad p_n > 0,$$

and that the sequences $\{\alpha_n\}$ *and* $\{p_n\}$ *are almost periodic. Then system* (2) *admits an almost periodic impulsive "infection-free" solution* $(\bar{S}(t), 0)$*, which is asymptotically stable for* $T < T^*$*. In other words, relation* (5) *holds.*

3 Main results

We begin by recalling several fundamental concepts from the theory of almost periodic functions, whose theoretical foundations are developed in [11].

Definition 1 *A sequence* $\{\tau_i\}_{i\in\mathbb{Z}}$ *is called* **almost periodic** *if*

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \quad \forall k \in \mathbb{Z} \quad \exists p \in [k, k+N] \cap \mathbb{Z} : \forall i \in \mathbb{Z} \quad |\tau_{i+p} - \tau_i| < \varepsilon. \tag{6}$$

Every *almost periodic* sequence is bounded.

If the sequence $\{\tau_i\}_{i\in\mathbb{Z}}$ is periodic, i.e., if there exists an integer p such that

$$\tau_{i+p} = \tau_i \quad \text{for all } i \in \mathbb{Z},$$

then condition (6) is satisfied automatically.

A useful criterion for verifying condition (6) is provided by the following theorem.

Theorem 4 *[11] A sequence* $\{\tau_i\}_{i\in\mathbb{Z}}$ *is almost periodic if and only if for every subsequence* $\{m_k\} \subset \mathbb{Z}$ *there exists a further subsequence* $\{m_{k_j}\}$ *such that* $\left\{\tau_{i+m_{k_j}}\right\}$ *converges uniformly in* $i \in \mathbb{Z}$ *as* $j \to \infty$*.*

3.1 Example

Consider the sequence

$$\tau_i = \sin(i) - a\sin(i\sqrt{2}), \qquad i \in \mathbb{Z},\ a > 0.$$

Let us note that this sequence is defined only for integer values of i. Since the numbers 1 and $\sqrt{2}$ are not rationally related (they correspond to incommensurable frequencies), the sequence is non-periodic. Its behavior can be interpreted as quasi-periodic or even "pseudo-chaotic", depending on the value of the parameter a.

An increase in a amplifies the influence of the second harmonic and alters both the amplitude and the overall shape of the oscillations. For $a = 0.5$, the corresponding behavior is illustrated in Fig. 1.

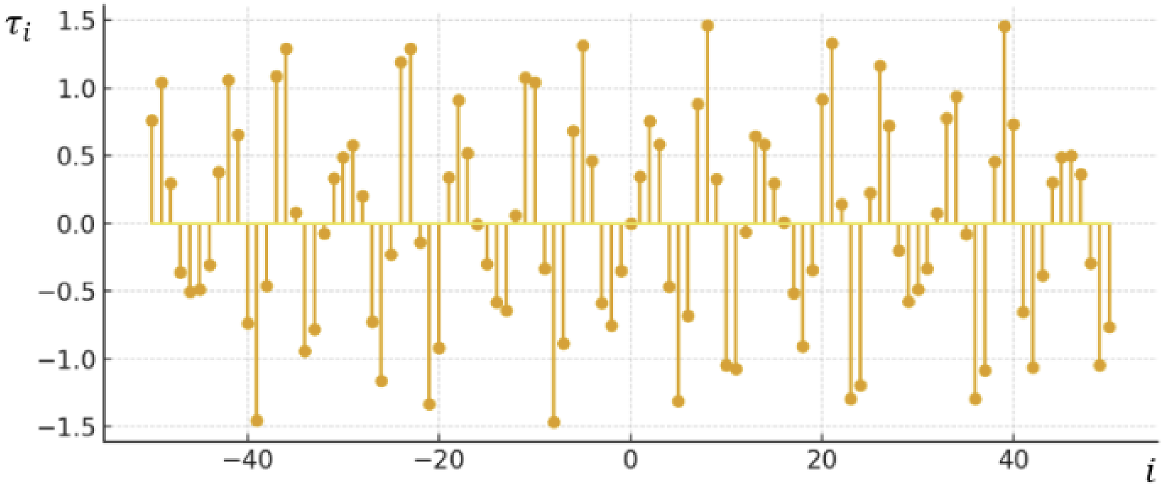

Figure 1: Sequence $\tau_i = \sin(i) - a\sin(i\sqrt{2})$, $i \in \mathbb{Z}$

Now consider a sequence $\{t_i\}_{i\in\mathbb{Z}}$ satisfying,

$$\cdots < t_i < t_{i+1} < \cdots, \quad t_i \to \pm\infty \quad \text{as} \quad i \to \pm\infty.$$

Let $N(t, t+T)$ denote the number of points $\{t_i\}$ belonging to the interval $(t, t+T]$.

Theorem 5 *[11] If the sequence $\{t_{i+1} - t_i\}_{i\in\mathbb{Z}}$ is almost periodic, then there exists a finite limit*

$$\lim_{T\to\infty} \frac{N(t, t+T)}{T} = \theta \tag{7}$$

uniformly in $t \in \mathbb{R}$. In particular, if there exist integers $\tilde{p}$ and a real number ω such that $t_{i+\tilde{p}} = t_i + \omega$, for all $i \in \mathbb{Z}$ (the periodic case), then

$$\theta = \frac{\tilde{p}}{\omega}.$$

To introduce the notion of an almost periodic impulsive solution, we require the following strengthening of Definition 1.

Definition 2 *For a given sequence* $\{\tau_i\}_{i\in\mathbb{Z}}$ *define*

$$\tau_i^j = \tau_{i+j} - \tau_i.$$

We say that the sequence $\{\tau_i\}_{i\in\mathbb{Z}}$ *is* ***uniformly almost periodic*** *if the sequences* $\{\tau_i^j\}$ *satisfy Definition 1 uniformly in* j*, i.e., if*

$$\forall\,\varepsilon>0\ \ \exists\,N=N(\varepsilon)\ \ \forall\,k\in\mathbb{Z}\ \ \exists\,p\in[k,\,k+N]\cap\mathbb{Z}\quad \text{such that}\quad \forall\,i,j\in\mathbb{Z},\ \ |\tau_{i+p}^j-\tau_i^j|<\varepsilon.$$

Lemma 1 *[11] Let the sequence* $\{t_i\}_{i\in\mathbb{Z}}$ *be defined by*

$$t_i = iT + \alpha_i,$$

where the subsequence $\{\alpha_i\}_{i\in\mathbb{Z}}$ *is almost periodic and*

$$\sup_i |\alpha_i| = \alpha < \frac{T}{2}.$$

Then the sequence $\{t_{i+1}-t_i\}_{i\in\mathbb{Z}}$ *is uniformly almost periodic.*

Definition 3 *A piecewise continuous function* $\varphi:\mathbb{R}\to\mathbb{R}$ *with jump discontinuities of the first kind at the points* $\{t_i\}_{i\in\mathbb{Z}}$ *is called* ***almost periodic*** *if the sequence* $\{t_{i+1}-t_i\}_{i\in\mathbb{Z}}$ *is uniformly almost periodic and the following properties hold:*

1. *For every* $\varepsilon>0$ *there exists* $\delta=\delta(\varepsilon)>0$ *such that for all* $i\in\mathbb{Z}$ *and all* $t',t''\in(t_i,t_{i+1})$*,*

   $$|t'-t''|<\delta \quad\Rightarrow\quad |\varphi(t')-\varphi(t'')|<\varepsilon.$$

2. *For every* $\varepsilon>0$ *there exists a set* Γ *of* ε*-almost periods such that there exists* $l>0$ *satisfying:*

   $$[a,a+l]\cap\Gamma\neq\varnothing \quad \text{for all } a\in\mathbb{R},$$

   and for every $\tau\in\Gamma$ *and every* $t\in\mathbb{R}$*,*

   $$|t-t_i|>\varepsilon \quad\Rightarrow\quad |\varphi(t+\tau)-\varphi(t)|<\varepsilon.$$

Remark 2 *Any periodic piecewise continuous function with jump discontinuities of the first kind at the points* $\{t_i\}$*, where*

$$t_{i+p} = t_i + \omega \quad \text{for some integer } p,$$

automatically satisfies Definition 3.

3.2 Example [11]

Consider the piecewise constant function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \mu_i, \qquad t \in [t_i, t_{i+1}),$$

where the sequence $\{\mu_i\}_{i\in\mathbb{Z}}$ is almost periodic and the sequence

$$\{t_{i+1} - t_i\}_{i\in\mathbb{Z}}$$

is uniformly almost periodic. Then φ satisfies Definition 3.

In what follows, we will say that a sequence $\{a_n\}_{n=0}^{\infty}$ is almost periodic if the corresponding two-sided sequence

$$\tilde{a}_i = \begin{cases} a_i, & i \geq 0, \\ a_{-i}, & i \leq 0, \end{cases}$$

is almost periodic as a sequence $\{\tilde{a}_i\}_{i\in\mathbb{Z}}$.

The main result of this work is the following.

Theorem 6 *In system* (2)*, assume that the impulsive moments are given by*

$$t_n = nT + \alpha_n,$$

where the sequence $\{\alpha_n\}$ *is almost periodic and satisfies*

$$\sup_n |\alpha_n| = \alpha < \frac{T}{2}.$$

Assume further that the sequence $\{p_n\} \subset (0, 1)$ *is almost periodic and*

$$\inf_n p_n = p > 0.$$

Then system (2) *admits an almost periodic (in the sense of Definition 3) "infection-free" solution* $(\bar{S}(t), 0)$*.*

Moreover, there exists a constant $T^* > 0$*, depending only on the parameters* $\lambda, \mu, \gamma, \alpha$*, and* p*, such that for all* $T < T^*$ *this solution is asymptotically stable. In particular, there exists* $\delta > 0$ *such that for all initial values* $(S(0), I(0)) \in \operatorname{Int} D$*,*

$$|S(0) - \bar{S}(0)| + I(0) < \delta \quad \Rightarrow \quad \begin{cases} |S(t) - \bar{S}(t)| \to 0, \\ I(t) \to 0, \end{cases} \qquad t \to \infty.$$

Proof. Let us consider the impulsive problem, which is obtained from (2) when $I = 0$:

$$\left\{ \begin{array}{rl} \dot{S} = & -\mu S + \mu, \quad t \neq t_n, \\ \Delta S|_{t=t_n} = & -p_n S\left(t_n\right). \end{array} \right. \tag{8}$$

Because

$$-\mu + \theta \cdot \ln\left(1 - p_n\right) < -\mu < 0,$$

then by Theorem 24.1 [11] there exists a unique almost-periodic (in the sense of Definition 3) solution (8) $S = \bar{S}(t)$ which is given by the

$$\bar{S}(t) = \mu \cdot \int_{-\infty}^{t} X(t,s)ds, \tag{9}$$

where at $t_k \le t < t_{k+1}$, $t_{m-1} \le s < t_m$, $s < t$

$$X(t,s) = e^{-\mu(t-t_k)} \prod_{s \le t_i < t} (1-p_i)\, e^{-\mu(t_i - t_{i-1})} \cdot (1-p_m)\, e^{-\mu(t_m - s)}.$$

From (9) it follows that $\forall t \in \mathbb{R}$

$$0 < \bar{S}(t) = \mu(1-p) \int_{-\infty}^{t} e^{-\mu(t-s)} ds = 1 - p. \tag{10}$$

Linearizing system (2) in the neighborhood $(\bar{S}, 0)$ we obtain the system

$$\begin{array}{rll} \dot{S} = - & \lambda\bar{S}(t)I(t) + \lambda S(t)I(t) - \mu S(t), & t \ne t_n, \\ \dot{I} = & \lambda\bar{S}(t)I(t) + \lambda S(t)I(t) - \gamma I(t) - \mu I(t), & t \ne t_n, \\ \Delta S|_{t=t_n} = - & p_n S\left(t_n\right) & \end{array} \tag{11}$$

The stability of the zero solution (11) is equivalent to the stability of the "infection free" solution $(\bar{S}, 0)$ of the original system (2) .

By introducing the vector $x(t) = \begin{pmatrix} S(t) \\ I(t) \end{pmatrix}$, system (11) can be rewritten as

$$\left\{ \begin{array}{rl} \dot{x} = & A(t)x + g(t,x), \quad t \ne t_n, \\ \Delta x|_{t=t_n} = & B_n x, \end{array} \right. \tag{12}$$

where

$$A(t) = \left(\begin{array}{cc} -\mu & -\lambda\bar{S}(t) \\ 0 & \lambda\bar{S}(t) - \gamma - \mu \end{array} \right), \quad g(t,x) = \left(\begin{array}{r} -\lambda S(t)I(t) \\ \lambda S(t)I(t) \end{array} \right), \quad B_n = \left(\begin{array}{rr} -p_n & 0 \\ 0 & 0 \end{array} \right).$$

Note that

$$\|g(t,x)\| \le \frac{\lambda}{\sqrt{2}} \cdot (|S(t)| + |I(t)|) \cdot \|x\|.$$

Therefore, for every $a > 0$ there exists $\delta = a\sqrt{2}/\lambda$ such that

$$\|x\| < \delta \quad \Rightarrow \quad \|g(t,x)\| \le a \cdot \|x\|. \tag{13}$$

This inequality allows us to use the following theorem.

Theorem 7 *[11] Consider the impulsive system*

$$\left\{ \begin{array}{rl} \dot{x} = & A(t)x + g(t,x), \quad t \ne t_n, \\ \Delta x|_{t=t_n} = & B_n x, \end{array} \right. \tag{14}$$

where $g(t,0)=0$, *and* $\|g(t,x)\| \le a\|x\|$ *for* $\|x\|<h$, *uniformly in* $t>0$. *Assume that*

$$\lim_{T\to\infty}\frac{N(t,t+T)}{T}=\theta \tag{15}$$

exists, and that the largest eigenvalue of the matrix

$$\frac{1}{2}\big(A(t)+A^{+}(t)\big) \qquad \big(\textit{here}\quad A^{+}(t)=A^{\mathrm{T}}(t)\big)$$

satisfies the inequality

$$\sup_{t\ge 0}\Lambda(t)\le\gamma.$$

Assume also that the largest eigenvalue of the matrices

$$(E+B_n^T)(E+B_n)$$

satisfies $\sup_{n\ge 1}\Lambda_n=\alpha$. *Then, if the condition* $\gamma+\theta\ln\alpha<0$ *holds, the zero solution of system* (14) *is asymptotically stable for all sufficiently small* $a>0$.

Let us verify that the conditions of this theorem (Theorem 7) are satisfied for (12).

First, the smallness of $a>0$ is ensured, according to (13), by choosing a sufficiently small δ-neighborhood, which will serve as the domain of attraction for $(\bar S,0)$.

The existence of the limit in (15) follows directly from (7).

We have that

$$E+B_n=\begin{pmatrix}1-p_n & 0\\ 0 & 1\end{pmatrix}.$$

Hence $\alpha=1$, and the condition will be satisfied if we assume that

$$\sup_{t\ge 0}\Lambda(t)<0. \tag{16}$$

We obtain:

$$\frac{1}{2}\big(A(t)+A^{+}(t)\big)=\begin{pmatrix}-\mu & -\frac{\lambda}{2}\bar S(t)\\ -\frac{\lambda}{2}\bar S(t) & \lambda\bar S(t)-\gamma-\mu\end{pmatrix}$$

A sufficient condition for the validity of (16) is

$$4\mu(\lambda\bar S(t)-\gamma-\mu)<-\lambda^2\bar S^2(t),$$

$$\lambda^2\bar S^2(t)+4\mu\lambda\bar S(t)-4\mu(\gamma+\mu)<0,$$

that is,

$$\bar S(t)\le\frac{-2\mu\lambda+2\lambda\sqrt{2\mu^2+\mu\gamma}}{\lambda^2}\quad\Longleftrightarrow\quad\bar S(t)\le\frac{2\left(\sqrt{2\mu^2+\mu\gamma}-\mu\right)}{\lambda}.$$

From this and from (9), the statement of the theorem 6 follows. ■

Taking (10) into account, we conclude that under condition

$$1-p<\frac{2(\sqrt{2}-1)\mu}{\lambda}$$

the “infection-free” almost periodic impulsive solution of system (2) is asymptotically stable.

4 Conclusions

This work continues and complements recent research on impulsive dynamical systems, where qualitative properties, attracting sets, and solvability of impulsive problems have been investigated in different functional settings [15–17]. The present paper investigates an impulsive SIR model with almost periodic sequences of impulses. It is shown that, under certain conditions on the model parameters and the characteristics of the impulses, the system possesses an almost periodic "infection-free" solution which is asymptotically stable. The derived stability condition generalizes the classical periodic case and ensures the stability of the infection-free state even under irregular external perturbations.

The proposed approach combines the theory of almost periodic functions with methods for the analysis of impulsive differential equations, offering a unified methodology for studying nonlinear systems with discrete perturbations. The results contribute to the development of stability theory and can be applied to the control of impulsive processes in epidemiology, information security [14], and related fields. Further research may focus on numerical modeling and the application of adaptive or artificial intelligence-based control strategies to enhance the robustness of such systems.

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