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SOLUTION OF AN INVERSE PROBLEM FOR THE HEAT EQUATION WITH A DISCONTINUOUS COEFFICIENT AND A DIRICHLET BOUNDARY CONDITION

The problems where the coefficients or the right-hand side of a differential equation are determined simultaneously with its solution are called inverse problems of mathematical physics. Such problems frequently arise in a wide variety of fields, which makes them one of the most pressing issues in modern mathematics. This paper considers a class of problems modeling the process of determining the temperature and density of heat sources with given initial and final temperatures. Their mathematical formulation includes inverse problems for the heat equation, where it is necessary not only to solve the equation, but also to find an unknown right-hand side depending only on the spatial variable. In such inverse problems for the heat equation with a discontinuous coefficient, the existence and uniqueness of a classical and generalized solution can be established. The problem considered in this paper can arise in describing the diffusion of particles in a turbulent plasma as well as in modeling temperature field of heat propagation in a thin rod of finite length consisting of two sections with different thermophysical properties. In such problems, at the interface between two media with different thermophysical properties, it is necessary to specify not only boundary conditions but also conjugation conditions.

Key words: heat equation, Fourier method, spectral problem, orthonormal basis, classical solution, generalized solution.

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Коэффициенті үзілісті жылуөткізгіштік теңдеуі үшін Дирихле шеттік шартымен қойылған кері есепті шешу

Дифференциалдық теңдеудің коэффициенттерін немесе оң жақ бөлігін оның шешімімен бір мезгілде анықтауға бағытталған есептер математикалық физиканың кері есептері деп аталады. Мұндай есептер адам қызметінің сан алуан салаларында жиі кездеседі, сондықтан олар қазіргі математикадағы өзекті мәселелер қатарына жатады. Бұл жұмыста берілген бастапқы және соңғы мәндері бойынша температураны және жылу көздерінің тығыздығын анықтау үдерісін модельдейтін есептердің бір класы қарастырылады. Оларды математикалық тұрғыдан тұжырымдау барысында жылуөткізгіштік теңдеуі үшін кері есептер туындайды: теңдеудің шешімімен қатар кеңістіктік айнымалыға ғана тәуелді белгісіз оң жақ бөлікті табу талап етіледі. Сонымен бірге, үзілісті коэффициенті бар жылуөткізгіштік теңдеуі үшін қойылған мұндай кері есептерде есептің классикалық және жалпыланған шешімнің бар болуы мен бірмәнділігін (жалғыздығын) анықтауға болады. Біз қарастыратын есеп турбуленттi плазмада бөлшектердің диффузиясы үдерісін сипаттайтын есептерді шешуде, сондай-ақ жылу-физикалық сипаттамалары әртүрлі екі бөліктен тұратын, шекті ұзындықтағы жіңішке өзекшеде температуралық өріс арқылы жылудың таралу үдерісін модельдеуде туындауы мүмкін. Шеттік шарттармен қатар, жылу-физикалық қасиеттері әртүрлі екі ортаның жанасу шекарасында түйісу (сәйкестендіру) шарттары да беріледі.

Түйін сөздер: жылуөткізгіштік теңдеуі, Фурье әдісі, спектрлік есеп, ортонормалдық базис, классикалық шешім, жалпыланған шешім.

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Решение одной обратной задачи для уравнения теплопроводности с разрывным коэффициентом и с граничным условием Дирихле

Задачи определения коэффициентов или правой части дифференциального уравнения одновременно с его решением носят название обратных задач математической физики. Такие задачи достаточно часто возникают в самых различных областях человеческой деятельности, что ставит их в ряд актуальных проблем современной математики. В работе рассматривается один класс задач, моделирующих процесс определения температуры и плотности источников тепла по заданным начальной и конечной температурам. При их математической формулировке возникают обратные задачи для уравнения теплопроводности, в которых вместе с решением уравнения требуется найти неизвестную правую часть, зависящую только от пространственной переменной. Также, при таких обратных задачах для уравнения теплопроводности с разрывным коэффициентом можно установить существование и единственность классического и обобщенного решения задачи. Рассматриваемая нами задача может возникнуть при решении задач описывающих процесс диффузии частиц в турбулентной плазме, а также при моделировании процесса распространения тепла температурного поля в тонком стержне конечной длины, состоящем их двух участков с различными теплофизическими характеристиками. Дополнительно к краевым условиям задаются условия сопряжения на границе контакта двух сред с различными теплофизическими характеристиками.

Ключевые слова: уравнение теплопроводности, метод Фурье, спектральная задача, ортонормированный базис, классическое решение, обобщенное решение.

1 Introduction

The solvability of various inverse problems for parabolic equations has been considered in a large number of papers [1]–[10]. In [11]–[12], the formulation and the subject of the problem are similar to those used in the present work. However, unlike the previous papers, we study the inverse problem for the heat equation with a discontinuous coefficient and a Dirichlet boundary condition. In this work, the unknown quantity is the function $f(x)$ (source/parameter in the right-hand side), which must be reconstructed using an additional condition from the formulation of the inverse problem. We present an exact formulation of the inverse problem in the specified functional spaces and discuss the existence and uniqueness of the classical and generalized solutions. In particular, inverse problems for differential equations with multiple characteristics have been studied in several works, where questions of existence and solvability in Sobolev spaces were investigated[13].

The discontinuities in the coefficients significantly affect the spectral properties of the corresponding differential operators. Taking into account the possible loss of smoothness at the discontinuity point x_0, the solution with respect to the spatial variable is considered piecewise smooth: separately on $(0, x_0)$ and (x_0, l). A particular challenge in such problems is the basis property of the system of eigenfunctions [14]. It is known that for non-self-adjoint operators arising in the problems with discontinuous coefficients, the system of eigenfunctions may not form a basis in the classical sense, but form a Riesz basis. The authors use the method of separation of variables and the spectral method [15] to construct solutions and justify their smoothness.

2 Statement of the problem and its solution

In the domain $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 = \{(x,t) : 0 < x < x_0,\ 0 < t < T\}$, $\Omega_2 = \{(x,t) : x_0 < x < l, 0 < t < T\}$, $(i = 1, 2)$, we consider the problem of finding the right-hand side $f(x)$ of the heat conductivity equation with the discontinuous coefficient

$$Lu = \begin{cases} u_t - k_1^2 u_{xx}, & 0 < x < x_0, \\ u_t - k_2^2 u_{xx}, & x_0 < x < l, \end{cases} = f(x), \tag{1}$$

and its solution $u(x,t)$ satisfying the initial and final conditions

$$u(x,0) = \varphi(x), \quad u(x,T) = \psi(x), \quad 0 \le x \le l, \tag{2}$$

boundary conditions

$$u(0,t) = u(l,t) = 0, \quad 0 \le t \le T, \tag{3}$$

and conjugation conditions

$$\begin{cases} u(x_0 - 0, t) = u(x_0 + 0, t), \\ k_1 u_x(x_0 - 0, t) = k_2 u_x(x_0 + 0, t), \end{cases} \tag{4}$$

where the coefficients $k_1, k_2 > 0$.
By W we denote a linear manifold of functions from the class $u\,(x,t) \in C\left(\overline{\Omega}\right) \cap C^{2,1}\left(\overline{\Omega_1}\right) \cap C^{2,1}\left(\overline{\Omega_2}\right)$ satisfying conditions (2)-(4).
Functions $u\,(x,t)$ from the class $u\,(x,t) \in W$ and f (x) ∈C$[0, l]$ will be called a *classical solution* of problem (1)-(4) if the function $u\,(x,t)$: 1) is continuous in the domain $\bar{\Omega}$; 2) has continuous first-order derivatives with respect to t and continuous second-order derivatives with respect to x in the domain Ω; 3) satisfies equation (1) and all conditions (2)-(4) in the usual, continuous sense and the function $f(x)$ is continuous in the interval $[0, l]$.
The application of the Fourier method (in the case $f(x) = 0$) to solving problems (1)-(4) leads to the following spectral problem:

$$L_0 X(x) = \begin{cases} -k_1^2 X''(x), & 0 < x < x_0, \\ -k_2^2 X''(x), & x_0 < x < l, \end{cases} = \lambda X(x), \tag{5}$$

$$X(0) = X(l) = 0, \quad 0 \le t \le T, \tag{6}$$

$$\begin{cases} X(x_0 - 0) = X(x_0 + 0), \\ k_1 X'(x_0 - 0) = k_2 X'(x_0 + 0), \end{cases} \tag{7}$$

The function $T(t)$ is a solution to the equation

$$T'(t) + \lambda T(t) = 0. \tag{8}$$

In [16] the eigenvalues and eigenfunctions of the spectral problem (5)-(7) were found:

$$\lambda_n = (\pi n r)^2, n = 1, 2, ..., \tag{9}$$

where

$$r = \frac{k_1 k_2}{k_1(l - x_0) + k_2 x_0}. \tag{10}$$

$$X_n(x) = \sqrt{2r} \begin{cases} \sin\left(\frac{\pi n r x}{k_1}\right), & 0 < x < x_0, \\ (-1)^{n+1} \sin\left(\frac{\pi n r (l-x)}{k_2}\right), & x_0 < x < l, \end{cases} \tag{11}$$

The authors of [16] also proved the following propositions.
Proposition 1. Spectral problem (5)-(7) is non-self-adjoint.
The adjoint problem to problem (5)-(7) has the following form:

$$L_0 Y(x) = \left\{ \begin{array}{ll} -k_1^2 Y''(x), & 0 < x < x_0 \\ -k_2^2 Y''(x), & x_0 < x < l \end{array} \right\} = \lambda Y(x), \tag{12}$$

$$Y(0) = Y(l) = 0, \quad 0 \le t \le T, \tag{13}$$

$$k_1 Y(x_0 - 0) = k_2 Y(x_0 + 0), \quad k_1^2 Y'(x_0 - 0) = k_2^2 Y'(x_0 + 0), \tag{14}$$

The eigenfunctions of the spectral problem (12)-(14) have the form:

$$Y_n(x) = \sqrt{2r} \begin{cases} \frac{1}{k_1} \sin\left(\frac{\pi n r x}{k_1}\right), & 0 < x < x_0, \\ \frac{(-1)^{n+1}}{k_2} \sin\left(\frac{\pi n r (l-x)}{k_2}\right), & x_0 < x < l, \end{cases} \tag{15}$$

$\lambda_n,\ r$ are determined by formulas (9)-(10).
Proposition 2. The system of eigenfunctions $\{X_n(x)\}$ and $\{Y_n(x)\}$ is biorthogonal, i.e.

$$\int_0^l X_n(x) Y_m(x) dx = \begin{cases} 1, & if \quad n = m, \\ 0, & if \quad n \neq m, \end{cases}$$

Proposition 3. The system of eigenfunctions $\{X_n(x)\}$ forms a Riesz basis $L_2(0, l)$.
We derive the solution to the inhomogeneous problem (1)-(4) (in the case $f(x) \neq 0$) in the form:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x), f(x) = \sum_{n=1}^{\infty} f_n X_n(x).$$

Substituting $u(x,t)$,$f(x)$ into (1) we get

$$\sum_{n=1}^{\infty} u_n'(t) X_n(x) = \sum_{n=1}^{\infty} k_i^2 u_n(t) X_n''(x) + \sum_{n=1}^{\infty} f_n X_n(x) \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} u_n'(t) X_n(x) = -\sum_{n=1}^{\infty} \lambda_n u_n(t) X_n(x) + \sum_{n=1}^{\infty} f_n X_n(x).$$

Now we obtain an ordinary differential equation of the first order

$$u_n'(t) + \lambda_n u_n(t) = f_n, \tag{16}$$

where λ_n is determined by formula (9). Condition (2) will take the form: $u_n(0) = \varphi_n, \quad u_n(T) = \psi_n$, where

$$\varphi_n = \int_0^l \varphi(x) Y_n(x) dx, \quad \psi_n = \int_0^l \psi(x) Y_n(x) dx \tag{17}$$

The solution of equation (16) satisfying the condition $u_n(0) = \varphi_n$ has the form:

$$u_n(t) = \varphi_n e^{-\lambda_n t} + \int_0^t f_n e^{-\lambda_n (t-\tau)} d\tau = \varphi_n e^{-\lambda_n t} + \frac{f_n}{\lambda_n}\left(1 - e^{-\lambda_n t}\right). \tag{18}$$

Substituting the last expression into the condition $u_n(T) = \psi_n$ we get $u_n(T) = \varphi_n e^{-\lambda_n T} + \frac{f_n}{\lambda_n}\left(1 - e^{-\lambda_n T}\right) = \psi_n$. From where we obtain

$$f_n = \frac{\lambda_n \left(\psi_n - \varphi_n e^{-\lambda_n T}\right)}{1 - e^{-\lambda_n T}}. \tag{19}$$

After substituting f_n into formula (18) we get

$$u_n(t) = \varphi_n e^{-\lambda_n t} + \frac{\left(\psi_n - \varphi_n e^{-\lambda_n T}\right)\left(1 - e^{-\lambda_n t}\right)}{1 - e^{-\lambda_n T}}. \tag{20}$$

Further, it is not difficult to find $u(x,t)$ and $f(x)$. Taking into account formula (17), we have

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^l \psi(\xi) Y_n(\xi) d\xi\right) \frac{X_n(x)}{1 - e^{-\lambda_n T}} - \sum_{n=1}^{\infty} \left(\int_0^l \psi(\xi) Y_n(\xi) d\xi\right) \frac{X_n(x) e^{-\lambda_n t}}{1 - e^{-\lambda_n T}} + \tag{21}$$

$$+ \sum_{n=1}^{\infty} \left(\int_0^l \varphi(\xi) Y_n(\xi) d\xi\right) \frac{X_n(x) e^{-\lambda_n T}}{1 - e^{-\lambda_n T}} - \sum_{n=1}^{\infty} \left(\int_0^l \varphi(\xi) Y_n(\xi) d\xi\right) \frac{X_n(x) e^{-\lambda_n t}}{1 - e^{-\lambda_n T}}.$$

$$f(x) = \sum_{n=1}^{\infty} \left(\int_0^l \psi(\xi) Y_n(\xi) d\xi\right) \frac{\lambda_n X_n(x)}{1 - e^{-\lambda_n T}} + \sum_{n=1}^{\infty} \left(\int_0^l \varphi(\xi) Y_n(\xi) d\xi\right) \frac{\lambda_n e^{-\lambda_n T} X_n(x)}{1 - e^{-\lambda_n T}}. \tag{22}$$

3 Main results

Theorem 1. Let the functions $\varphi(x), \psi(x)$ from the class $\varphi(x) \in C^3(0,l),\ \psi(x) \in C^4(0,l)$, satisfy the conditions

$$\varphi(0) = \varphi(l) = 0, \quad \varphi(x_0 - 0) = \varphi(x_0 + 0), \quad k_1\varphi'(x_0 - 0) = k_2\varphi'(x_0 + 0), \tag{23}$$

$$\psi(0) = \psi(l) = 0, \psi''(0) = \psi''(l) = 0, \quad \psi(x_0 - 0) = \psi(x_0 + 0), \tag{24}$$

$$k_1\psi'(x_0-0) = k_2\psi'(x_0+0), k_1^2\psi''(x_0-0) = k_2^2\psi''(x_0+0), k_1^3\psi'''(x_0-0) = k_2^3\psi'''(x_0+0), \tag{25}$$

then there is a unique classical solution $u(x,t) \in C(\bar{\Omega}) \bigcap C^{2,1}_{x,t}(\bar{\Omega}_1) \bigcap C^{2,1}_{x,t}(\bar{\Omega}_2),\ f(x) \in C[0,l]$ to problem (1)-(4) and it is represented by series (21)-(22), where $\lambda_n,\ r, X_n(x), Y_n(x)$ are determined by formulas (9), (10), (11), (15), respectively.

Proof. First, let us prove the existence of a function $u(x,t)$ determined by formula (21). As $\{X_n(x)\}$are the eigenfunctions and λ_n- the eigenvalues of problem (5)-(7), it is easy to verify that the function $u(x,t)$ determined by formula (21) satisfies equation (1), the initial condition (2), the boundary conditions (3), and the conjugation conditions (4). Series (21) is the sum of the functions
$u(x,t) = P_1(x) + P_2(x,t) + P_3(x) + P_4(x,t)$, where

$$P_1(x) = \sum_{n=1}^{\infty}\left(\int_0^l \psi(\xi)Y_n(\xi)d\xi\right)\frac{X_n(x)}{1-e^{-\lambda_n T}},$$

$$P_2(x,t) = -\sum_{n=1}^{\infty}\left(\int_0^l \psi(\xi)Y_n(\xi)d\xi\right)\frac{X_n(x)e^{-\lambda_n t}}{1-e^{-\lambda_n T}},$$

$$P_3(x) = \sum_{n=1}^{\infty}\left(\int_0^l \varphi(\xi)Y_n(\xi)d\xi\right)\frac{X_n(x)e^{-\lambda_n T}}{1-e^{-\lambda_n T}},$$

$$P_4(x,t) = -\sum_{n=1}^{\infty}\left(\int_0^l \varphi(\xi)Y_n(\xi)d\xi\right)\frac{X_n(x)e^{-\lambda_n t}}{1-e^{-\lambda_n T}}.$$

Using d'Alembert's criterion, we can show that for $t \geq \varepsilon > 0$ (ε is any positive number), the series $P_2,\ P_4$ converge uniformly. The series $P_1,\ P_3$ do not depend on t, and the series $P_3(x)$ converges uniformly everywhere in $\bar{\Omega}$. Let us consider the series $P_1(x)$. Separately, we will analyze the integral:

$$\int_0^l \psi(\xi)Y_n(\xi)d\xi = \frac{\sqrt{2r}}{k_1}\int_0^{x_0}\psi(\xi)\sin\left(\frac{\pi n r\xi}{k_1}\right)d\xi + \frac{\sqrt{2r}}{k_2}(-1)^{n+1}\int_{x_0}^l \psi(\xi)\sin\left(\frac{\pi n r(l-\xi)}{k_2}\right)d\xi.$$

Integrating by parts we obtain

$$\int_0^l \psi(\xi)Y_n(\xi)d\xi = \frac{\sqrt{2}}{\pi n\sqrt{r}}\left(\psi(0) - \psi(x_0-0)\cos\left(\frac{\pi n r x_0}{k_1}\right)\right) +$$
$$+\frac{\sqrt{2}}{\pi n\sqrt{r}}\int_0^{x_0}\psi'(\xi)\cos\left(\frac{\pi n r\xi}{k_1}\right)d\xi + \frac{\sqrt{2}}{\pi n\sqrt{r}}\left((-1)^{n+1}\psi(l) + (-1)^n\psi(x_0+0)\cos\left(\frac{\pi n r(l-x_0)}{k_2}\right)\right) +$$
$$+\frac{\sqrt{2}}{\pi n\sqrt{r}}(-1)^n\int_{x_0}^l \psi'(\xi)\cos\left(\frac{\pi n r(l-\xi)}{k_2}\right)d\xi.$$

Taking into account the conditions $\psi(0)=\psi(l)=0, \quad \psi(x_0-0)=\psi(x_0+0),$ and the reduction formula
$\cos\left(\frac{\pi n r x_0}{k_1}\right)=(-1)^n\cos\left(\frac{\pi n r(l-x_0)}{k_2}\right)$, we obtain

$$\int_0^l \psi(\xi)Y_n(\xi)d\xi=\frac{\sqrt{2}}{\pi n\sqrt{r}}\int_0^{x_0}\psi'(\xi)\cos\left(\frac{\pi n r\xi}{k_1}\right)d\xi+\frac{\sqrt{2}}{\pi n\sqrt{r}}(-1)^n\int_{x_0}^{l}\psi'(\xi)\cos\left(\frac{\pi n r(l-\xi)}{k_2}\right)d\xi.$$

Integrating by parts, we have

$$\int_0^l \psi(\xi)Y_n(\xi)d\xi=\frac{\sqrt{2r}k_1\psi'(x_0-0)}{(\pi n r)^2}\sin\left(\frac{\pi n r x_0}{k_1}\right)-\frac{\sqrt{2r}k_1}{(\pi n r)^2}\int_0^{x_0}\psi''(\xi)\sin\left(\frac{\pi n r\xi}{k_1}\right)d\xi-$$
$$-\frac{\sqrt{2r}k_2\psi'(x_0+0)}{(\pi n r)^2}(-1)^{n+1}\sin\left(\frac{\pi n r(l-x_0)}{k_2}\right)-\frac{\sqrt{2r}k_2}{(\pi n r)^2}(-1)^{n+1}\int_{x_0}^{l}\psi''(\xi)\sin\left(\frac{\pi n r(l-\xi)}{k_2}\right)d\xi.$$

Taking into account the conditions $k_1\psi'(x_0-0)=k_2\psi'(x_0+0)$ and the reduction formula
$\sin\left(\frac{\pi n r x_0}{k_1}\right)=(-1)^{n+1}\sin\left(\frac{\pi n r(l-x_0)}{k_2}\right)$, we obtain

$$\int_0^l \psi(\xi)Y_n(\xi)d\xi=-\frac{\sqrt{2r}k_1}{(\pi n r)^2}\int_0^{x_0}\psi''(\xi)\sin\left(\frac{\pi n r\xi}{k_1}\right)d\xi-\frac{\sqrt{2r}k_2}{(\pi n r)^2}(-1)^{n+1}\int_{x_0}^{l}\psi''(\xi)\sin\left(\frac{\pi n r(l-\xi)}{k_2}\right)d\xi,$$

or

$$\int_0^l \psi(\xi)Y_n(\xi)d\xi=-\frac{k_1\sqrt{k_1}\sqrt{2r}}{(\pi n r)^2\sqrt{k_1}}\int_0^{x_0}\psi''(\xi)\sin\left(\frac{\pi n r\xi}{k_1}\right)d\xi-$$
$$-\frac{k_2\sqrt{k_2}\sqrt{2r}}{(\pi n r)^2\sqrt{k_2}}(-1)^{n+1}\int_{x_0}^{l}\psi''(\xi)\sin\left(\frac{\pi n r(l-\xi)}{k_2}\right)d\xi=-\frac{C}{(\pi n r)^2}\int_0^l\psi''(\xi)Z_n(\xi)d\xi,$$

where $C=\begin{cases} k_1\sqrt{k_1}, & 0<x<x_0,\\ k_2\sqrt{k_2}, & x_0<x<l,\end{cases}$

$$Z_n(x)=\sqrt{2r}\begin{cases}\frac{1}{\sqrt{k_1}}\sin\left(\frac{\pi n r x}{k_1}\right), & 0<x<x_0,\\ \frac{(-1)^{n+1}}{\sqrt{k_2}}\sin\left(\frac{\pi n r(l-x)}{k_2}\right), & x_0<x<l,\end{cases}$$

eigenfunctions of the following self-adjoint problem [15]:

$$L_0Z(x)=\left\{\begin{array}{ll}-k_1^2Z''(x), & 0<x<x_0\\ -k_2^2Z''(x), & x_0<x<l\end{array}\right\}=\lambda Z(x), Z(0)=Z(l)=0, \quad 0\le t\le T,$$

$$\sqrt{k_1}Z(x_0-0)=\sqrt{k_2}Z(x_0+0), \quad k_1^{\frac{3}{2}}Z'(x_0-0)=k_2^{\frac{3}{2}}Z'(x_0+0),$$

$\lambda_n,\ r$ are defined by formulas (9)-(10).
It is clear that under conditions (24) and (25) the series $P_1(x)$ converges uniformly. Similarly, it can be proved that the series $P_2(x,t),\ P_4(x,t)$ converge uniformly everywhere in $\bar{\Omega}$.
Now it is sufficient to show the uniform convergence of the series $u_{xx}(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n''(x)$.
The uniform convergence of the series $u_t(x,t)=\sum_{n=1}^{\infty}u_n'(t)X_n(x)$ is obtained from equation (1).
It is not difficult to show the uniform convergence of the series $\frac{\partial^2 P_2}{\partial x^2}=-\sum_{n=1}^{\infty}\left(\int_0^l\psi(\xi)Y_n(\xi)d\xi\right)\frac{X_n''(x)e^{-\lambda_n t}}{1-e^{-\lambda_n T}}$, $P_3''(x)=\sum_{n=1}^{\infty}\left(\int_0^l\varphi(\xi)Y_n(\xi)d\xi\right)\frac{X_n''(x)e^{-\lambda_n T}}{1-e^{-\lambda_n T}}$, $\frac{\partial^2 P_4}{\partial x^2}=-\sum_{n=1}^{\infty}\left(\int_0^l\varphi(\xi)Y_n(\xi)d\xi\right)\frac{X_n''(x)e^{-\lambda_n t}}{1-e^{-\lambda_n T}}$,

using d'Alembert's criterion. The uniform convergence of the series

$$P_1{}''(x) = \sum_{n=1}^{\infty} \left(\int_0^l \psi(\xi) Y_n(\xi) d\xi \right) \frac{X_n''(x)}{1 - e^{-\lambda_n T}} = - \sum_{n=1}^{\infty} \left(\int_0^l \psi(\xi) Y_n(\xi) d\xi \right) \frac{\lambda_n X_n(x)}{k_i^2 (1 - e^{-\lambda_n T})},$$

can be proved similarly by integrating by parts four times and requiring that conditions (24), (25) be satisfied. The uniform convergence of the series

$$f(x) = \sum_{n=1}^{\infty} \left(\int_0^l \psi(\xi) Y_n(\xi) d\xi \right) \frac{\lambda_n X_n(x)}{1 - e^{-\lambda_n T}} + \sum_{n=1}^{\infty} \left(\int_0^l \varphi(\xi) Y_n(\xi) d\xi \right) \frac{\lambda_n e^{-\lambda_n T} X_n(x)}{1 - e^{-\lambda_n T}},$$

is proved similarly. For uniform convergence of the series $\sum_{n=1}^{\infty} \left(\int_0^l \psi(\xi) Y_n(\xi) d\xi \right) \frac{\lambda_n X_n(x)}{1 - e^{-\lambda_n T}}$, conditions (24) and (25) must be satisfied. For the second series $\sum_{n=1}^{\infty} \left(\int_0^l \varphi(\xi) Y_n(\xi) d\xi \right) \frac{\lambda_n e^{-\lambda_n T} X_n(x)}{1 - e^{-\lambda_n T}}$, only continuity is required. Thus, the existence of the solution has been proved.

Now, let us prove its uniqueness. Suppose there are two solutions: $\left\{ \tilde{u}(x,t), \tilde{f}(x) \right\}, \left\{ \hat{u}(x,t), \hat{f}(x) \right\}$. Then for the function $v(x,t) = \tilde{u}(x,t) - \hat{u}(x,t),\ g(x) = \tilde{f}(x) - \hat{f}(x)$ we have the following problem:

$$Lv = \begin{cases} v_t - k_1^2 v_{xx}, & 0 < x < x_0, \\ v_t - k_2^2 v_{xx}, & x_0 < x < l, \end{cases} = g(x), \tag{26}$$

$$v(x,0) = 0, \quad v(x,T) = 0, \quad 0 \le x \le l, \tag{27}$$

$$v(0,t) = v(l,t) = 0, \quad 0 \le t \le T, \tag{28}$$

$$\begin{cases} v(x_0 - 0, t) = v(x_0 + 0, t), \\ k_1 v_x(x_0 - 0, t) = k_2 v_x(x_0 + 0, t), \end{cases} \tag{29}$$

We derive the solution to problem (26)-(29) in the form:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) X_n(x), g(x) = \sum_{n=1}^{\infty} g_n X_n(x).$$

Substituting $v(x,t)$,$g(x)$ into equation (26), we obtain an ordinary first-order differential equation

$$v_n'(t) + \lambda_n v_n(t) = g_n, \tag{30}$$

where λ_n is determined by formula (9). Condition (27) takes the form: $v_n(0) = 0,\ \ v_n(T) = 0.$

The solution of equation (30) satisfying the condition $v_n(0) = 0$ is written as

$$v_n(t) = \int_0^t g_n e^{-\lambda_n(t-\tau)} d\tau = \frac{g_n}{\lambda_n}\left(1 - e^{-\lambda_n t}\right).$$

Substituting the last expression in the condition $v_n(T) = 0$ we obtain
$v_n(T) = \frac{g_n}{\lambda_n}\left(1 - e^{-\lambda_n T}\right) = 0$. This expression gives that $g_n = 0 \;\Rightarrow\; v_n(t) = 0$.
It means that $v(x,t) = 0$ and $g(x) = 0$. Thus, the theorem is proved.
Let $\|u(x,t)\|_{L_2}$ be the norm of the space $L_2(\Omega)$. By $W_2^{2,1}(\Omega)$ we will denote the space of functions $u(x,t)$ for which the generalized derivatives $u_{xx}(x,t)$, $u_t(x,t)$ exist almost everywhere, belonging to $L_2(\Omega)$ with the norm

$$\|u(x,t)\|_{2,1}^2 = \|u(x,t)\|_{L_2}^2 + \|u_{xx}(x,t)\|_{L_2}^2 + \|u_t(x,t)\|_{L_2}^2.$$

As a generalized solution of problem (1)–(4) we will consider a pair of functions $u(x,t) \in L_2(0,l) \cap W_2^{2,1}(\Omega_1) \cap W_2^{2,1}(\Omega_2)$, $f(x) \in L_2(0,l)$ [17].
Theorem 2. If the functions $\varphi(x) \in W_2^2(0,l)$, $\psi(x) \in W_2^4(0,l)$ and satisfy conditions (6), then there exists a unique generalized solution $u(x,t) \in L_2(0,l) \cap W_2^{2,1}(\Omega_1) \cap W_2^{2,1}(\Omega_2)$, $f(x) \in L_2(0,l)$ of problem (1)–(4).
Proof. As the functions $\varphi(x) \in W_2^2(0,l)$, $\psi(x) \in W_2^4(0,l)$ and satisfy all the conditions of problem (1)-(4), then they are expanded into uniformly convergent Fourier series in the system $\{X_k(x)\}$. Using the formulas
$X''(x) = -\lambda_k X(x)$ and $X^{IV}(x) = \lambda_k^2 X(x)$
we obtain

$$\varphi''(x) = -\sum_{k=1}^{\infty} \lambda_k \varphi_k X_k(x), \psi^{IV}(x) = \sum_{k=1}^{\infty} \lambda_k^2 \psi_k X_k(x). \tag{31}$$

As the system $\{X_k(x)\}$ is the Riesz basis of the space $L_2(0,l)$, then by virtue of the two-sided Parseval inequality we obtain

$$\sum_{k=1}^{\infty} |\lambda_k \varphi_k|^2 \le C\|\varphi''(x)\|_{L2}, \quad \sum_{k=1}^{\infty} \left|\lambda_k^2 \psi_k\right|^2 \le C\left\|\psi^{IV}(x)\right\|_{L2}. \tag{32}$$

From (21), (22) we obtain estimates uniform in k :

$$|u_k(t)| \le C\left(|\varphi_k| e^{-\lambda_n T} + |\psi_k|\right),$$

$$|u_k'(t)| \le C\left(|\varphi_k| e^{-\lambda_n T} + |\psi_k|\right)|\lambda_k|,$$

$$|f_k| \le C\left(|\varphi_k| e^{-\lambda_n T} + |\psi_k|\right)|\lambda_k|,$$

From here, due to the absolute convergence of series (31) and estimates (32), it follows that series (19), (20) converge and the solution of problem (1)–(4) belongs to the classes $u(x,t) \in L_2(0,l) \cap W_2^{2,1}(\Omega_1) \cap W_2^{2,1}(\Omega_2)$, $f(x) \in L_2(0,l)$.
As the system $\{X_k(x)\}$ forms a Riesz basis for the space $L_2(0,l)$, any solution to problem (1)–(4) from this class can be represented by the series (19), (20). From the uniqueness of the construction of solutions (21), (22) of problem (16), it follows that the solution to problem (1)–(4) is unique.

4 Conclusion

In this paper, we considered a class of problems modeling the process of determining the temperature and density of heat sources given initial and final temperatures. Their mathematical formulation leads to inverse problems for the heat equation, where it is necessary not only to solve along the equation, but also to find an unknown right-hand side depending only on the spatial variable.

To solve the equation, the method of separation of variables is used. It reduces the problem to a spectral boundary value problem for the second-order differential operator with a discontinuous coefficient at the highest derivative. We used the following results obtained by one of the authors [16] of this paper:

- explicit eigenvalues and eigenfunctions for both the original and adjoint spectral problems,
- the proof that the original spectral problem is not self-adjoint,
- the auxiliary self-adjoint problem with the same eigenvalues constructed in [16],
- the proof that the eigenfunctions of the self-adjoint problem form an orthonormal basis in the space $L_2(0, l)$,
- the proof that the eigenfunctions of the original spectral problem form a Riesz basis in $L_2(0, l)$.

Using this property of the basis, we obtained a solution to the initial boundary value problem in the form of a Fourier series. From the condition of a given final temperature, we found the unknown right-hand side. We also reasonably differentiated the series by its terms and proved the existence and uniqueness of the solution under natural conditions of smoothness and consistency for the initial and final data. The results provide a rigorous justification for the Fourier method for the diffusion model in a composite (two-phase) medium with discontinuous coefficients.

Further research may include extension of the analysis to multilayer media with multiple interfaces, more general coupling or boundary conditions (including nonlocal ones), and inverse problems on determining unknown interface coefficients or parameters.

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