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**K.T. Konisbayeva , M.K. Dauylbayev* , A.E. Mirzakulova**
Al-Farabi Kazakh National University, Almaty, Kazakhstan
*e-mail: mdauylbayev@gmail.com

# ASYMPTOTIC ESTIMATES OF THE SOLUTION OF THE BVP FOR SINGULARLY PERTURBED INTEGRO-DIFFERENTIAL EQUATIONS

In this paper, we study a boundary value problem for a higher-order linear singularly perturbed integro-differential equation, focusing on the case where all roots of the associated "additional characteristic equation" have negative real parts. The main objective is to derive an asymptotic estimate for the solution of this boundary value problem as the perturbation parameter tends to zero. To achieve this, we construct an explicit analytical representation of the solution and rigorously prove a theorem describing its asymptotic behavior. Additionally, a modified degenerate boundary value problem is formulated, which serves as the limiting case of the original perturbed problem. It is shown that the solution of the singularly perturbed problem converges uniformly to the solution of this degenerate problem. The study also addresses the determination of the magnitude of the initial jump in the integral terms, which plays a crucial role in the convergence analysis. The results provide a clear understanding of the solution structure and its dependence on the perturbation parameter.
**Key words**: singular perturbation, integro-differential equations, small parameter, asymptotic, initial jump.

К.Т. Конисбаева, М.К. Дауылбаев[1*], А.Е. Мирзакулова
Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан
*e-mail: mdauylbayev@gmail.com

**Сингулярлы ауытқыған интегро-дифференциалдық теңдеу үшін шеттік есеп шешімінің асимптотикалық бағалаулары**

Бұл жұмыста жоғары ретті сызықты сингулярлы ауытқыған интегро-дифференциалдық теңдеуге арналған шекаралық есеп қарастырылады, оның ішінде "қосымша сипаттауыш теңдеудің"барлық түбірлерінің нақты бөліктері теріс болатын жағдайға баса назар аударылады. Негізгі мақсат - ауытқу параметрі нөлге ұмтылғанда осы шекаралық есептің шешіміне асимптотикалық бағалау алу. Бұл мақсатқа жету үшін шешімнің айқын аналитикалық формуласы құрылады және оның асимптотикалық сипаты туралы теорема қатаң дәлелденеді. Сонымен қатар, берілген ауытқыған шеттік есеп үшін өзгертілген шеттік есеп ұсынылады. Көрсетілгендей, сингулярлық ауытқыған есеп шешімі осы өзгертілген ауытқымаған есеп шешіміне бірқалыпты жинақталады. Жұмыс интегралдық мүшенің бастапқы секіріс шамасын анықтауды да қарастырады, ол жинақтылықты талдауда маңызды рөл атқарады. Алынған нәтижелер шешім құрылымын және оның ауытқу параметріне тәуелділігін анық түсінуге мүмкіндік береді.
**Түйін сөздер**: сингулярлы ауытқу, интегро-дифференциалдық теңдеу, кіші параметр, асимптотика, бастапқы секіріс.

К.Т. Конисбаева, М.К. Дауылбаев*, А.Е. Мирзакулова
Казахский национальный университет имени аль-Фараби, Алматы, Казахстан
*e-mail: mdauylbayev@gmail.com

**Асимптотические оценки решения краевой задачи для сингулярно возмущенных интегро-дифференциальных уравнений**

В данной работе рассматривается краевая задача для линейного интегро-дифференциального уравнения высокого порядка с сингулярным возмущением, с акцентом на случай, когда все корни связанного "дополнительного характеристического уравнения"имеют отрицательные действительные части. Основная цель заключается в том, чтобы получить асимптотическую оценку решения этой краевой задачи при стремлении параметра возмущения к нулю. Для достижения этой цели строится явное аналитическое представление решения и строго доказывается теорема, описывающая его асимптотическое поведение. Кроме того, формулируется модифицированная вырожденная краевая задача, которая служит предельным случаем исходной возмущенной задачи. Показывается, что решение сингулярно возмущенной задачи равномерно сходится к решению этой вырожденной задачи. Работа также затрагивает определение величины начального скачка в интегральных членах, который играет ключевую роль в анализе сходимости. Полученные результаты обеспечивают ясное понимание структуры решения и его зависимости от параметра возмущения.
**Ключевые слова**: сингулярное возмущение, интегро-дифференциальное уравнение, малый параметр, асимптотика, начальный скачок.

## 1 Introduction

In the modern theory of differential equations, the study of singularly perturbed problems occupies a central place. The relevance of this topic stems from the widespread use of such equations in the mathematical modeling of processes characterized by the presence of variables of different scales—for example, in fluid dynamics, semiconductor theory, and gyroscopic stabilization problems. The combination of a singular perturbation and an integral operator introduces additional complexity into the analysis.

The foundations of the theory of singularly perturbed differential equations were laid in the pioneering works of Academician A. N. Tikhonov. He proved his famous fundamental theorems on the limit transition in singularly perturbed differential equations [1]. For a broad class of singularly perturbed differential problems, a variety of asymptotically efficient techniques have been developed, allowing one to approximate the solution with an arbitrary degree of accuracy. In this context, it is appropriate to highlight the contributions of M.I. Vishik and L.A. Lyusternik [2], A.B. Vasilyeva [3], and M.I. Imanaliev [4]. These approaches are presently referred to as the boundary function method.

Problems with initial jumps occupy a important place in the theory of singular perturbations of differential equations. In the classical works of M.I. Vishik, L.A. Lyusternik [5] and K.A. Kasymov [6], it was first proven that solutions of singularly perturbed problems converge to solutions of degenerate equations with modified conditions as a small parameter tends to zero, which were called Cauchy problems with initial jumps.

In recent years, boundary value problems with initial jumps for singularly perturbed differential and integro-differential equations have been actively studied [7–14]. The results of these studies showed that the nature of the asymptotic behavior of solutions is essentially determined by the differential order of the kernel, which indicates the deep specificity of the class of problems under consideration.

The object of the study is a boundary value problem for an integro-differential equation of order $n + m$ with a small parameter at the highest derivatives. The key condition of the analysis is the negativity of the real parts of the roots of the "additional characteristic equation,"which determines the structure and type of singularity in the problem under consideration. Earlier, in [15], a similar integro-differential equation was studied when the order of the highest derivative under the integral was equal to $n_1 + 1$. In this paper we investigate the case where the order of the highest derivative in the integrand is exactly equal to $n_1 + m$. It should be noted that the problem under consideration is not a simple generalization of the previously studied formulation. In the paper it will be shown that the asymptotic behavior of its solution differs significantly from the behavior of the solution of the boundary value problem studied in [15].

## 2 Statement of the problem and Preliminary materials

Let us move on to considering the boundary value problem for the integro-differential equation.

$$L_\varepsilon y \equiv \sum_{r=1}^{m} \varepsilon^r A_{n+r}(t)\frac{d^{n+r}y}{dt^{n+r}} + \sum_{k=0}^{n} A_k(t)\frac{d^k y}{dt^k} = F(t) + \int_0^1 \sum_{i=0}^{n_1+m} H_i(t,x)y^{(i)}(x,\varepsilon)dx, \tag{1}$$

$$h_{i+1}y(t,\varepsilon) \equiv y^{(i)}(0,\varepsilon) = \alpha_i, \quad i = \overline{0,l-1}, \quad h_{l+1+i}y(t,\varepsilon) \equiv y^{(i)}(1,\varepsilon) = \beta_i, \quad i = \overline{0,p-1}, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, $\alpha_i, \beta_i$ are known constants, $A_{n+m}(t) = 1$, $m + n = l + p$, $m < l$, $n_1 = l - m$, $p = n - n_1 = n_2$.

In what follows, we will assume that the following assumptions are valid for the boundary value problem under consideration:

C1. $A_i(t) \in C^{n+m-1}([0,1])$, $i = \overline{0,n+m}$, $F(t) \in C([0,1]), H_i(t,x) \in C(D)$, $i = \overline{0,n_1+m}$, $D = (0 \le t, x \le 1)$.

C2. The "additional characteristic equation" $\mu^m(t) + A_{n+m-1}(t)\mu^{m-1}(t) + ... + A_{n+1}(t)\mu + A_n(t) = 0$ have $m$ different roots $\mu_1(t), ..., \mu_m(t)$ with negative real parts

C3.

$$\overline{J}_0 = \begin{vmatrix} h_1 y_{10}(t) & h_1 y_{20}(t) & \dots & h_1 y_{n0}(t) \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ h_{n_1} y_{10}(t) & h_{n_1} y_{20}(t) & \dots & h_{n_1} y_{n0}(t) \\ h_{l+1} y_{10}(t) & h_{l+1} y_{20}(t) & \dots & h_{l+1} y_{n0}(t) \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ h_{n+m} y_{10}(t) & h_{n+m} y_{20}(t) & \dots & h_{n+m} y_{n0}(t) \end{vmatrix} \neq 0,$$

where $y_{i0}(t)$, $i = \overline{1,n}$ is fundamental system of solutions of the homogeneous degenerate equation $L_0 y = 0$.

C4. The number 1 is not an eigenvalue of the kernel $H(t,s,\varepsilon)$ for sufficiently small $\varepsilon$.

C5.

$$\overline{\delta} = \begin{vmatrix} 1 + \overline{d}_{l+1}(1) & \overline{d}_{l+2}(1) & \dots & \overline{d}_{l+p}(1) \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ \overline{d}_{l+1}^{(p-1)}(1) & \overline{d}_{l+2}^{(p-1)}(1) & \dots & 1 + \overline{d}_{l+p}^{(p-1)}(1) \end{vmatrix} \neq 0,$$

where $\overline{d}_{l+i}^{(j)}(1), i = \overline{1,p},\ j = \overline{0,p-1}$ are expressed in a certain way in terms of the coefficients of (1).

In the work [15], for a singularly perturbed linear homogeneous differential equation $L_\varepsilon y = 0$, a fundamental system of solutions and, accordingly, the Cauchy function $K(t,s,\varepsilon), 0 \le s \le t \le 1$ were constructed, which is the solution to the problem

$$L_\varepsilon K(t,s,\varepsilon)=0, K^{(i)}(s,s,\varepsilon)=0, \quad i=\overline{0,n+m-2}, K^{(n+m-1)}(s,s,\varepsilon)=1,$$

and also boundary functions $\Phi_i(t,\varepsilon), i=\overline{1,n+m}$, which are solutions to the problem

$$L_\varepsilon \Phi_i(t,s,\varepsilon)=0, \quad h_k\Phi_i(t,\varepsilon)=\delta_{ki}, \quad i,k=\overline{1,n+m},$$

are constructed. For the functions $K(t,s,\varepsilon)$, $\Phi_i(t,\varepsilon), \overline{1,n+m}$ the following asymptotic representations as $\varepsilon\to 0$ was obtained:

$$K^{(j)}(t,s,\varepsilon)=\varepsilon^m\left[(-1)^m\frac{\overline{W}_n^{(j)}(t,s)}{\overline{W}(s)A_n(s)}+\varepsilon^{n-1-j}\sum_{k=1}^{m}(-1)^{m+k}\frac{y_{n+k,0}(t)\mu_k^j(t)\omega_{mk}(s)}{y_{n+k,0}(s)\mu_k^n(s)\omega(s)}\,e^{\frac{1}{\varepsilon}\int\limits_s^t\mu_k(x)dx}+\right.$$

$$\left.+O\left(\varepsilon+\varepsilon^{n-j}\sum_{k=1}^{m}e^{\frac{1}{\varepsilon}\int\limits_s^t\mu_k(x)dx}\right)\right], \quad j=\overline{0,n+m-1}, \tag{3}$$

and

$$\Phi_i^{(j)}(t,\varepsilon)=\frac{\overline{J}_i^{(j)}(t)}{\overline{J}_0}+\varepsilon^{n_1-j}\frac{\overline{J}_i^{(n_1)}(0)}{\overline{J}_0}\sum_{k=1}^{m}(-1)^k\frac{y_{n+k,0}(t)\mu_k^j(t)\omega_{1k}(0)}{\mu_k^{n_1}(0)\omega(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}+$$

$$+O\left(\varepsilon+\varepsilon^{n_1+1-j}\sum_{k=1}^{m}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}\right), \quad i=\overline{1,n_1},$$

$$\Phi_{n_1+i}^{(j)}(t,\varepsilon)=\varepsilon^i\left(-\frac{\overline{J}_{n_1}^{(j)}(t)}{\overline{J}_0}\cdot\frac{\omega_i(0)}{\omega(0)}+(-1)^i\varepsilon^{n_1-1-j}\sum_{k=1}^{m}(-1)^k\frac{y_{n+k,0}(t)\mu_k^j(t)\omega_{ik}(0)}{\mu_k^{n_1}(0)\omega(0)}\,e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}+\right.$$

$$\left.+\,O\left(\varepsilon+\varepsilon^{n_1-j}\sum_{k=1}^{m}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}\right)\right), \quad i=\overline{1,m},$$

$$\Phi_{n_1+i+m}^{(j)}(t,\varepsilon)=\frac{\overline{J}_{n_1+i}^{(j)}(t)}{\overline{J}_0}+\varepsilon^{n_1-j}\frac{\overline{J}_{n_1+i}^{(n_1)}(0)}{\overline{J}_0}\sum_{k=1}^{m}(-1)^k\frac{y_{n+k,0}(t)\mu_k^j(t)\omega_{1k}(0)}{\mu_k^{n_1}(0)\omega(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}+$$

$$+O\left(\varepsilon+\varepsilon^{n_1-j+1}\sum_{k=1}^{m}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}\right), \quad i=\overline{1,n_2}. \tag{4}$$

Similar to the work [15], when conditions C1-C5 are satisfied, the existence and uniqueness of the solution to the boundary value problem (1), (2) on the interval $[0,1]$ is established, and the desired solution can be represented in the form of the following analytical expression:

$$y^{(j)}(t,\varepsilon)=\sum_{i=1}^{l}\alpha_{i-1}Q_i^{(j)}(t,\varepsilon)+ \tag{5}$$

$$+\sum_{i=1}^{p}C_{l+i}Q_{l+i}^{(j)}(t,\varepsilon)+P^{(j)}(t,\varepsilon),\, j=\overline{0,n+m-1},$$

where

$$Q_i(t,\varepsilon)=\Phi_i(t,\varepsilon)+\frac{1}{\varepsilon^m}\int_0^t K(t,s,\varepsilon)\varphi_i(s,\varepsilon)ds,$$

$$P(t,\varepsilon)=\frac{1}{\varepsilon^m}\int_0^t K(t,s,\varepsilon)F(s,\varepsilon)ds, \tag{6}$$

$C_{l+i}, i=\overline{1,p}$ are determined by the following system of algebraic equations

$$\begin{cases} C_{l+1}(1+d_{l+1}(1,\varepsilon))+\ldots+C_{l+p}d_{l+p}(1,\varepsilon)=\beta_0-P(1,\varepsilon)-\sum\limits_{i=1}^{l}\alpha_{i-1}d_i(1,\varepsilon),\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ C_{l+1}d_{l+1}^{(p-1)}(1,\varepsilon)+\ldots+C_{l+p}(1+d_{l+p}^{(p-1)}(1,\varepsilon))=\beta_{p-1}-P^{(p-1)}(1,\varepsilon)-\sum\limits_{i=1}^{l}\alpha_{i-1}d_i^{(p-1)}(1,\varepsilon), \end{cases} \tag{7}$$

in which

$$d_i^{(j)}(1,\varepsilon)=\frac{1}{\varepsilon^m}\int_0^1 K^{(j)}(1,s,\varepsilon)\varphi_i(s,\varepsilon)ds,\quad i=\overline{1,l+p},\ j=\overline{0,p-1}. \tag{8}$$

Formulas (6)-(8) are similar to the formulas in paper [15], but the difference is in formula for the functions $\varphi_i(t,\varepsilon), i=\overline{1,l+p}$. In this work, these functions are expressed by the formula

$$\varphi_i(t,\varepsilon)=\int_0^1\sum_{j=0}^{n_1+m}H_j(t,x,\varepsilon)\Phi_i^{(j)}(x,\varepsilon)dx, i=\overline{1,l+p}, \tag{9}$$

i.e. the right-hand side of (9) includes derivatives of the boundary functions $\Phi_i(t,\varepsilon)$ up to the order $n_1+m$, and the functions $H_j(t,x,\varepsilon)$ have the same representations as in work [15]:

$$H_j(t,x,\varepsilon)=\overline{H}_j(t,x)+O(\varepsilon), \tag{10}$$

where

$$\overline{H}_j(t,x)\equiv H_j(t,x)+\int_0^1\overline{R}(t,s)H_j(s,x)ds.$$

Therefore, the asymptotic representation of the functions $\varphi_i(t,\varepsilon), i=\overline{1,l+p}$ differs from the representation of similar functions $\varphi_i(t,\varepsilon), i=\overline{1,l+p}$ in paper [15].

## 3 Main results

Let us turn to formula (9) for the functions $\varphi_i(t,\varepsilon), i=\overline{1,l+p}$. From (9) by virtue of representations (4), we obtain

$$\varphi_i(s,\varepsilon)=\int\limits_0^1\sum_{j=0}^{n_1}\overline{H}_j(s,x)\frac{\overline{J}_i^{(j)}(x)}{\overline{J}_0}dx+\left(\sum_{k=1}^{m}(-1)^{k+1}\overline{H}_{n_1+k}^{(k-1)}(s,0)\right)\frac{\overline{J}_i^{(n_1)}(0)}{\overline{J}_0}+$$

$$+\int_0^1\left(\sum_{k=1}^{m}(-1)^{k+1}\overline{H}_{n_1+k}^{(k-1)}(s,x)\right)\frac{\overline{J}_i^{(n_1+1)}(x)}{\overline{J}_0}dx+\sum_{k=2}^{m}\sum_{q=2}^{k}(-1)^{q}\overline{H}_{n_1+k}^{(q-2)}(s,1)\frac{\overline{J}_i^{(n_1+k-q+1)}(1)}{\overline{J}_0}+O(\varepsilon)\equiv \tag{11}$$

$$\equiv\overline{\varphi}_i(s)+O(\varepsilon),\quad i=\overline{1,n_1},$$

$$\varphi_{n_1+i}(s,\varepsilon)=\sum_{k=0}^{m-i}(-1)^{k+1}\overline{H}_{n_1+i+k}^{(k)}(s,0)+O(\varepsilon)\equiv\overline{\varphi}_{n_1+i}(s)+O(\varepsilon),\quad i=\overline{1,m}, \tag{12}$$

$$\varphi_{n_1+m+i}(s,\varepsilon)=\int\limits_0^1\sum_{j=0}^{n_1}\overline{H}_j(s,x)\frac{\overline{J}_{n_1+i}^{(j)}(x)}{\overline{J}_0}dx+\left(\sum_{k=1}^{m}(-1)^{k+1}\overline{H}_{n_1+k}^{(k-1)}(s,0)\right)\frac{\overline{J}_{n_1+i}^{(n_1)}(0)}{\overline{J}_0}+$$

$$+\int_0^1\left(\sum_{k=1}^{m}(-1)^{k+1}\overline{H}_{n_1+k}^{(k-1)}(s,x)\right)\frac{\overline{J}_{n_1+i}^{(n_1+1)}(x)}{\overline{J}_0}dx+\sum_{k=2}^{m}\sum_{q=2}^{k}(-1)^{q}\overline{H}_{n_1+k}^{(q-2)}(s,1)\frac{\overline{J}_{n_1+i}^{(n_1+k-q+1)}(1)}{\overline{J}_0}+O(\varepsilon)\equiv$$

$$(13)$$

$$\equiv \overline{\varphi}_{n_1+m+i}(s)+O(\varepsilon), \quad i=\overline{1,n_2}.$$

From formulas (11)-(13) it follows that the functions $\varphi_i(t,\varepsilon), i=\overline{1,l+p}$ are bounded as $\varepsilon\to 0$, i.e. $\varphi_i(t,\varepsilon)=O(1), \varepsilon\to 0, i=\overline{1,l+p}$. Note that in paper [15] the functions $\varphi_i(t,\varepsilon), i=\overline{1,l+p}$ had different asymptotic representations

$$\varphi_i(t,\varepsilon)=O(1), i=\overline{1,n_1+1}, \ \ \varphi_{n_1+i}(t,\varepsilon)=O(\varepsilon^i), i=\overline{2,m},$$
$$\varphi_{n_1+m+i}(t,\varepsilon)=O(1), i=\overline{1,n_2}, \varepsilon\to 0.$$

Note that in paper [15] the functions $\varphi_{n_1+1}(t,\varepsilon)$ is expressed only through $H_{n_1+1}(t,0)$, and in the present work the $\varphi_{n_1+1}(t,\varepsilon)$ is expressed not only through $H_{n_1+1}(t,0)$, but also through $H_{n_1+2}(t,0),...,H_{n_1+m}(t,0)$. The asymptotic representations of the functions $\varphi_{n_1+i}(t,\varepsilon), i=\overline{2,m}$ also change.

These differences in the asymptotic behavior of the functions $\varphi_i(t,\varepsilon), i=\overline{1,n+m}$ subsequently play a significant role in describing the behavior of solutions to the problem (1), (2). Then, in view of (11)-(13), for the coefficients of system (7) $P^{(j)}(1,\varepsilon), d_i^{(j)}(1,\varepsilon), i=\overline{1,n+m}$, the following asymptotic representation is valid

$$d_i^{(j)}(1,\varepsilon)=(-1)^m\int\limits_0^1\frac{\overline{W}_n^{(j)}(1,s)}{\overline{W}(s)A_n(s)}\overline{\varphi}_i(s)ds\equiv \overline{d}_i^{(j)}(1)+O(\varepsilon), i=\overline{1,n+m}, \ j=\overline{0,p-1},$$

$$P^{(j)}(1,\varepsilon)=(-1)^m\int\limits_0^1\frac{\overline{W}_n^{(j)}(1,s)}{\overline{W}(s)A_n(s)}\overline{F}(s)ds\equiv \overline{P}^{(j)}(1)+O(\varepsilon), \quad j=\overline{0,p-1}. \tag{14}$$

Taking into account condition C5, solutions $C_{l+i}, i=\overline{1,p}$ of the system (7) can be presented in the form of the following asymptotic expression

$$C_{l+i}=\overline{C}_{l+i}+O(\varepsilon), \tag{15}$$

where

$$\overline{C}_{l+i}=\sum_{k=1}^{p}(-1)^{k+i}\frac{\overline{\delta}_{ki}}{\overline{\delta}}\left[\beta_{k-1}+(-1)^{m+1}\int\limits_0^1\frac{\overline{W}_n^{(k-1)}(1,s)}{\overline{W}(s)A_n(s)}\cdot\right.$$
$$\left.\cdot\left(\overline{F}(s)+\sum_{j=1}^{n_1}\alpha_{j-1}\overline{\varphi}_j(s)+\sum_{j=1}^{m}\alpha_{n_1+j-1}\overline{\varphi}_{n_1+j}(s)\right)ds\right], \quad i=\overline{1,p}, \tag{16}$$

$\overline{\delta}_{ki}$ is the determinant that is determined by removing $\overline{\delta}$ from the $k$-th row and $i$-th column.

From (6), taking into account (3), (4) and (11)-(13), we obtained an asymptotic as $\varepsilon \to 0$ representation for the functions $Q_i(t,\varepsilon), i = \overline{1, n+m}$ and $P(t,\varepsilon)$ :

$$Q_i^{(j)}(t,\varepsilon) = \frac{\overline{J}_i^{(j)}(t)}{\overline{J}_0} + (-1)^m \int\limits_0^t \frac{\overline{W}_n^{(j)}(t,s)}{\overline{W}(s)A_n(s)}\overline{\varphi}_i(s)ds + \varepsilon^{n_1-j}\frac{\overline{J}_i^{(n_1)}(0)}{\overline{J}_0}\cdot$$

$$\cdot\sum_{k=1}^m(-1)^k\frac{y_{n+k,0}(t)\mu_k^j(t)\omega_{1k}(0)}{y_{n+k,0}(0)\mu_k^{n_1}(0)\omega(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx} + O\left(\varepsilon + \varepsilon^{n_1+1-j}\sum_{k=1}^m e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}\right), \quad i = \overline{1, n_1},$$

$$Q_{n_1+i}^{(j)}(t,\varepsilon) = (-1)^m\int\limits_0^t \frac{\overline{W}_n^{(j)}(t,s)}{\overline{W}(s)A_n(s)}\overline{\varphi}_{n_1+i}(s)ds +$$

$$+\varepsilon^{n_1+i-1-j}\sum_{k=1}^m(-1)^{i+k}\frac{y_{n+k,0}(t)\mu_k^j(t)\omega_{ik}(0)}{\mu_k^{n_1}(0)\omega(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx} + O\left(\varepsilon + \varepsilon^{n_1+i-j}\sum_{k=1}^m e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}\right), i = \overline{1, m}, \tag{17}$$

$$Q_{n_1+m+i}^{(j)}(t,\varepsilon) = \frac{\overline{J}_{n_1+i}^{(j)}(t)}{\overline{J}_0} + (-1)^m\int\limits_0^t \frac{\overline{W}_n^{(j)}(t,s)}{\overline{W}(s)A_n(s)}\overline{\varphi}_{n_1+m+i}(s)ds + \varepsilon^{n_1-j}\frac{\overline{J}_{n_1+i}^{(n_1)}(0)}{\overline{J}_0}\cdot$$

$$\cdot\sum_{k=1}^m(-1)^k\frac{y_{n+k,0}(t)\mu_k^j(t)\omega_{1k}(0)}{y_{n+k,0}(0)\mu_k^{n_1}(0)\omega(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx} + O\left(\varepsilon + \varepsilon^{n_1-j+1}\sum_{k=1}^m e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}\right), i = \overline{1, n_2},$$

$$P^{(j)}(t,\varepsilon) = (-1)^m\int\limits_0^t \frac{\overline{W}_n^{(j)}(t,s)}{\overline{W}(s)A_n(s)}\overline{F}(s)ds +$$

$$+\varepsilon^{n-j}\sum_{k=1}^m(-1)^{m+k}\frac{y_{n+k,0}(t)\mu_k^j(t)\omega_{mk}(0)\overline{F}(0)}{\mu_k^{n+1}(0)\omega(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx} + O\left(\varepsilon + \varepsilon^{n+1-j}\sum_{k=1}^m e^{\frac{1}{\varepsilon}\int\limits_0^t\mu_k(x)dx}\right)$$

### 3.1 Asymptotic estimates of the solution

**Theorem 1** *If the conditions C1–C5 hold. Then, for sufficiently small $\varepsilon > 0$, the following asymptotic estimates hold for the solution to the boundary value problem* (1), (2) *:*

$$|y^{(j)}(t,\varepsilon)| \le C\left[\sum_{i=1}^{n_1}|\alpha_{i-1}| + \sum_{i=1}^{p}|\beta_{i-1}| + \max_{0\le t\le 1}\left|F(t) + \sum_{i=1}^m \alpha_{n_1+i-1}\overline{\varphi}_{n_1+i}(t)\right|\right] +$$

$$+C\varepsilon^{n_1-j}\left|\sum_{k=1}^{m}(-1)^k\frac{\mu_k^j(t)\omega_{1k}(0)}{\mu_k^{n_1}(0)\omega(0)}\right|\left[\sum_{i=1}^{n_1}|\alpha_{i-1}|+\sum_{i=1}^{p}|\beta_{i-1}|+\right.$$

$$\left.+\max_{0\le t\le 1}\left|F(t)+\sum_{i=1}^{m}\alpha_{n_1+i-1}\overline{\varphi}_{n_1+i}(t)\right|\right]e^{-\gamma\frac{t}{\varepsilon}}, j=\overline{0,n+m-1}, \tag{18}$$

*where* $\overline{\varphi}_{n_1+i}(t), i=\overline{1,m}$ *have the form* (12), *and* $\omega(s)$ *is the Vandermonde determinant, i.e.*

$$\omega(s)=\begin{vmatrix}1 & 1 & \dots & 1\\ \mu_1(s) & \mu_2(s) & \dots & \mu_m(s)\\ \dots & \dots & \dots & \dots\\ \mu_1^{m-1}(s) & \mu_2^{m-1}(s) & \dots & \mu_m^{m-1}(s)\end{vmatrix}$$

*and* $\omega_{mk}(s)$ *is the determinant obtained by removing* $\omega(s)$ *from the m-th row and the k-th column. It should be noted that*

$$\sum_{k=1}^{m}(-1)^k\frac{\mu_k^j(t)\omega_{1k}(0)}{\mu_k^{n_1}(0)\omega(0)}\Bigg|_{t=0}\equiv 0,\quad j=\overline{n_1+1,n_1+m-1}. \tag{19}$$

The proof of the theorem 1 is derived from (5) considering (11)-(13), (15),(16),(17). From Theorem 1 using (19) we obtain

$$y^{(i)}(0,\varepsilon)=O\left(1\right), i=\overline{0,n_1+m-1},$$

$$y^{(n_1+m)}(0,\varepsilon)=O\left(\frac{1}{\varepsilon^m}\right),\dots,y^{(n+m-1)}(0,\varepsilon)=O\left(\frac{1}{\varepsilon^{n_2+m-1}}\right),\quad \varepsilon\to 0. \tag{20}$$

The obtained result demonstrates a fundamental difference in the asymptotic behavior of the solution to the problem under study compared to that previously studied in [15] in which $y^{(i)}(0,\varepsilon),\ \ i=\overline{0,n_1}$ is bounded and $y^{(n_1+1)}(0,\varepsilon)$ is infinitely large of the order $O(\frac{1}{\varepsilon}), \varepsilon\to 0$.In the present work, the dependence on the small parameter exhibits growth starting from the order $O(\frac{1}{\varepsilon^m})$.

This phenomenon is called *the initial jump of the* $n_1$*-th order* $m$*-th degree.*

### 3.2 Modified degenerate problem

Let us define a degenerate problem where the solution to the original perturbed problem tends to the solution of this problem. Since in equation (1) the derivative of the order $n_1+m$, which is under the integral, by virtue of (20) is an infinitely large at the point $t=0$, then we should construct a degenerate equation of the following form

$$L_0\overline{y}\equiv\sum_{k=0}^{n}A_k(t)\frac{d^k\overline{y}}{dt^k}=F(t)+\int_0^1\sum_{i=0}^{n_1+m}H_i(t,x)\overline{y}^{(i)}(x)dx+\Delta(t), \tag{21}$$

where $\Delta(t)$ is an unknown function called the initial jump of the integral terms. Now, we will determine the boundary conditions for equation (21). Since $\alpha_i$, $i=\overline{0,n_1-1}$ and $\beta_i$, $i=\overline{0,p-1}$ are bounded then for the degenerate equation (21) we leave these conditions. By virtue of the presence of the initial jump $n_1-th$ order $m-th$ degree in the solution to problem (1), (2) we add to the initial conditions for the degenerate equation the initial values $\alpha_{n_1+i-1}+\Delta_i, i=\overline{1,m}$. Then, the degenerate equation (21) is solved under the following boundary conditions:

$$\overline{y}^{(i)}(0)=\alpha_i,\ \ i=\overline{0,n_1-1},\ \ \overline{y}^{(n_1+i-1)}(0)=\alpha_{n_1+i-1}+\Delta_i,\ \ i=\overline{1,m},\ \ \overline{y}^{(i)}(1)=\beta_i,\ \ i=\overline{0,p-1}, \tag{22}$$

where $\Delta_i$ are the initial jumps of the solutions $\overline{y}^{(i)}(t), i=\overline{n_1,n_1+m-1}$ at the point $t=0$.

To find $\Delta(t)$ in (21), we denote the difference between the solution $y(t,\varepsilon)$ of the original boundary value problem (1), (2) and the solution $\overline{y}(t)$ of the modified degenerate boundary value problem (21), (22) by $u(t,\varepsilon)$, and substitute the solution $y(t,\varepsilon)=\overline{y}(t)+u(t,\varepsilon)$ into (1), (2). Taking into account (21), (22) we obtain the singularly perturbed integro-differential equation for $u(t,\varepsilon)$:

$$L_\varepsilon u = f(t,\varepsilon)+\int\limits_0^1\sum_{i=0}^{n_1+m}H_i(t,x)u^{(i)}(x,\varepsilon)dx-\Delta(t) \tag{23}$$

with boundary conditions

$$\begin{aligned} &u^{(i)}(0,\varepsilon)=0,\quad i=\overline{0,n_1-1},\quad u^{(n_1+i-1)}(0,\varepsilon)=-\Delta_i,\quad i=\overline{1,m},\\ &u^{(i)}(1,\varepsilon)=0,\quad i=\overline{0,p-1}, \end{aligned} \tag{24}$$

where

$$f(t,\varepsilon)=-(\varepsilon^m\overline{y}^{(n+m)}+\ldots+\varepsilon A_{n+1}(t)\overline{y}^{(n+1)})=O(\varepsilon).$$

Since the problem (23), (24) is similar in type to the problem (1), (2), then the estimate (18) can be used to solve the problem (23), (24), and by virtue of (12) for $\varepsilon\to 0$ we obtain the following asymptotic estimates:

$$|u^{(j)}(t,\varepsilon)|\leq C\max_{0\leq t\leq 1}\left|\Delta(t)-\sum_{i=1}^{m}\left(\sum_{k=0}^{m-i}(-1)^kH^{(k)}_{n_1+i+k}(t,0)\right)\Delta_i\right|+ \tag{25}$$

$$+C\varepsilon^{n_1-j}e^{-\gamma\frac{t}{\varepsilon}}\left|\sum_{k=1}^{m}(-1)^k\frac{\mu_k^j(t)\omega_{1k}(0)}{\mu_k^{n_1}(0)\omega(0)}\right|\max_{0\leq t\leq 1}\left|\Delta(t)-\sum_{i=1}^{m}\left(\sum_{k=0}^{m-i}(-1)^kH^{(k)}_{n_1+i+k}(t,0)\right)\Delta_i\right|,$$

$j=\overline{0,n+m-1}$

From estimates (25) it follows that $u^{(j)}(t,\varepsilon),\quad j=\overline{0,n+m-1}$ tends to zero as $\varepsilon\to 0$ if we define the initial jump of the integral terms $\Delta(t)$ in the form

$$\Delta(t)=\sum_{i=1}^{m}\left(\sum_{k=0}^{m-i}(-1)^k H^{(k)}_{n_1+i+k}(t,0)\right)\Delta_i. \tag{26}$$

Therefore, the validity of the following theorem is established.

**Theorem 2** *Let conditions C1-C5 and equality* (26) *be satisfied. Then the singularly perturbed boundary value problem* (1), (2) *for sufficiently small* $\varepsilon$ *has a unique solution* $y(t,\varepsilon)$ *such that the following limit equalities are satisfied:*

$$\lim_{\varepsilon\to 0} y^{(j)}(t,\varepsilon)=\overline{y}^{(j)}(t),\quad j=\overline{0,n_1-1},\quad 0\le t\le 1,$$

$$\lim_{\varepsilon\to 0} y^{(j)}(t,\varepsilon)=\overline{y}^{(j)}(t),\quad j=\overline{n_1,n+m-1},\quad 0< t\le 1,$$

*where* $\overline{y}(t)$ *is the solution of the modified degenerate boundary value problem* (21), (22).

## 4 Conclusion

This paper presents a comprehensive study of the influence of integral terms on the asymptotic behavior of solutions to singularly perturbed problems. Key points include the introduction and justification of the concepts of the order and degree of the initial jump. It was established that the solution to the studied class of equations in the neighborhood of the initial point is characterized by an initial jump of order $n_1$ and degree $m$. The paper theoretically substantiates that, as a small parameter tends to zero, the solution to the original problem asymptotically converges to the solution of the modified degenerate equation. A specific feature of the limit transition is the appearance of an additional term on the right-hand side—an initial jump in the integral terms, reflecting the influence of the integral constraint on the limit state of the problem. The initial jump value has been determined and compared with the results of a previously studied problem. Fundamental differences in the asymptotic behavior of the solution to this problem were shown compared to the behavior of the solution considered earlier.

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### References

[1] Tihonov, A. N. *On the Dependence of Solutions of Differential Equations on a Small Parameter.* [Mathematical Collection] 22, no. 2 (1948): 193–204. [In Russian].

[2] Vishik, M. I., and L. A. Lyusternik. "Regular Degeneration and the Boundary Layer for Linear Differential Equations with Small Parameter." *Usp. Mat. Nauk* 12, no. 5 (1957): 3–122. [In Russian].

[3] Vasil'eva, A., V. Butuzov, and L. Kalachev. "The Boundary Function Method for Singular Perturbation Problems." *SIAM Studies in Applied Mathematics. Philadelphia: SIAM* 14 (1995).

[4] Imanaliev, M. I. *Asymptotic Methods in the Theory of Singularly Perturbed Integro-Differential Systems*. Frunze: Ilim, 1972. [In Russian].

[5] Vishik, M. I., and L. A. Lyusternik. "On the Initial Jump for Nonlinear Differential Equations Containing a Small Parameter." *Dokl. USSR Academy of Sciences* 132, no. 6 (1960): 1242–1245. [In Russian].

[6] Kassymov, K. A. *On the Asymptotic of the Solutions of Cauchy Problem with Boundary Conditions for Non-linear Ordinary Differential Equations Containing a Small Parameter*. [Usp. Mat. Nauk] 17, no. 5 (1962): 187–188. [In Russian].

[7] Kassymov, K. A., and D. Nurgabyl. "Asymptotic Estimates of Solution of a Singularly Perturbed Boundary Value Problem with an Initial Jump for Linear Differential Equations." *Differential Equations* 40, no. 5 (2004): 641–651.

[8] Dauylbayev, M. K., and N. Atakhan. "The Initial Jumps of Solutions and Integral Terms in Singular BVP of Linear Higher Order Integro-Differential Equations." *Miskolc Math. Notes, Hungary* 16, no. 2 (2015): 747–761. https://doi.org/10.18514/MMN.2015.1437.

[9] Dauylbayev, M. K., and A. E. Mirzakulova. "Asymptotic Behavior of Solutions of Singular Integro-Differential Equations." *Journal of Discontinuity, Nonlinearity, and Complexity. L&H Scientific Publishing, LLC, USA* 5, no. 2 (2016): 147–154. https://doi.org/10.5890/DNC.2016.06.004.aspx.

[10] Dauylbayev, M. K., and A. E. Mirzakulova. "Boundary-Value Problems with Initial Jumps for Singularly Perturbed Integrodifferential Equation." *Journal of Mathematical Sciences* 222, no. 3 (2017): 214–225. https://doi.org/10.1007/s10958-017-3294-7.

[11] Dauylbayev, M. K., and A. B. Uaissov. "Asymptotic Behavior of the Solutions of Boundary-Value Problems for Singularly Perturbed Integrodifferential Equations." *Ukrainian Mathematical Journal* 71, no. 11 (2020): 1677–1691. https://doi.org/10.1007/s11253-020-01741-w.

[12] Dauylbayev, M. K., and A. B. Uaissov. "Integral Boundary-Value Problem with Initial Jumps for a Singularly Perturbed System of Integrodifferential Equations." *Chaos, Solitons and Fractals: The Interdisciplinary Journal of Nonlinear Science, and Nonequilibrium and Complex Phenomena* 141 (2020). https://doi.org/10.1016/j.chaos.2020.110328.

[13] Bukanay, N. U., A. E. Mirzakulova, M. K. Dauylbayev, and K. T. Konisbayeva. "A Boundary Jumps Phenomenon in the Integral Boundary Value Problem for Singularly Perturbed Differential Equations." *Bulletin of the Karaganda University. Mathematics Series* 98, no. 2 (2020): 46–58. https://doi.org/10.31489/2020m2/46-58.

[14] Dauylbayev, M. K., Zh. Artykbayeva, and K. T. Konisbayeva. "Asymptotic Behavior of the Solution of a Singularly Perturbed Three-Point Boundary Value Problem with Boundary Jumps."*International Journal of Mathematics and Physics* 10, no. 2 (2019): 47–52. https://doi.org/10.26577/ijmph-2019-i2-7.

[15] Dauylbayev, M.K., and K.T. Konisbayeva. "Asymptotical Convergence of Solutions of Boundary Value Problems for Singularly Perturbed Higher-Order Integro-Differential Equations."*Quaestiones Mathematicae* 47 (2024): 2231–2246. https://doi.org/10.2989/16073606.2024.2364297.

***Авторлар туралы мәлімет:***

*Қонысбаева Куралай Талгатовна — магистр, әл-Фараби атындагы Қазақ ұлттық университетінің математика кафедрасының ага оқытушысы (Алматы, Қазақстан, электрондық пошта: kuralaimm7@gmail.com).*

*Дауылбаев Мұратхан Құдайбергенұлы (корреспондент-автор) — физика-математика ғылымдарының докторы, әл-Фараби атындагы Қазақ ұлттық университетінің математика кафедрасының профессоры (Алматы, Қазақстан, электрондық пошта: mdauylbayev@gmail.com).*

*Мирзакулова Азиза Еркомековна — PhD, әл-Фараби атындагы Қазақ ұлттық университетінің математика кафедрасының доценті (Алматы, Қазақстан, электрондық пошта: mirzakulovaaziza@gmail.com).*

***Сведения об авторах:***

*Конысбаева Куралай Талгатовна — магистр, старший преподаватель кафедры математики Казахского национального университета имени аль-Фараби (Алматы, Казахстан, электронная почта: kuralaimm7@gmail.com).*

*Даулбаев Муратхан Кудайбергенович (корреспондент-автор) — доктор физико-математических наук, профессор кафедры математики Казахского национального университета имени аль-Фараби (Алматы, Казахстан, электронная почта: mdauylbayev@gmail.com).*

*Мирзакулова Азиза Еркомековна — PhD, доцент кафедры математики Казахского национального университета имени аль-Фараби (Алматы, Казахстан, электронная почта: mirzakulovaaziza@gmail.com).*

***Information about authors:***

*Konisbayeva Kuralay Talgatovna — Master of Science, senior lecture of the Department of Mathematics, Al-Farabi Kazakh National University (Almaty, Kazakhstan, email: kuralaimm7@gmail.com).*

*Dauylbayev Muratkhan Kudaybergenovish (corresponding author) — Doctor of Physical and Mathematical Sciences, Professor of the Department of mathematics, Al-Farabi Kazakh National University (Almaty, Kazakhstan, email: mdauylbayev@gmail.com).*

*Mirzakulova Aziza Erkomekovna — PhD, Associate Professor of the Department of Mathematics, Al-Farabi Kazakh National University (Almaty, Kazakhstan, email: mirzakulovaaziza@gmail.com).*

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