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TIME OPTIMAL CONTROL FOR A DUBINS CAR WITH STATE CONSTRAINS

This paper addresses the time-optimal control problem for a kinematic model known as the Dubins car. The main goal of the study is to determine control inputs that enable the system to reach a given final state in minimum time. Such problems play an important role in optimal control theory and have applications in robotics, autonomous vehicle navigation, and trajectory planning tasks. Particular attention is given to analyzing the optimal trajectories of the Dubins car. It is known that, in the absence of state constraints, optimal trajectories have a specific structure and consist of segments of circular arcs and straight lines. These trajectories allow the system to move most efficiently while respecting the vehicle's turning radius limitations. The study also considers the problem under state constraints, which may be represented as forbidden regions in the workspace. The presence of such constraints significantly affects the shape of optimal trajectories and complicates the control problem. Analyzing these conditions allows a more complete investigation of the properties of optimal solutions and helps identify feasible modes of system motion.
Keywords: Dubins car, optimal trajectories, state constraints, time-optimal control problem, minimum time.

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Күйлік шектеулері бар Дубинс автомобилі үшін уақыт бойынша оңтайлы басқару

Бұл мақалада уақыт бойынша оңтайлы басқару мәселесі Дубинс автомобилі деп аталатын кинематикалық модель үшін қарастырылады. Зерттеудің негізгі мақсаты – жүйеге берілген соңғы күйге ең аз уақыт ішінде жетуге мүмкіндік беретін басқару әрекеттерін анықтау. Мұндай есептер оңтайлы басқару теориясында маңызды рөл атқарады және робототехникада, автономды көліктерді навигациялауда және траекторияларды жоспарлау мәселелерінде қолданылады. Арнайы назар Дубинс автомобилінің оңтайлы траекторияларын талдауға аударылады. Күйлік шектеулер жоқ жағдайда оңтайлы траекториялардың белгілі бір құрылымы бар екені белгілі және олар шеңбер доғалары мен түзу сызықтардан тұрады. Мұндай траекториялар жүйеге көліктің бұрылу радиусын ескере отырып, ең тиімді қозғалу мүмкіндігін береді. Зерттеуде сонымен қатар күйлік шектеулері бар жағдай қарастырылады, олар кеңістіктегі тыйым салынған аймақтар арқылы берілуі мүмкін. Мұндай шектеулер оңтайлы траекториялардың формасына айтарлықтай әсер етеді және басқару есептерін күрделендіреді. Бұл жағдайларды талдау оңтайлы шешімдердің қасиеттерін толық зерттеуге және жүйенің мүмкін қозғалыс режимдерін анықтауға мүмкіндік береді.
Түйін сөздер: Дубинс автомобилі, оңтайлы траекториялар, күйлік шектеулер, уақыт бойынша оңтайлы басқару есебі, ең аз уақыт.

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Задача оптимального по времени управления автомобилем Дубинса при наличии ограничений на состояние

В данной работе рассматривается задача оптимального по времени управления для кинематической модели, известной как автомобиль Дубинса. Основной целью исследования является определение таких управлений, которые позволяют системе достичь заданного конечного состояния за минимальное время. Подобные задачи играют важную роль в теории оптимального управления и находят применение в робототехнике, навигации автономных транспортных средств и задачах планирования траекторий. Особое внимание уделяется анализу оптимальных траекторий движения автомобиля Дубинса. Известно, что при отсутствии ограничений на состояние оптимальные траектории имеют специальную структуру и состоят из участков дуг окружностей и прямых линий. Такие траектории позволяют системе двигаться наиболее эффективно с учётом ограничений на радиус поворота. В работе также рассматривается задача при наличии ограничений на состояние, которые могут задаваться в виде запрещённых областей пространства. Наличие таких ограничений существенно влияет на форму оптимальных траекторий и усложняет задачу управления. Анализ этих условий позволяет более полно исследовать свойства оптимальных решений и определить возможные режимы движения системы.
Ключевые слова: автомобиль Дубинса, оптимальные траектории, ограничения на состояние, задача оптимального по времени управления, минимальное время.

1 Introduction

In 1887, A.A. Markov considered the problem described in [1], which is referred to as the time-optimal control problem for Dubins' car. In [1], the motion of the car on a plane is studied. The car moves forward with a constant linear velocity while simultaneously turning with a bounded angular velocity. The initial and final positions and orientations of the car on the plane are given. The problem consists in moving the car from the initial configuration to the final configuration in minimum time.

The development of time-optimal control for systems with bounded curvature, such as the Dubins car, originates from the early studies of optimal control and geometric path planning in the mid-twentieth century. A key milestone was the work of Lester E. Dubins, who in 1957 investigated the problem of finding the shortest path between two configurations of a vehicle that moves forward with a bounded turning radius. In his seminal article "On Curves of Minimal Length with a Constraint on Average Curvature," Dubins proved that the shortest feasible paths consist of combinations of circular arcs and straight-line segments. These trajectories are now known as Dubins paths and form the theoretical basis for many motion-planning problems involving nonholonomic vehicles ([2]). Around the same period, the foundations of optimal control theory were established through the work of Lev S. Pontryagin and his collaborators. Their formulation of the Pontryagin Maximum Principle in the late 1950s provided necessary conditions for optimality in control systems. This principle made it possible to systematically analyze time-optimal trajectories for dynamical systems with control constraints, including vehicle motion models similar to the Dubins car ([3]). In later decades, researchers extended the classical Dubins problem by incorporating additional practical considerations such as obstacles, environmental restrictions, and state constraints. Studies in the 1980s and 1990s analyzed the structure of optimal trajectories and switching conditions for curvature-constrained vehicles in constrained environments. With the rapid development of robotics, autonomous vehicles, and unmanned aerial systems in the 21st century, the Dubins vehicle model and its time-optimal variants have become central tools in motion planning and optimal trajectory generation in [4].

As a result, we obtain a time-optimal control problem for a nonlinear system.

$$\begin{cases} \dot{x}(t) = \cos\theta(t), \\ \dot{y}(t) = \sin\theta(t), \\ \dot{\theta}(t) = u, \end{cases} \tag{1}$$

$$(x, y) \in \mathbb{R}^2, \qquad \theta \in S^1, \qquad |u| \leq 1.$$

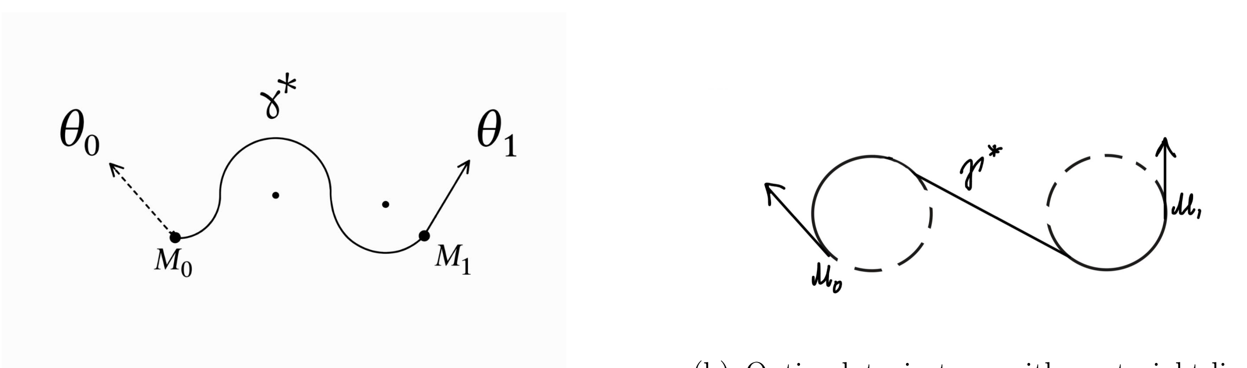
The initial and final configurations $x(0),\ y(0), x(t_1), y(t_1)$ are fixed. It is required to minimize the time t_1 by choosing the control $u = u(t) \in U$ for $0 \leq t \leq t_1$. From system (1) it follows that the linear velocity is

$$v = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2} = 1.$$

Therefore, the trajectory of the car is

- either a circle of radius 1 when $u(t) \neq 0$;
- or a straight line when $u(t) = 0$.

To minimize the time t_1, it is necessary to increase the angular velocity as much as possible, i.e., the angular velocity can take the maximum values either $u(t) = +1$ or $u(t) = -1$. When $u(t) = +1$, the car turns counterclockwise. If $u(t) = -1$, this corresponds to the car turning clockwise. If $u(t) = 0$, this corresponds to a straight-line motion with a constant linear velocity equal to one.



(a) Optimal trajectory under relay control

(b) Optimal trajectory with a straight-line segment

Figure 1: Optimal trajectory

The existence of solutions follows from Filippov's theorem [5]. The existence of solutions follows from Filippov's theorem [5] This paper describes the trajectories and corresponding controls that allow the car to be moved from the initial configuration to the final configuration.

Theorem 1 *The time-optimal control problem has a solution for any initial and final configurations.*

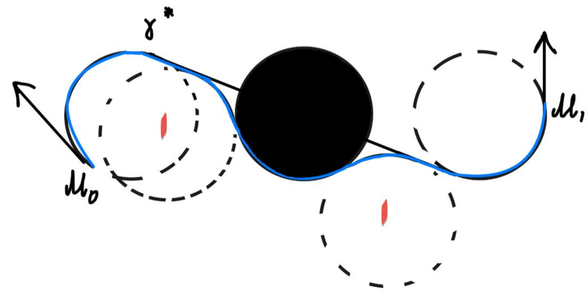


Figure 2: Optimal trajectory in a time-optimal control problem with a forbidden region

The proof of Theorem 1 can be found in the monograph [5]. Let us denote the initial configuration of the car by $M_0(x_0, y_0, \theta_0) = M_0$, and the final configuration by $M_1(x_1, y_1, \theta_1) = M_1$.

Let us introduce the following notation: let $u^*(t; M_0; M_1)$ denote the optimal control corresponding to (x_0, y_0, θ_0) as the initial position and (x_1, y_1, θ_1) as the final position of the car; let $t_1^*(M_0, M_1)$ denote the shortest time; Let $(x^*(t; u^*(t; M_0; M_1), y^*(t; u^*(t; M_0; M_1), \theta^*(t; u^*(t; M_0; M_1))$ denote the optimal trajectory of the car at $0 \le t \le t_1^*$. The following statements are known about the optimal solution [6].

Lemma 1 *For any* $t_1 \le t_1^*(M_0, M_1)$ *the equality*

$$u^*(t, M_0, M(t_1)) = u^*(t, M_0, M_1), \qquad 0 \le t \le t_1$$

holds, where

$$M(t_1) = \big(x^*(t_1, u^*(t_1, M_0, M_1)), y^*(t_1, u^*(t_1, M_0, M_1)), \theta^*(t_1, u^*(t_1, M_0, M_1))\big).$$

Moreover, part of the optimal trajectory is also optimal.

Let us present the algorithm for constructing the trajectory connecting the points M_0 and M_1. On the plane at the point (x_0, y_0) we mark a unit vector in the direction θ_0. Then we draw a perpendicular to the obtained unit vector $\vec{V_0}$. On this perpendicular, at a distance of one from the point M_0, we choose the point Q_0. Taking Q_0 as the center, we draw a circle of radius 1. The orientation of the circle is chosen according to the unit vector in the direction θ_0.Now let's move on to the endpoint M_1. On the plane at point (x_1, y_1), let's plot a unit vector in the direction θ_1. Then draw a perpendicular line to the obtained unit vector $\vec{V_1}$. On this perpendicular line, at a distance of one from the point (x_1, y_1), select the point Q_1. Taking Q_1 as the center, draw a circle of radius 1. The orientation of the circle is chosen according to the unit vector in the direction θ_1.

Next, draw a common tangent to the obtained circles. The orientation of the tangent is chosen to match the orientation of the circles.

Finally, denote the points of tangency on the circles by Q_3 and Q_4.

Thus, we obtain a trajectory connecting the initial configuration M_0 and the final configuration M_1. The circular motion begins at the point (x_0, y_0) in the direction θ_0 and continues until the point of tangency Q_3. Then the motion proceeds along the tangent to the point Q_4. At the final stage, the motion continues along the circle from the point Q_4 to the point (x_1, y_1). The corresponding graphs are shown in Figures 1a and 1b.

Now we determine which control corresponds to the obtained trajectory. On the first section of the trajectory, the control is $u(t) = 1$ if the motion along the circle occurs counterclockwise. Otherwise, the control is $u(t) = -1$. Motion along the tangent corresponds to the control $u(t) = 0$.

Thus, the optimal trajectory is selected from the trajectories connecting M_0 with M_1. The shortest of all trajectories connecting M_0 with M_1 is considered optimal.

2 Time-optimal control problem with state constraints

The second part of the article examines the time-optimal control problem for a Dubins car with phase constraints. The problem is to move the Dubin car from the initial configuration to the final configuration in the minimum time if certain trajectories are prohibited. Only permissible trajectories may be used. We assume that prohibited trajectories represent phase constraints in the form of the inequality

$$R^2 - (x-a)^2 - (y-b)^2 \leq 0 \tag{2}$$

where a, b, c, R are certain geometric parameters.

Several different cases may arise depending on the relative position of the optimal trajectory of the first time-optimal control problem and the circle (2). A permissible trajectory is one that connects M_0 to M_1 and satisfies inequality (2).

If the optimal trajectory of the time-optimal control problem without phase constraints is an admissible trajectory, then the optimal trajectory of the time-optimal control problem with constraints (2) coincides with the optimal trajectory of the time-optimal control problem without phase constraints.

In this case, we say that constraints (2) are not active.

If the optimal trajectory of the time-optimal control problem without phase constraints intersects the circle

$$(x-a)^2 + (y-b)^2 < R^2 \tag{3}$$

then constraints (2) are considered active.

In this case, it is possible to prove the existence of an optimal trajectory of the time-optimal control problem with phase constraints of the form (2).

Theorem 2 *Let $(x(t), y(t), \theta(t))$, $0 \leq t \leq t_1$ be points on the optimal trajectory of the time-optimal control problem without phase constraints that satisfy inequality (3). Then, for any initial and final configurations, there exists an optimal trajectory for the time-optimal control problem with phase constraints of the form (2), if the constraint parameters a, b, R are finite.*

Proof. For the existence of an optimal trajectory for the time-optimal control problem with phase constraints of the form (2), it is sufficient to prove the existence of at least one admissible trajectory. To do this, we select an intermediate configuration $M_2(x_2, y_2, \theta_2)$. First, we move from the initial configuration M_0 to the intermediate configuration M_2. Then we construct a trajectory from M_2 to M_1.

In this case, the intermediate configuration M_2 is chosen so that the optimal trajectory from M_0 to M_2 has no points satisfying inequality (3). This is possible because the forbidden zone satisfying inequality (3) is bounded.

Then we assume that the optimal trajectory from M_2 to M_1 also does not intersect the forbidden zone. This requirement is also feasible, since the dimensions of the forbidden zone are finite.

Thus, there exist admissible trajectories. Now it is necessary to select the optimal trajectory from all admissible trajectories. Such a selection is possible because, according to Filippov's theorem, admissible trajectories form a compact set. Theorem 2 is completely proven.

In the concluding part of the article, we present examples with different locations of the state constraints .

Figure 2 shows an example of movement from point M_0 to point M_1, where the dark spot represents the forbidden zone. Here, the forbidden zone represents a unit circle. As can be seen in Figure 3, the optimal trajectory intersects with the boundary of the forbidden zone. Similar effects are noted in the Kohn-Tucker theorem [10]. Constraints in the form of inequalities become equalities on some parts of the boundary of the forbidden zone.

3 Conclusion

The investigated Dubins problem belongs to the relevant field of mathematical modeling for controlling power engineering and robotic systems. Each system operates normally within a predefined workspace. In mathematical models, this workspace is defined by state constraints on the control inputs [7–9]. The control problems with state constraints have been systematically studied in the works of Aisagaliev [7,8], where profound theoretical results were obtained for optimal control problems with constraints in the form of both inequalities and equalities. This article presents specific examples where optimal controls are derived based on the physical interpretation of the original problem.

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