Families without Friedberg but with positive numberings in the Ershov hierarchy

Mustafa Manat
Kazakh National University, Al-Farabi ave., 71, Almaty, 050038, Kazakhstan
e-mail: manat.mustafa@gmail.com

Аннотация

Abstract

We point out that for every ordinal notation a of a nonzero ordinal, there are families of Σ_a^{-1} sets having computable positive numberings, but no computable Friedberg numberings: this answers for all levels (whether finite or infinite) of the Ershov hierarchy.

Introduction

The results of Talasbaeva's paper [9] and of this paper are partly motivated by the observation that for the arithmetical hierarchy, questions about the existence of Friedberg numberings for a family may be reduced to the existence of positive numberings. Indeed, Goncharov and Sorbi [6] show that if a family of Σ_n^0 sets, $n \geq 2$, has positive numberings then it has Friedberg numberings as well. A natural problem is to see to what extent this, or similar circumstances, carry over to the Ershov hierarchy. It is shown in [9] that for every finite level n of the Ershov hierarchy, every infinite family containing $\emptyset$ if n is even, or ω if n is odd, has infinitely many positive undecidable numberings, which are pairwise incomparable with respect to Rogers reducibility of numberings. (For $n = 1$ this was first proved by Badaev [1].) We prove something similar for all levels Σ_a^{-1} of the Ershov hierarchy, where a is the ordinal notation of any nonzero computable ordinal: in particular we show that if a is notation of an infinite computable ordinal, and $\mathcal{A}$ is an infinite family of Σ_a^{-1} sets, containing some set A which belongs to some finite level of the Ershov hierarchy, then $\mathcal{A}$ has infinitely many positive undecidable numberings, which are pairwise incomparable with respect to Rogers reducibility. As a consequence, the family of all Σ_a^{-1} sets has positive undecidable numberings, verifying Conjecture 15 of [2] for all levels of the Ershov hierarchy. (Of course, for finite levels this conjecture had been verified by Talasbaeva's theorem). A straightforward observation, derived as a consequence of Ospichev's theorem on the existence, at all levels, of families without Friedberg numberings, allows us to show also that at every level there exist families with positive numberings but without Friedberg numberings, answering negatively Question 17 of [2].

We refer to Kleene's system O of ordinal notations for computable ordinals, as presented in[8]. If $a \in O$, then the symbol $|a|_O$ indicates the ordinal denoted by a. We begin by recalling the definition of the Ershov hierarchy, [3, 4, 5]. Our characterization below is due to Ospichev [7].

Definition 1. If a is a notation for a computable ordinal, then a set of numbers A is said to be Σ_a^{-1} (or $A \in \Sigma_a^{-1}$) if there are a computable function $f(z,t)$ and a partial computable function $\gamma(z,t)$ such that, for all z,

1. $A(z) = \lim_t f(z,t)$, with $f(z,0) = 0$; (here and in the following, for a given set X, $X(x)$ denotes the value of the characteristic function of X on x;)

2. (a) $\gamma(z,t)\downarrow \Rightarrow \gamma(z,t+1)\downarrow$, and $\gamma(z,t+1) \leq_O \gamma(z,t) <_O a$;

(b) $f(z,t+1) \neq f(z,t) \Rightarrow \gamma(z,t+1) \downarrow \neq \gamma(z,t)$.

We call the partial function γ the *mind–change function* for A, relatively to f.

A Σ^{-1}_a*–approximation* to a Σ^{-1}_a–set A, is a pair $\langle f, \gamma\rangle$, where f and γ are respectively a computable function and a partial computable function satisfying 1. and 2., above, for A.

If the ordinal $|a|_O = n$ is finite, we also write Σ^{-1}_n instead of Σ^{-1}_a, as notations for finite ordinals are unique.

Following the general approach to the theory of numberings proposed by [6], we can give the following definition:

Definition 2. A Σ^{-1}_a*–computable numbering*, or simply a *computable numbering*, of a family $\mathcal{A}$ of Σ^{-1}_a–sets is an onto function $\pi : \omega \to \mathcal{A}$, such that the set

$$\{\langle k, x\rangle : x \in \pi(k)\} \in \Sigma^{-1}_a.$$

Therefore it is easy to see that a computable numbering of a family $\mathcal{A}$ of Σ^{-1}_a–sets is an onto function $\pi : \omega \to \mathcal{A}$ for which there exist a computable function $f(k, x, t)$ and a partial computable function $\gamma(k, x, t)$, such that for all k, x, t,

1. $\pi(k)(x) = \lim_t f(k, x, t)$, with $f(k, x, 0) = 0$;
2. $\gamma(k, x, t) \downarrow \Rightarrow \gamma(k, x, t+1) \downarrow$; $\gamma(k, x, t+1) \leq_O \gamma(k, x, t) <_O a$; and $f(k, x, t+1) \neq f(k, x, t) \Rightarrow \gamma(k, x, t+1) \downarrow \neq \gamma(k, x, , t)$.

We recall (see e.g. [7]) that there is an effective indexing $\{\nu_e\}_{e\in\omega}$ of all computable numberings of families of Σ^{-1}_a sets, i.e. an indexing satisfying

$$\{\langle e, k, x\rangle : x \in \nu_e(k)\} \in \Sigma^{-1}_a;$$

and from e one has (see [7]) an effective way of getting a computable function f_e and a partial computable function γ_e witnessing that the set $\{\langle k, x\rangle : x \in \nu_e(k)\}$ is Σ^{-1}_a, as in Definition 2.

We will write $Comp^{-1}_a(\mathcal{A})$ to denote the set of computable numberings of a family $\mathcal{A} \in \Sigma^{-1}_a$. A family $\mathcal{A} \in \Sigma^{-1}_a$ is *computable* if $Comp^{-1}_a(\mathcal{A}) \neq \emptyset$. If α, β are numberings of a same family, let $\alpha \leq \beta$ if there is a computable function f such that $\alpha = \beta \circ f$. The relation $\leq$ is a reducibility (called *Rogers reducibility)*, and gives rise to a degree structure, where a degree (called a *Rogers degree*) is the equivalence class of a numbering under the equivalence relation $\equiv$ generated by $\leq$: the set of Rogers degrees of the elements in $Comp^{-1}_a(\mathcal{A})$ is denoted by $\mathfrak{R}^{-1}_a(\mathcal{A})$, and called the *Rogers semilattice* of $\mathcal{A}$: it is well known that if $\mathfrak{R}^{-1}_a(\mathcal{A}) \neq \emptyset$ then $\mathfrak{R}^{-1}_a(\mathcal{A})$ is an upper semilattice. An infinite subset $X \subseteq \mathfrak{R}^{-1}_a(\mathcal{A})$ is an *antichain* if for every pair of Rogers degrees $\mathbf{a}, \mathbf{b} \in X$ we have $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{b} \not\leq \mathbf{a}$.

Definition 3. a numbering of a family $\mathcal{A}$. Then α is called a *Friedberg* numbering, if $\alpha(i) \neq \alpha(j)$ for every $i \neq j$; α is called *decidable* if $\{\langle i, j\rangle : \alpha(i) = \alpha(j)\}$ is a decidable set; α is *positive* if $\{\langle i, j\rangle : \alpha(i) = \alpha(j)\}$ is a computably enumerable (c.e.) set.

Of course, if α is a Friedberg numbering, then α is decidable; and every decidable numbering is positive. Moreover, the following obvious and well known fact holds:

Lemma. If $\mathcal{A}$ is infinite, and α is a decidable numbering of $\mathcal{A}$, then $\mathcal{A}$ has a Friedberg numbering β with $\alpha \equiv \beta$.

proof. Let $\mathcal{A}$ be infinite, and suppose that $\alpha \in Com^{-1}_a(\mathcal{A})$ is decidable. Then define $\beta \in Com^{-1}_a(\mathcal{A})$ by:

- $\beta(0) = \alpha(0)$;

- suppose that $\beta(j) = \alpha(i_j)$, all $j \leq n$, and define $\beta(n+1) = \alpha(i)$, where i is the least number such that $\alpha(i) \neq \alpha(i_j)$, for all $j \leq n$.

It follows that $\beta \in Com_a^{-1}(\mathcal{A})$, β is a Friedberg numbering, and $\beta \leq \alpha$. The converse reducibility $\alpha \leq \beta$ follows from the well known fact that the Rogers degree of every decidable (in fact, positive) numbering is minimal.

The main theorem

We now show that at each level of the Ershov hierarchy there are infinite families without Friedberg numberings, but with positive numberings.

Theorem. For every ordinal notation a, $a >_O 1$, there exists an infinite family $\mathcal{A}$ such that $Com_a^{-1}(\mathcal{A})$ has no Friedberg numberings but it has positive numberings.

proof. We show here that a slight modification of Ospichev's proof in [7] produces immediately an infinite Σ_a^{-1}–computable family without Friedberg numberings, but with a positive numbering. Let us fix a uniform effective listing $\{\nu_e\}_{e\in\omega}$ of all Σ_a^{-1}–numberings. We build a Σ_a^{-1}–computable family $\mathcal{A}$ without Friedberg numberings, by building a positive numbering α of the family. We define α by defining $f(e,x,s)$ (i.e. $\alpha(e)(x) = \lim_s f(e,x,s)$) and a corresponding mind-change function γ. (We refer again here for notations and notions, to Definition 2).

The construction is by stages. At each subsequent stage, all parameters maintain the same values as at the previous stage, unless explicitly redefined.

Stage 0. Define $f(e,x,0) = 0$, $\gamma(e,x,0) =\uparrow$, for all e, x.

Step 1. Define, for all e, e', k,

$$f(2e, 3e, 1) = f(2e+1, 3e, 1) = 1$$

and

$$\gamma(e', 3k, 1) = 1:$$

the definition $\gamma(e', 3k, 1) = 1$ (recall that $|1|_O = 0$) shows that these values $\alpha(e')(3k)$ will never be redefined. Also notice that if $k \neq e'$ then $\alpha(2e')(3k) = \alpha(2e'+1)(3k) = 0$. This will have the effect that, for $i \neq j$,

$$\alpha(i) = \alpha(j) \Rightarrow \{i,j\} = \{2e, 2e+1\}, \text{ for some } e. \quad (1)$$

Define also

$$\gamma(2e, 3e+2, 1) = \gamma(2e+1, 3e+1, 1) = 1:$$

thus the values

$$f(2e, 3e+2, 1) = f(2e+1, 3e+1, 1) = 0$$

will never be redefined. Hence eventually $3e+2 \notin \alpha(2e)$ and $3e+1 \notin \alpha(2e+1)$.

Finally define

$$f(2e, 3e+1, 1) = f(2e+1, 3e+2, 1) = 1$$

and

$$\gamma(2e+1, 3e+1, 1) = \gamma(2e+1, 3e+2, 1) = 2:$$

these values will be allowed to change at most once.

Thus at stage 1 we have reserved for $\alpha(e')$, with $e' \in \{2e, 2e+1\}$ (each e), three fixed coding locations, namely the numbers in the interval $[3e, 3e+2]$: notice that at this stage,

$$\begin{array}{lll} \alpha(2e)(3e) = 1 & \alpha(2e)(3e+1) = 1 & \alpha(2e)(3e+2) = 0 \\ \alpha(2e+1)(3e) = 1 & \alpha(2e+1)(3e+1) = 0 & \alpha(2e+1)(3e+2) = 1 : \end{array}$$

among these values, only $\alpha(2e)(3e+1)$ and $\alpha(2e+1)(3e+2)$ may change, and they are allowed to change at most once.

Stage $s > 1$. Consider all $e \leq s$. We *take action on* e if s is the first stage at which there are $i, j \leq s$, with $i \neq j$, such that

$$\nu_e(i, 3e+1, s) = \nu_e(j, 3e+2, s) = 1,$$

(we may assume without loss of generality that $\nu_e(k, t) = 0$ for all k, and all $t \leq 1$). Then pick the least such pair i, j, define

$$f(2e, 3e+1, s) = f(2e+1, 3e+2, s) = 0 :$$

these values will never be redefined and thus we set

$$\gamma(2e, 3e+1, s) = \gamma(2e+1, 3e+2, s) = 1.$$

Moreover, for all $e' \neq 2e, 2e+1$, perform the *diagonalization procedure* by letting

$$f(e', 3e+1, s) = 1 - \nu_e(i, 3e+1, s) \quad f(e', 3e+2), s) = 1 - \nu_e(j, 3e+2, s)$$

and

$$\gamma(e', 3e+1, s) = \gamma_e(i, 3e+1, s) \quad \gamma(e', 3e+2, s) = \gamma_e(j, 3e+2, s)$$

where γ_e is a mind-change function corresponding to ν_e.

If there is $t < s$ at which we took action on e, then for all $e' \neq 2e, 2e+1$ proceed with the diagonalization procedure, by letting

$$f(e', 3e+1, s) = 1 - \nu_e(i, 3e+1, s) \quad f(e', 3e+2), s) = 1 - \nu_e(j, 3e+2, s)$$

and

$$\gamma(e', 3e+1, s) = \gamma_e(i, 3e+1, s) \quad \gamma(e', 3e+2, s) = \gamma_e(j, 3e+2, s).$$

Verification. We follow closely [**?**]. Let $\mathcal{A}$ be the family numerated by α. It is straightfoward to see that $\mathcal{A}$ is Σ_a^{-1}–computable, as witnessed by the computable function f and the mind-changing function γ.

Next we show that no ν_e is a Friedberg numbering for $\mathcal{A}$. If there is no stage s at which there are distinct $i, j \leq s$ such that $\nu_e(i, 3e+1, s) = \nu_e(j, 3e+2), s) = 1$, then after stage 1 we never redefine $\alpha(2e)(3e+1)$ and $\alpha(2e+1)(3e+2)$, hence $3e+1 \in \alpha(2e)$ and $3e+2 \in \alpha(2e+1)$; on the other hand, the family enumerated by ν_e does not contain distinct sets, one of them containing $3e+1$, and the other one containing $3e+2$; thus ν_e does not enumerate $\mathcal{A}$.

Otherwise, there is a least stage at which we take action on e relatively to some pair i, j. We first observe that in this case, $\alpha(2e) = \alpha(2e+1)$. To see this let k be any number: if $k \in [3e, 3e+2]$ then $\alpha(2e)(k) = \alpha(2e+1)(k) = 0$; otherwise, let $k \in [3e', 3e'+2]$, with $e' \neq e$: if we never take action on e', then $\alpha(2e)(k) = \alpha(2e+1)(k) = 0$, as we never modify the default value taken at stage 0; if at some stage we take action on e' relatively to a pair i', j', then $\alpha(2e)(3e') = \alpha(2e+1)(3e') = 0$, $\alpha(2e)(3e'+1) = \alpha(2e+1)(3e'+1) = 1 - \nu_{e'}(i')(3e'+1)$ and $\alpha(2e)(3e'+2) = \alpha(2e+1)(3e'+2) = 1 - \nu_{e'}(j')(3e'+1)$.

There are now two cases to be considered:

1. $3e+1 \in \nu_e(i)$ or $3e+2 \in \nu_e(j)$. Assume for instance that $3e+1 \in \nu_e(i)$: the other case is treated similarly. Again we show that ν_e does not enumerate $\mathcal{A}$: if $e' \neq 2e, 2e+1$ then by diagonalization $\nu_e(i) \neq \alpha(e')$ as $\alpha(e')(3e+1) = 1 - \nu_e(i)(3e+1)$; on the other hand, $3e+1 \notin \alpha(2e+1)$ by what done at stage 1, and, by the action taken on e, $3e+1 \notin \alpha(2e)$; thus $\nu(i) \neq \alpha(2e), \alpha(2e+1)$.

2. $3e+1 \notin \nu_e(i)$ and $3e+2 \notin \nu_e(j)$. Then if ν_e numbers $\mathcal{A}$ we have that i, j are distinct indices of $\alpha(2e) = \alpha(2e+1)$, since by diagonalization $\nu_e(i)$ can only be $\alpha(2e)$ or $\alpha(2e+1)$, and similarly $\nu_e(j)$ can only be $\alpha(2e)$ or $\alpha(2e+1)$. Thus ν_e is not a Friedberg numbering.

Next, we show that $\mathcal{A}$ is infinite: this follows from the fact that if $e \neq e'$ then $\alpha(2e) \neq \alpha(2e')$ as $3e \in \alpha(2e)$ but $3e \notin \alpha(2e')$ (and symmetrically, $3e' \in \alpha(2e')$ but $3e' \notin \alpha(2e)$.

It remains to show that α is positive. Notice that, for distinct i, j, equation (1) holds, and on the other hand we have:

$$\alpha(2e) = \alpha(2e+1) \Leftrightarrow (\exists s)(\exists i, j)[i \neq j \text{ and } \nu_e(i, 3e+1, s) = \nu_e(j, 3e+2, s) = 1] :$$

indeed, we have already observed that the right-to-left implication holds, as if we take action on e at some stage, the eventually $\alpha(2e) = \alpha(2e+1)$. As to the opposite implication, the construction ensures for instance that if we never take action on e, then $\alpha(2e)(3e+1) = 1$ and $\alpha(2e+1)(3e+1) = 0$.

Question 17 of [2] asks whether, for any $n \geq 1$, families of Σ^{-1}_n sets with positive numberings have also decidable numberings. We show in fact that this is not so for every level (finite or infinite) of the Ershov hierarchy:

Corollary For every ordinal notation a of a nonzero computable ordinal, there exists a family $\mathcal{A}$ such that $Com^{-1}_a(\mathcal{A})$ has no decidable numberings but it has infinitely many positive numberings, whose Rogers degrees form an antichain.

*proof.*By Lemma, and the first proof of Theorem.

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