Description of positively existentially closed models in any class of Σ-structures with unary predicates axiomatizable by any h-universal sentence

A.M. Kungozhin
al-Farabi Kazakh National University, Almaty, Kazakhstan
e-mail: kalmas@mail.ru

Abstract

It was shown that any finitely h-universally axiomatized class of models in unary predicate signature has a finite number of positively existentially closed models, and all them are finite. Proposed the example of a class of models which described by the infinite number of h-universal sentences and its class of positively existential closed models is not elementary.

1 Introduction. Examples.

We take definitions of notions of the positive logic from [1] and repeat some of them.

We consider a signature Σ with an equality and relations, and we form as usually the first order formulas using $\neg$, $\wedge$, $\vee$ and $\exists$. *Positive* formulas are formed without negation, i.e. only using $\wedge$, $\vee$ and $\exists$; its can be written as $(\exists\overline{y})f(\overline{x},\overline{y})$, where $f(\overline{x},\overline{y})$ is a quantifier free positive formula.

A homomorphism from the Σ-structure $\mathcal{M}$ into the Σ-structure $\mathcal{N}$ is an application h from the underlying set M of $\mathcal{M}$ into the underlying set N of $\mathcal{N}$ such that, for each $\overline{a}$ from M, if $\overline{a}$ satisfies the atomic formula $A(\overline{x})$, so does $h(\overline{a})$; we do not assume the reciprocal, so that $h(\overline{a})$ may satisfy furthermore atomic formulas than $\overline{a}$, and in particular h may not be injective. If there exists an homomorphism from $\mathcal{M}$ to $\mathcal{N}$, we say that $\mathcal{N}$ is a *continuation* of $\mathcal{M}$, and that $\mathcal{M}$ is a *beginning* of $\mathcal{N}$. (We use the words extension/restriction only when h is an embedding, i.e. when $\overline{a}$ and $h(\overline{a})$ satisfy the same atomic formulas).

If h is an homomorphism, then every positive formula $(\exists\overline{y})f(\overline{x},\overline{y})$ satisfied by $\overline{a}$ is also satisfied by $h(\overline{a})$. We say that h is an *immersion* if we have the converse, that is if $\overline{a}$ and $h(\overline{a})$ satisfy the same positive formulas; we say then that $\mathcal{M}$ is *immersed*, or *positively existentially closed*, in $\mathcal{N}$. An immersion is an embedding, but positively existentially closed is weaker than the robinsonian notion, since we consider only positive existential formulas.

An *h-universal sentence* is by definition the negation of a positive sentence; it can be written $\neg(\exists\overline{y})f(\overline{y})$, or equivalently $(\forall\overline{y})\neg f(\overline{y})$, where $f(\overline{y})$ is free and positive.

If C is a class of Σ-structures, we say that an element $\mathcal{M}$ of C is *positively existentially closed* in C if every homomorphism from $\mathcal{M}$ into any member of C is an immersion.

In this work we consider the signature $\Sigma = \langle =, A_1, A_2, ..., A_m\rangle$. Where $=$ is a binary predicate of equality, and $\{A_i\}$ are unary predicates. An h-universal sentence φ may have any number of quantifiers. We consider examples and transformation of the sentence φ with only two quantifiers, but they can be easily generalized to any number of quantifiers.

First we omit the cases where Σ-structures axiomatized by false sentences:

1. $\neg\exists x\exists y(x = y) \sim \forall x\forall y(x \neq y) \sim False$.

2. $\neg\exists x\exists y[(x = y) \vee F(x, y)] \sim \forall x\forall y[(x \neq y) \wedge \neg F(x, y)] \sim False$, where $F(x, y)$ is any free formula with two variables.

Some cases could be reduced to the cases with one quantifier which author described before [2].

3. $\neg\exists x\exists yF(x) \sim \neg\exists xF(x)$, where $F(x)$ is any free formula with one variable.

4. $\neg\exists x\exists y[F(x) \vee G(y)] \sim \forall x\forall y\neg[F(x) \vee G(y)] \sim \forall x\forall y[\neg F(x) \wedge \neg G(y)] \sim \forall x[\neg F(x) \wedge \neg G(x)] \sim \neg\exists x[F(x) \vee G(x)]$, where $F(x)$ and $G(x)$ are free formulas with one variable.

5. $\neg\exists x\exists y[(x = y) \wedge F(x,y)] \sim \forall x\forall y[(x \neq y) \vee \neg F(x,y)] \sim \forall x[\neg F(x,x)] \sim \neg\exists x F(x,x)$, where $F(x,y)$ is any free formula with two variables.

We are interesting in this paper with cases which are not reduced to the case with one quantifier. For example:

6. $\neg\exists x\exists y[F(x) \wedge G(y)] \nsim \neg\exists x[F(x) \wedge G(x)]$.

But we can consider this sentence as a disjunction of two sentences since:

6'. $\neg\exists x\exists y[F(x) \wedge G(y)] \sim [\neg\exists x F(x)] \vee [\neg\exists x G(x)]$.

2 Transformations of the formula φ with unary predicates in general case.

We can present any h-universal sentence φ with n quantifiers and unary predicates in the following form:

$$\varphi = \neg\exists^{(n)}\overline{x} \bigvee [\bigwedge A_i(x_1) \wedge \bigwedge A_i(x_2) \wedge ... \wedge \bigwedge A_i(x_n) \wedge \bigwedge (x_s = x_s) \wedge \bigwedge (x_s = x_t)]$$

where $\{A_i\}$ - unary predicates, and some atomic subformulas may be absent.

If in some conjunction exists only identically true subformula(s) $x_s = x_s$ then this sentence φ is false (see Examples 1 and 2). Otherwise we can eliminate the identically true subformulas $x_s = x_s$ since $\psi \wedge True = \psi$.

If in some conjunction exist only identities $x_s = x_t$ $(s \neq t)$ then this sentence φ is false. Since $\neg\exists^{(n)}\overline{x}[[\bigwedge(x_s = x_t)] \vee F(\overline{x})] \sim \forall^{(n)}\overline{x}[[\bigvee(x_s \neq x_t)] \wedge \neg F(\overline{x})] \sim False$. Otherwise we can eliminate the identities $x_s = x_t$ with a little changing of some variables since, for example $[\bigwedge A_i(x_s) \wedge \bigwedge A_j(x_t) \wedge (x_s = x_t)] \sim [\bigwedge A_i(x_s) \wedge \bigwedge A_j(x_s)]$.

Consider the next equal transformations of the consistent sentence φ:

$$\varphi = \neg\exists^{(n)}\overline{x} \bigvee [\bigwedge A_i(x_1) \wedge \bigwedge A_i(x_2) \wedge ... \wedge \bigwedge A_i(x_n)] \sim$$

$$\sim \forall^{(n)}\overline{x} \bigwedge \neg[\bigwedge A_i(x_1) \wedge \bigwedge A_i(x_2) \wedge ... \wedge \bigwedge A_i(x_n)] \sim$$

$$\sim \bigwedge \forall^{(n)}\overline{x}\neg[\bigwedge A_i(x_1) \wedge \bigwedge A_i(x_2) \wedge ... \wedge \bigwedge A_i(x_n)] \sim$$

$$\sim \bigwedge \forall^{(n)}\overline{x}[\bigvee \neg A_i(x_1) \vee \bigvee \neg A_i(x_2) \vee ... \vee \bigvee \neg A_i(x_n)] \sim$$

$$\sim \bigwedge [\forall x \bigvee \neg A_i(x) \vee \forall x \bigvee \neg A_i(x) \vee ... \vee \forall x \bigvee \neg A_i(x)] \sim$$

$$\sim \bigvee \bigwedge [\forall x \bigvee \neg A_i(x)] \sim \bigvee \forall x \bigwedge [\bigvee \neg A_i(x)] \sim$$

$$\sim \bigvee \forall x \bigvee [\bigwedge \neg A_i(x)] \sim \bigvee \neg\exists x \bigwedge [\bigvee A_i(x)].$$

In a similar way for any finite number of h-universal axioms we can get an equivalent disjunction of h-universal sentences with one quantifier each.

3 Structure of the positively existentially closed models in the class of Σ-structures with unary predicates axiomatizable by any h-universal sentence (special case).

Before to start consideration of general case lets begin with special case to understanding better general case after.

Let axiomatizing sentence φ has a next structure:

$$\varphi = \bigvee \neg\exists x \bigwedge A_i(x) \sim \bigvee \forall x \bigvee \neg A_i(x).$$

According to the results of the case with one quantifier we know that if the model $\mathcal{M} = \langle M, =, A_1, A_2, ..., A_n \rangle$ is positively existentially closed in the class of Σ-structures axiomatizable by the h-universal subsentence $\neg\exists x \bigwedge_{i\in I} A_i(x) \sim \forall x \bigvee_{i\in I} \neg A_i(x)$ then the number of elements of the underlying set M is equal to $|I|$; for any element $a \in M$ exists a single relation A_{i_a} $(i_a \in I)$ which is not satisfied in the model $\mathcal{M}$ by the element a and other relations are satisfied in the model $\mathcal{M}$ by this element; for any relation A_i $(i \in I)$ exists a single element $a_i \in M$ by which this relation is not satisfied in the model $\mathcal{M}$ and this relation is satisfied in the model $\mathcal{M}$ by other elements.

But the sentence is a disjunction of such subsentences. It means that each Σ-structure axiomatizable at least by one of these subsentences. Each subsentence has the own single positively existentially closed model. But some of them are continuations of others (the corresponding e.c. model of the subsentence $\forall x \bigvee_{i\in I} \neg A_i(x)$ is a continuation of the corresponding e.c. model of the subsentence $\forall x \bigvee_{i\in J} \neg A_i(x)$ if and only if $J \subset I$).

Proposition If the model $\mathcal{M} = \langle M, =, A_1, A_2, ..., A_n \rangle$ is positively existentially closed in the class of Σ–structures with unary predicates axiomatizable by any h-universal sentence $\varphi = \bigvee_I \forall x \bigvee_{i\in I} \neg A_i(x)$ then it has the corresponding subsentence $\forall x \bigvee_{i\in I} \neg A_i(x)$ with maximal set of indexes I; the number of elements of the underlying set M is equal to the number of predicates met in the corresponding subsentence ($|I|$); for any element $a \in M$ exists a single relation A_{i_a} $(i_a \in I)$ which is not satisfied in the model $\mathcal{M}$ by the element a and other relations are satisfied in the model $\mathcal{M}$ by this element; for any relation A_i met in the corresponding subsentence exists a single element $a_i \in M$ which does not satisfy this relation in the model $\mathcal{M}$ and this relation is satisfied in the model $\mathcal{M}$ by other elements. The number of such positively existentially closed models is equal to the number of such maximal sets I.

Proof: Let $\mathcal{M} = \langle M, =, A_1, A_2, ..., A_n \rangle$ be a positively existentially closed model in the class of Σ-structures axiomatizable by the h-universal sentence φ.

Since the sentence φ is satisfied in the model $\mathcal{M}$ then for any element $a \in M$ there is at least one relation A_{i_a} which is not satisfied in the model $\mathcal{M}$ by the element a. Suppose that there is one more relation A_{j_a} which is not satisfied in the model $\mathcal{M}$ by the element a. Consider the model $\mathcal{N}$ with the same underlying set where predicates same defined like in the model $\mathcal{M}$ except that relation A_{j_a} which is satisfied in the model $\mathcal{N}$ by the element a. Then the identical map is a homomorphism. But the formula $A_{j_a}(x)$ is satisfied in the model $\mathcal{N}$ and not satisfied in the model $\mathcal{M}$ by the element a. It means that the model $\mathcal{M}$ is not positively existentially closed, contradiction.

Suppose that some predicate A_i is not satisfied by at least two elements a_1 and a_2 from the underlying set M. One can easily construct a homomorphism h from the model $\mathcal{M}$ into the model $\mathcal{N} = \langle N, =, A_1, A_2, ..., A_n \rangle$ where a_1 and a_2 correspond to one element, other elements are saved, all relations are saved. Since it is not an injunction the formula $x_1 = x_2$ could not be satisfied in the models $\mathcal{M}$ and $\mathcal{N}$ by $\overline{a} = (a_1, a_2)$ and $h(\overline{a})$ the same time, contradiction.

Let S be a set of indexes such that $i \in S$ iff $A_i(a)$ is not satisfied by some element $a \in M$. If exists $i \in S$ which does not belong to some set I of indexes of subsentence $\forall x \bigvee_{i\in I} \neg A_i(x)$ then this subsentence is not satisfied in the model $\mathcal{M}$. If the set S is not a subset of each maximal set I of indexes then the sentence φ is not satisfy in the model $\mathcal{M}$, contradiction. Then S is a subset of some maximal set I of indexes.

Let the set S be a proper subset of maximal set I of indexes. Consider the existentially closed model $\mathcal{N}$ corresponding to the subsentence $\forall x \bigvee_{i\in I} \neg A_i(x)$. One can easily construct a homomorphism h from the model $\mathcal{M}$ into the model $\mathcal{N}$. But the positive formula $\exists y \bigwedge_{i\in S} A_i(y)$ is satisfied in the model $\mathcal{N}$ and not satisfied in the model $\mathcal{M}$, contradiction.

Finally it is easy to show that each existentially closed model $\mathcal{M}$ corresponding to the subsentence $\forall x \bigvee_{i\in I} \neg A_i(x)$ of maximal set I of indexes is existentially closed in the class of Σ-structures axiomatizable by the h-universal sentence φ.

4 Structure of the positively existentially closed model in the class of Σ-structures with unary predicates axiomatizable by any h-universal sentence (general case).

Let axiomatizing sentence φ has a next structure: $\varphi = \bigvee \neg\exists x \bigwedge[\bigvee A_i(x)] \sim \bigvee \forall x \bigvee[\bigwedge \neg A_i(x)]$.

According to the results of the case with one quantifier we know that if the model $\mathcal{M} = \langle M, =, A_1, A_2, ..., A_n\rangle$ is positively existentially closed in the class of Σ-structures axiomatizable by the h-universal subsentence $\neg(\exists x) \bigwedge_{i\in I}[\bigvee_{j\in J_i} A_j(x)]$ (remember that such presentation should be minimal, i.e. any disjunction could not be a subformula of any other disjunction, otherwise it can be reduced by the rule $\phi \wedge (\phi \vee \psi) = \phi$) then the number of elements of the underlying set M is equal to $m = |I|$: $\{a_1, a_2, ..., a_m\}$; for any element $a_i \in M$ the relation $A_l(a_i)$ is not satisfied in the model $\mathcal{M}$ by the element a_i if and only if the predicate A_l presented in the i-th disjunction of the subsentence, otherwise the relation $A_l(a_i)$ is satisfied in the model $\mathcal{M}$.

Actually each element a_i has an own type: a maximal set of atomic formulas $A_l(x)$ which is consistent with a subsentence. Each subsentence defines the set of types. Types of each subsentence could not be subsets of each other.

But the sentence φ is a disjunction of such subsentences. It means that each Σ-structure axiomatizable at least by one of these subsentences. Each subsentence has the own single positively existentially closed model.

Some of them are continuations of others. The corresponding e.c. model of the subsentence $\neg(\exists x) \bigwedge_{i\in I_1}[\bigvee_{j\in J_i} A_j(x)]$ is a continuations of the corresponding e.c. model of the subsentence $\neg(\exists x) \bigwedge_{i\in I_2}[\bigvee_{j\in J_i} A_j(x)]$ if and only if the set of types I_1 dominates the set of types of I_2.

Set of sets $\{S_i\}_{i\in I}$ *dominates* the set of sets $\{T_j\}_{j\in J}$ iff for any set T_j exists a set S_i such that $T_j \subset S_i$.

Proposition If the model $\mathcal{M} = \langle M, =, A_1, A_2, ..., A_n\rangle$ is positively existentially closed in the class of Σ-structures with unary predicates axiomatizable by any h-universal sentence φ then it has the corresponding subsentence $\neg(\exists x) \bigwedge_{i\in I}[\bigvee_{j\in J_i} A_j(x)]$ with a dominating set of indexes $\{J_i\}_{i\in I}$; the number of elements of the underlying set M is equal to $m = |I|$: $\{a_1, a_2, ..., a_m\}$; for any element $a_i \in M$ the relation $A_l(a_i)$ is not satisfied in the model $\mathcal{M}$ by the element a_i if and only if the predicate A_l presented in the i-th disjunction of the subsentence, otherwise the relation $A_l(a_i)$ is satisfied in the model $\mathcal{M}$. The number of such positively existentially closed models is equal to the number of such dominating sets of types.

Proof: Let $\mathcal{M} = \langle M, =, A_1, A_2, ..., A_n\rangle$ be a positively existentially closed model in the class of Σ-structures axiomatizable by the h-universal sentence φ.

Each element from the underlying set M has a type. These types should be different for each element, otherwise we can construct noninjective homomorphism from the model $\mathcal{M}$ into the model

$\mathcal{N}$ with different types of elements.

Since the sentence φ is satisfied in the model $\mathcal{M}$ then some subsentences $\neg(\exists x)\bigwedge_{i\in I}[\bigvee_{j\in J_i} A_j(x)]$ are satisfied in the model $\mathcal{M}$. Then the set of types of the positively existentially closed model corresponding to each subsentence $\neg(\exists x)\bigwedge_{i\in I}[\bigvee_{j\in J_i} A_j(x)]$ should dominate the set of types of the model $\mathcal{M}$, i.e. we can construct homomorphisms from the model $\mathcal{M}$ to the e.c. models corresponding to these subsentences. Since the model $\mathcal{M}$ is existentially closed then all these homomorphisms should be immersions, i.e. dominations should be noninjective. Otherwise we can not construct an injective homomorphism and formula $x_1 = x_2$ is satisfied in continuation and not satisfied in the model $\mathcal{M}$.

Since the model $\mathcal{M}$ is existentially closed the set of types of its elements could not be strictly dominated by the set of types of the e.c. model corresponding to any subsentence $\neg(\exists x)\bigwedge_{i\in I}[\bigvee_{j\in J_i} A_j(x)]$ of the sentence φ. Otherwise the sentence $\exists^{(|I|)}\overline{y}\bigwedge_{i\in I}\bigwedge_{j\in\overline{J_i}} A_j(y_i)$ is satisfied in the corresponding e.c. model but not satisfied in the model $\mathcal{M}$, contradiction.

Finally it is easy to show that each existentially closed model $\mathcal{M}$ corresponding to the subsentence $\neg(\exists x)\bigwedge_{i\in I}[\bigvee_{j\in J_i} A_j(x)]$ of dominating set of types is existentially closed in the class of Σ-structures axiomatizable by the h-universal sentence φ.

From the last proposition easily follows the following theorem.

Theorem 1 *Any finitely h-universally axiomatized class of models in unary predicate signature has a finite number of positively existentially closed models, and all them are finite.*

5 The example of the class of models axiomatizabled by the h-universal sentences which class of positively existential closed models is not elementary.

The next example shows that there is a class of models which described by the infinite number of h-universal sentences and its class of positively existential closed models is not elementary. This example proposed by professor Bruno Poizat which improves proposed example by author and contains only unary predicates.

Example Let A_i - unary relations, $i \in \omega$. Let this class is described by the following sentences: some elements have properties A_i but only one of them: $(\neg\exists x(A_i(x)\&A_j(x)),\ i \neq j \in \omega)$. Then the subclass of positively existential closed models of this class is not elementary.

Proof: It is easy to understand that that the class of positively existential closed models of this class consists of models $\mathcal{M}$ such that: 1) M contains infinite number of elements with property A_i for each $i \in \omega$, 2) there are not other elements in the model $\mathcal{M}$.

Let understand that this class is not axiomatizable. Indeed, all such models does not contain elements which have not properties A_i. However, the model $\mathcal{N}$, which is the extension of a positively existential closed model $\mathcal{M}$ by a single isolated element c will be consistent with the theory T of the model $\mathcal{M}$. Lets prove it. It is enough to realize that the model $\mathcal{N}$ is compatible with any finite part of the theory T. Since the finite part of the theory T contains a limited number of signature symbols A_i and to describe the isolation of the element c necessary infinite number of formulas, then the model $\mathcal{N}$ is consistent with this final part of a theory T.

REFERENCE

[1] *Ben Yaacov I., Poizat B.*, Fondaments de la Logique Positive, The Journal of Symbolic Logic, vol. 82 (2007), pp. 1141-1162.

[2] *Kungozhin A.* Description of positively existentially closed models in any class of Σ-structures axioma-tizable by the finite number of h-universal sentences with one quantifier, Preprint.