# Systems of first order partial differential equations with singular lines[1]

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**Abstract**

The main purpose of present paper consists in investigation of an systems of partial differential equations with singular lines in the plane. For such systems the initial value problem is solved.

## 1 Introduction

Let $0 < \varphi_0 \leq 2\pi,\ 0 < \varphi_1 < \varphi_2 < \varphi_0,\ 0 < R < \infty,\ k_1 = tg\varphi_1,\ k_2 = tg\varphi_2,\ 0 < \alpha < 1,\ \nu > 0$ is real numbers and $G = \{z = re^{i\varphi} : 0 \leq r \leq R,\ \ 0 \leq \varphi \leq \varphi_0\}$. $W_p^1(G)$ is the Sobolev space (see [1]).

We consider the equation

$$2\bar{z}a_1(\varphi)\frac{\partial w}{\partial \bar{z}} + 2za_2(\varphi)\frac{\partial w}{\partial z} + \frac{r^\alpha a_3(\varphi)w}{|y - k_1x|^\alpha} + b(\varphi)\bar{w} = \frac{f(r,\varphi)\cdot r^\alpha}{|y - k_2x|^\alpha}, \qquad (1)$$

in $G$, where $a_1(\varphi)$, $a_2(\varphi)$, $a_3(\varphi)$, $b(\varphi) \in C\,[0,\varphi_0]$ and the function $f(r,\varphi)$ satisfies condition (A).

Condition (A): the function $f(r,\varphi)$ is represented as

$$f(r,\varphi) = \sum_{k=0}^{\infty} f_k(\varphi)r^{\nu k},\ \nu > 0,\ f_k(\varphi) \in C[0,\varphi_0],$$

and $\sum\limits_{k=0}^{\infty} |f_k|_0\, r^{\nu k}$ is convergent series in $G$, where $|f|_0 = \|f\|_{C[0,\varphi_0]}$.

Let $1 < p < \frac{1}{1-\nu}$, if $\nu < 1$ and $p \geq 1$, if $\nu \geq 1$. We find the solution of equation (1) in the class

$$W_p^1(G)\bigcap C(\overline{G}) \qquad (2)$$

If we divide the equation (1) by $2\bar{z}a_1(\varphi)$, then it becomes the elliptic equation under $|a_2(\varphi)| < |a_1(\varphi)|$. For $a_2(\varphi) \equiv 0$, $f(r,\varphi) \equiv 0$ and $\alpha = 0$ the obtained these equation is studied in the articles $[2-6]$ and has important application in the theory of infinitesimal bending of surfaces of positive curvature with a point of flattening [4]. For $\alpha \neq 0$ the equation (1) is not study.

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Using the formulas

$$\frac{\partial}{\partial \bar{z}} = \frac{e^{i\varphi}}{2}(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}), \quad \frac{\partial}{\partial z} = \frac{e^{-i\varphi}}{2}(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}),$$

the equation (1) can be written in polar coordinates

$$r(a_1(\varphi) + a_2(\varphi))\frac{\partial w}{\partial r} + i(a_1(\varphi) - a_2(\varphi))\frac{\partial w}{\partial \varphi} + \frac{a_3(\varphi)w}{|\sin\varphi - k_1\cos\varphi|^\alpha}$$
$$+b(\varphi)\bar{w} = \frac{f(r,\varphi)}{|\sin\varphi - k_2\cos\varphi|^\alpha}. \tag{3}$$

We will search a solution of equation (3) in the form

$$w(r,\varphi) = \sum_{k=0}^{\infty} w_k(\varphi) r^{\nu k}, \tag{4}$$

where $w_k(\varphi)$, $\left(k = \overline{0,\infty}\right)$ are new unknown functions from $C^1[0,\varphi_0]$, such that $w(r,\varphi)$ is satisfying condition (A) and $\sum\limits_{k=0}^{\infty} w_k'(\varphi) r^{\nu k}$ is convergent series in $G$.

By substituting the expression (4) for $w(r, \varphi)$ into the formula (3), we get

$$i(a_1(\varphi) - a_2(\varphi))w_k' + (k\nu(a_1(\varphi) + a_2(\varphi)) + \frac{a_3(\varphi)}{|\sin\varphi - k_1\cos\varphi|^\alpha})w_k + b(\varphi)\bar{w}_k =$$

$$= \frac{f_k(\varphi)}{|\sin\varphi - k_2\cos\varphi|^\alpha}.$$

From the last equation it follows, then for $a_1(\varphi) = a_2(\varphi)$ the function $w_k(\varphi)$ is not bounded in $[0, \varphi_0]$ and hence is not exist from the class (2) the solution for the equation (3) in the form (4). Let us assume $a_1(\varphi) \neq a_2(\varphi)$ from last the equation we get

$$w_k' + A_k(\varphi) \cdot w_k = b_1(\varphi) \cdot \overline{w_k} + F_k(\varphi), \tag{5}$$

where

$$A_k(\varphi) = a_4(\varphi) - i\nu k a_5(\varphi), a_4(\varphi) = \frac{-ia_3(\varphi)}{|\sin\varphi - k_1\cos\varphi|^\alpha (a_1(\varphi) - a_2(\varphi))},$$

$$a_5(\varphi) = \frac{a_1(\varphi) + a_2(\varphi)}{a_1(\varphi) - a_2(\varphi)}, b_1(\varphi) = \frac{ib(\varphi)}{a_1(\varphi) - a_2(\varphi)},$$

$$F_k(\varphi) = -\frac{if_k(\varphi)}{|\sin\varphi - k_2\cos\varphi|^\alpha \cdot (a_1(\varphi) - a_2(\varphi))}.$$

Using the transformation

$$w_k = \psi_k \cdot \exp\left(\int_0^\varphi A_k(\gamma)d\gamma\right) \tag{6}$$

the equation (5) is transferred into the form

$$\psi_k' = g_k(\varphi) \cdot \overline{\psi_k} + h_k(\varphi), \tag{7}$$

where

$$g_k(\varphi) = b_1(\varphi) \cdot \exp(-2i \int_0^\varphi ImA_k\,(\gamma)\,d\gamma), h_k(\varphi) = F_k(\varphi) \cdot \exp(-\int_0^\varphi A_k\,(\gamma)\,d\gamma).$$

Integrating the equation (7) we get

$$\psi_k(\varphi) = \int_0^\varphi g_k(\gamma) \cdot \overline{\psi_k(\gamma)}\,d\gamma + \int_0^\varphi h_k(\gamma)\,d\gamma + c_k,$$

where $c_k$ is any complex number.

As $0 < \alpha < 1$, $a_1(\varphi) \neq a_2(\varphi)$ the integrals of the last equation are convergent.

Defining

$$(B_k f)(\varphi) = \int_0^\varphi g_k(\gamma)\overline{f(\gamma)}d\gamma,\ H_k(\varphi) = \int_0^\varphi h_k(\gamma)d\gamma$$

the last equation can be written in the form

$$\psi_k(\varphi) = (B_k\psi_1)(\varphi) + H_k(\varphi) + c_k. \tag{8}$$

For solving the equation (8) we use the iterated scheme

$$(B_k^0 f)(\varphi) = f(\varphi),\ (B_k^n\psi)(\varphi) = (B_k(B_k^{n-1}\psi))(\varphi),\ n = (\overline{1,\infty}),$$

and the family of functions $\{I_{k,n}(\varphi)\}$, $(n = \overline{1,\infty})$ which are defined by

$$I_{k,n}(\varphi) = \int_0^\varphi g_k(\gamma)\overline{I_{k,n-1}(\gamma)}d\gamma,\ I_{k,1}(\varphi) = \int_0^\varphi g_k(\gamma)d\gamma,\ n = (\overline{2,\infty}).$$

Using the algorithm solving the solutions, such described in [2, 3] we have get

$$\psi_k(\varphi) = \bar{c}_k P_{k,1}(\varphi) + c_k P_{k,2}(\varphi) + H_{k,1}(\varphi), \tag{9}$$

where

$$P_{k,1}(\varphi) = \sum_{j=1}^\infty I_{k,2j-1}(\varphi),\ P_{k,2}(\varphi) = 1 + \sum_{j=1}^\infty I_{k,2j}(\varphi),\ H_{k,1}(\varphi) = \sum_{j=0}^\infty (B_k^j H_k)(\varphi).$$

Taking into consideration the definition of the operators $(B_k^j f)(\varphi)$ and the function $I_{k,n}(\varphi)$ the following estimates are obtained:

$$\begin{gathered} |P_{k,1}(\varphi)| \leq |b_1|_0 \cdot \cosh(|b_1|_0 \cdot \varphi),\ |P_{k,2}(\varphi)| \leq 1 + |b_1|_0 \cdot \sinh(|b_1|_0 \cdot \varphi), \\ |H_{k,1}(\varphi)| \leq |H_k|_0 \exp(|b_1|_0\,\varphi). \end{gathered} \tag{10}$$

From (4), (6), (8) and (9) it follows

$$w\,(r,\varphi)=\sum_{k=0}^{\infty} r^{\nu k}\cdot \exp(-\int_0^{\varphi} A_\nu(\gamma)d\gamma)\cdot(\bar{c}_k\ P_{k,1}(\varphi)+c_k\ P_{k,2}(\varphi)+H_{k,1}(\varphi))\,. \qquad (11)$$

Here $c_k$, $k=\overline{(1,\infty)}$ are any complex numbers, such that the series $\sum\limits_{k=0}^{\infty} c_k\cdot r^{\nu k}$ is convergent. Let $Im\left(\frac{a_1(\varphi)+a_2(\varphi)}{a_1(\varphi)-a_2(\varphi)}\right)=0$. From (10) it follows that the function $w(r,\,\varphi)$ given by the formula (11) is satisfying condition (A) and belong to the class (2). Using the estimates (10) we get

$$|w\,(r,\varphi)|\le e_1\cdot\sum_{k=0}^{\infty}|c_k|\,r^{\nu k}+e_2\cdot\exp(|b_1|_0)\sum_{k=0}^{\infty}|H_k|_0\,r^{\nu k},$$

where

$$e_1=\max_{0\le\varphi\le\varphi_0}\left(|sh(|b_1|_0\cdot\varphi)|+|ch(|b_1|_0\cdot\varphi)|+1\right),$$

$$e_2=\max_{0\le\varphi\le\varphi_0}|\int_0^{\varphi}\frac{\exp\left(a_4(\gamma)\right)d\gamma}{|\sin\gamma-k_2\cos\gamma|^{\alpha}\left(a_1(\gamma)-a_2(\gamma)\right)}|.$$

Summarizing the following theorem is proved.

**Theorem 1** *The equation (1) as $a_1(\varphi)\neq a_2(\varphi)$ and $Im a_5(\varphi)=0$ has an infinite number of solutions from the class (2). These solutions are given by (11).*

Now let us consider the following initial value problem.

**Problem C.** *It is necessary to find a solution $w(r,\varphi)$ of the equation (1) from the class (2) satisfying the conditions*

$$w(r,0)=\beta_1 r^{\nu},\ \frac{\partial^k}{\partial\rho^k}w(r,\varphi)|_{\substack{r=0\\ \varphi=0}}=a_k,\ k=\overline{(1,\,\infty)}, \qquad (12)$$

*where $\rho=r^{\nu}$, $a_k$ are given complex numbers, such that the series $\sum\limits_{k=1}^{\infty}\frac{a_k}{k!}$ is convergent.*

Substituting the formula (11) in the conditions (12) we have get

$$c_k=\frac{a_k}{k!},\ (k=\overline{0,\infty}). \qquad (13)$$

Therefore the following theorem holds.

**Theorem 2** *For $a_1(\varphi)\neq a_2(\varphi)$, $Im a_5(\varphi)=0$ the problem C has only the solution, which can be found by the formulas (11) and (13).*

## 2 Conclusion

In this article the manyfold of solution for one systems of first order partial differential equations with singular lines is obtained in explicit form in any angular domain and the initial value problem for this systems is solved.

## Reference

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*Д.К. Ахмед-Заки*, *Система дифференциальных уравнений в частных производных первого порядка с сингулярными прямыми*, Вестник КазНУ, сер. мат., мех., инф. 2011, №4(71), 3 – 7

В статье получено многообразие непрерывных решений одного класса систем дифференциальных уравнений в частных производных первого порядка с оператором Фукса в дифференциальной части и с сингулярными прямыми. Для таких систем решена задача типа Коши.

*Д.К. Ахмед-Заки*, *Сингулярлы сызықтарымен бірінші ретті дербес туындылы дифференциалды теңдеулер жүйелері.*, ҚазҰУ хабаршысы, мат., мех., инф. сериясы 2011, №4(71), 3 – 7

Мақалада сингулярлы сызықтары және дифференциалды бөлігінде Фукс операторы бар бірінші ретті дербес туындылы дифференциалды теңдеулер жүйелерінің үзіліссіз шешімдері алынды. Осы жүйелер үшін Коши есебі шешілген.