Note on cardinality of Rogers semilattice in the Ershov hierarchy[1]

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Abstract

The paper is aimed to show that we can easily construct a family consisting of any given number of elements whose Rogers semilattice consists of one element.

Introduction

Computable numberings of a family of c.e. sets can be considered as the uniform enumeration procedures for the sets of the family. Monotonicity is the most salient features of such computations. This means that every number (as a piece of information) enumerated in any set via an uniform enumeration procedure never leaves this set in the future. Thus, information accumulated in every set of the family is growing more and more, i.e. monotonously with respect to growth by the time. In contrary with the classical case, during computations via computable numbering of a family in the Ershov hierarchy, any number could enter a set and, later, it could leave the set, and after that again enter it and *etc.* Number of these 'enter-leaves' is bounded by the level of the hierarchy.

For unexplained notions and results on the theory of numberings, the reader is referred to [6]. In a nutshell, Goncharov and Sorbi's proposal, [7], for generalizing the theory of numberings to different notions of computability consists in the following. Let $\mathcal{C}$ be an abstract "notion" of computability, i.e. a countable class of sets of numbers, and let $\mathcal{A} \subseteq \mathcal{C}$: then a numbering $\pi : \omega \to \mathcal{A}$ is $\mathcal{C}$*–computable*, if $\{\langle k, x\rangle : x \in \pi(k)\} \in \mathcal{C}$. On numberings α, β of a family $\mathcal{A}$, one defines $\alpha \leq \beta$ if there is a computable function f such that $\alpha = \beta \circ f$; and $\alpha \equiv \beta$ if $\alpha \leq \beta$ and $\beta \leq \alpha$; for $\mathcal{A} \subseteq \mathcal{C}$, we denote by $Com_{\mathcal{C}}(\mathcal{A})$ the set of $\mathcal{C}$–computable numberings of $\mathcal{A}$; we say that $\mathcal{A}$ is $\mathcal{C}$*–computable* if $Com_{\mathcal{C}}(\mathcal{A}) \neq \emptyset$; finally we denote by $\mathfrak{R}_{\mathcal{C}}(\mathcal{A})$ the set of *Rogers degrees* of the elements of $Com_{\mathcal{C}}(\mathcal{A})$, i.e. the set $Com_{\mathcal{C}}(\mathcal{A})/\equiv$; it can be shown that $\mathfrak{R}_{\mathcal{C}}(\mathcal{A})$, if nonempty, is an upper semilattice.

Historically, the first two problems on the Rogers semilattices of the families of c.e. sets were raised by Yu.L. Ershov: What is a possible cardinality of a Rogers semilattice? Can a Rogers semilattice be a lattice? There is a sharp distinction in the Rogers semilattices of the families of Σ^{-1}_n–sets for $n > 1$ and Rogers semilattices of Σ^0_n–sets for $n > 0$. It should be noted that nothing on a possible cardinality of the Rogers semilattices in the case of Ershov's hierarchy. Also the methods employed in the theorems of Khutoretsky [9], and Goncharov–Sorbi [7] are of no use in the case of computability in the hierarchy of Ershov. Non-monotonicity of computations in this hierarchy prevents anybody to use these methods for resolution the problem of cardinality as well as many other problems. The question on a cardinality of the Rogers semilattices in the case of Ershov's hierarchy seems to be non-trivial.

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We now briefly review the basic notions concerning ordinal notations, and the Ershov hierarchy. We refer to Kleene's system O of ordinal notations for computable ordinals: for details, see [11]. In particular, for $a \in O$, the symbol $|a|_O$ represents the ordinal of which a is a notation; the symbol $<_O$ denotes Kleene's partial ordering relation on O; moreover, the symbol $+_0$ denotes a partial computable function, defined on O, such that $|a +_O b|_O = |a|_O + |b|_O$, and $a \leq_O a +_O b$. We now briefly recall the definition of the Ershov hierarchy, introduced in [3, 4, 5]. Our presentation is based on [8].

Definition 1.1 *If a is a notation for a computable ordinal, then a set of numbers A is said to be Σ^{-1}_a if there are a computable function $f(z,t)$ and a partial computable function $\gamma(z,t)$ such that, for all z,*

1. *$A(z) = \lim_t f(z,t)$, with $f(z,0) = 0$; (here, given a set X, and a number z, the symbol $X(z)$ denotes the value of the characteristic function of X on z);*
2. *(a) $\gamma(z,t)\downarrow \Rightarrow \gamma(z,t+1)\downarrow \& \gamma(z,t+1) \leq_O \gamma(z,t) <_O a$;*

   (b) $f(z,t+1) \neq f(z,t) \Rightarrow \gamma(z,t+1)\downarrow \neq \gamma(z,t)$.

We call the partial function γ the mind–change function for A, relatively to f.

A Σ^{-1}_a–approximation *to a Σ^{-1}_a–set A, is a pair $\langle f, \gamma\rangle$, where f and γ are respectively a computable function and a partial computable function satisfying 1. and 2., above, for A.*

Following [7], we give the following:

Definition 1.2 A Σ^{-1}_a–computable numbering *of a family $\mathcal{A}$ of Σ^{-1}_a–sets is an onto function $\pi : \omega \longrightarrow \mathcal{A}$, such that*

$$\{\langle k, x\rangle : x \in \pi(k)\} \in \Sigma^{-1}_a.$$

Hence there exist a computable function $f(z,t)$ and a partial computable function $\gamma(z,t)$, such that $\pi(k)(x) = \lim_t f(\langle k, x\rangle, t)$, with $f(z,0) = 0$ for all z; and γ is the mind-change function for $\{\langle k, x\rangle : x \in \pi(k)\}$ relatively to f.

In the rest of the paper we will write $Com^{-1}_a(\mathcal{A})$ for $Com_{\Sigma^{-1}_a}(\mathcal{A})$, and $\mathfrak{R}^{-1}_a(\mathcal{A})$ for $\mathfrak{R}_{\Sigma^{-1}_a}(\mathcal{A})$.

We recall (see e.g. [4]) that there is an indexing $\{A_z\}_{z\in\omega}$ of the family of all Σ^{-1}_a sets, such that $\{\langle x, z\rangle : x \in A_z\} \in \Sigma^{-1}_a$. From this, it is possible (for more details, see [8]) to define an indexing $\{\pi_e\}_{e\in\omega}$ of all computable numberings of families of Σ^{-1}_a sets, for which

$$\{\langle e, k, x\rangle : x \in \pi_e(k)\} \in \Sigma^{-1}_a,$$

i.e. the set $\{\langle e, k, x\rangle : x \in \pi_e(k)\}$ has a Σ^{-1}_a–approximation $\langle f, \gamma\rangle$: an indexing satisfying this property is called a Σ^{-1}_a*–computable indexing* of all Σ^{-1}_a–computable numberings. Clearly, from e, k one has an effective way of getting a Σ^{-1}_a–approximation $\langle f_{\pi_e(k)}, \gamma_{\pi_e(k)}\rangle$ to the set $\pi_e(k)$.

2 The theorem

The following theorem (which follows along the lines of a similar theorem proved by Badaev and Talasbaeva in [2] for all finite levels of the Ershov hierarchy) shows that there is no problem when we consider families without any structural restrictions: we can easily construct a family consisting of any given number of elements whose Rogers semilattice consists of one element.

Theorem 2.1 *For every $n \in \omega \cup \{\omega\}$, and for every ordinal notation a of a nonzero ordinal, there exists a Σ^{-1}_a–computable family $\mathcal{A}$ of exactly n sets, such that $|\mathfrak{R}^{-1}_a(\mathcal{A})| = 1$.*

proof : Suppose that we are given a notation a for a computable ordinal ≥ 1. We begin with building a Σ^{-1}_a–computable Friedberg (i.e. injective) numbering α of an infinite family $\mathcal{A}$ such that, for every k, the requirement R_k,

$$R_k : \quad \pi_k \in Comp{-}1a(\mathcal{A}) \Rightarrow \pi_k \leqslant \alpha$$

is satisfied, where, as usual, we refer to some computable listing $\{\pi_k\}_{k\in\omega}$ of all Σ^{-1}_a – computable numberings. Without loss of generality, we may assume that $\pi_0(0) = \emptyset$. We write

$$\pi^s_k(x) = \{y : f_{\pi_k(x)}(y, s) = 1\}.$$

We also assume that $\pi^s_0(0) = \emptyset$ for all s. We use $\langle f_{\pi_k(x)}, \gamma_{\pi_k(x)}\rangle$ to denote a Σ^{-1}_a – approximation (uniform in k, x) to $\pi_k(x)$. In the construction we will build an auxiliary sequence $\{g_k\}_{k\in\omega}$ of partial computable functions, each g_k aiming at reducing π_k to α, if π_k is a numbering of $\mathcal{A}$. Finally, let $a(k, x, i)$ be the values of some fixed computable injective function.

The construction. We begin with describing the strategy to meet R_k. The requirement will be spread into subrequirements $R_{k,x}$: subrequirement $R_{k,x}$ aims at defining $g_k(x)$, if $\pi_k \in Comp{-}1a(\mathcal{A})$. In defining $\mathcal{A}$, we will have care to achieve that if $\pi_k \in Comp{-}1a(\mathcal{A})$ then there will be a unique i such that $\pi_k(x) = \alpha(i)$, and we will let in this case $g_k(x) = i$.

Module. A reasonable module for satisfying $R_{k,x}$ is the following, carried out for each i:

1. let $a(k, x, i) \in \alpha(i)$;
2. await $a(k, x, i) \in \pi_k(x)$;
3. define $g_k(x) = i$ and extract $a(k, x, i)$ from all $\alpha(j)$, $j \neq i$, if it lies in these sets;
4. await $a(k, x, i) \notin \pi_k(x)$: if $a(k, x, i)$ gets extracted from $\pi_k(x)$;
5. enumerate $a(k, x, i)$ in all $\alpha(j)$, and go to (2)

Outcomes. Notice that we can not loop infinitely many times from (2) to (4), since $\pi_k \in \Sigma^{-1}_a$. Thus we distinguish the following outcomes:

1. for every i we wait forever at (2), without ever passing through (4): then π_k is not a numbering of $\mathcal{A}$, since for every i, $a(k, x, i) \in \alpha(i) - \pi_k(x)$;
2. there is some i such that we move at some time from (2) to (4), and for this i:
   - (w_1) we wait forever at (2) (after being at (4)): then $a(k, x, i) \in \alpha(j)$, all j, but $a(k, x, i) \notin \pi_k(x)$, thus π_k is not a numbering of $\mathcal{A}$;
   - (w_2) we wait forever at (4): then $a(k, x, i) \in \alpha(i) \cap \pi_k(x)$, and $a(k, x, i) \notin \alpha(j)$, if $j \neq i$; thus is π_k is a numbering of the family, then $\pi_k(x) = \alpha(i)$, consistently with our definition of $g_k(x) = i$, made at (3).

We give the detailed construction of α by stages: at stage t we define $\alpha_t(e)$, or, by Definition , the values $f(e,z,t)$ of a suitable computable function, together with the values $\gamma(e,z,t)$ of a partial computable mind-change function for f: f and γ will witness that α is a Σ_a^{-1}–computable numbering. At each stage, each parameter will retain the same value as at the preceding stage, unless otherwise explicitly redefined. Given a, j, at stage s we say that we *enumerate* a *in* $\alpha(j)$, if we define $a \in \alpha_s(j)$; similarly, we say that we *extract* a *from* $\alpha(j)$, if we define $a \notin \alpha_s(j)$

Stage 0**:** Let $\alpha_0(e) = \emptyset$ and let $g_{k,0}(e)$ be undefined for all k and e. Moreover, let $f(e,z,0) = 0$, and $\gamma(e,z,0)\uparrow$. Go to the next stage.

Stage $t+1$**:** Let $m = (t)_0$ and suppose that $m = \langle k,x,i\rangle$. Carry out the instructions of Case 1 and Case 2, in the given order, and act accordingly: after acting go to next stage.

Case 1: If $t = \langle m, 0\rangle$ then enumerate the number $a(k,x,i)$ into $\alpha(i)$: define $f(i, a(k,x,i), t+1) = 1$ and $\gamma(z, t+1) = 1$. (Recall that 1 is a notation of the ordinal 0.)

Case 2: If there exists $t' < t$ such that $(t')_0 = m$ then carry out one the following mutually exclusive subcases 2.1–2.3.

Subcase 2.1: $a(k,x,i) \in \pi_k^{t+1}(x)$. Let $g_{k,t+1}(x) = i$ if $g_{k,t}(x)$ is undefined; for every $j \neq i$ extract $a(k,x,i)$ from $\alpha(j)$, defining $f(j, a(k,x,i), t+1) = 0$ and

$$\gamma(j, a(k,x,i), t+1) = \gamma_{\pi_k(x)}(a(k,x,i), t+1).$$

Subcase 2.2: $g_k^t(x) = i$ and $a(k,x,i) \notin \pi_k^{t+1}(x)$. For every $j \neq i$, enumerate $a(k,x,i)$ into $\alpha(j)$, define $f(j, a(k,x,i), t+1) = 1$ and

$$\gamma(j, a(k,x,i), t+1) = \gamma_{\pi_k(x)}(a(k,x,i), t+1).$$

Subcase 2.3: If Subcases 2.1, 2.2 do not hold then do nothing.

Verification. Notice that the only numbers that go into any of the sets numbered by α are numbers in the range of the function $a(k,x,i)$: the element $a(k,x,i)$ is enumerated once for all in $\alpha(i)$, whereas its membership status in $\alpha(j)$, for $j \neq i$, is determined by the equation $\alpha(j)(a(k,x,i)) = 1 - \pi_k(x)(a(k,x,i)))$ at all stages following the least stage $\langle\langle k,x,i\rangle, 0\rangle$ at which we attack $R_{k,x}$. The pair $\langle f, \gamma\rangle$ is a Σ_a^{-1}–computable approximation, since we redefine $\gamma(j, a(k,x,i))$ only after seeing that $f_{\pi_k(x)}(a(k,x,i))$ has changed, thus $\gamma_{\pi_k(x)}(a(k,x,i))$ has dropped, and in this case we let $\gamma(j, a(k,x,i)) = \gamma_{\pi_k(x)}(a(k,x,i))$. Moreover $a(0,0,i)$ is only contained in $\alpha(0)$. Finally, for every k, if π_k is a numbering of $\mathcal{A}$, then g_k is total and $\pi_k = \alpha \circ g_k$, since $\alpha(g_k(x))$ is the only set of the family containing $a(k,x,i)$.

The above construction builds in fact an infinite family. As in [2] we can show that for every finite n there is a family of n sets: it suffices to build $\alpha(0), \ldots, \alpha(n-1)$ as above, and $\alpha(j) = \alpha(n-1)$, for every $j \geq n-1$.

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Данная статья показывает, что мы можем легко построить семейство, состоящее из любого заданного числа элементов, такое, что полурешетка Роджерса состоит из одного элемента.

Манат Мұстафа, *Ершов иерархиясындағы Рожерс жарты торының қуаты туралы* , ҚазҰУ хабаршысы, мат., мех., инф. сериясы 2011, №4(71), 8 – 12

Бұл мақаланың мақсаты, Рожерс жарты торының қуаты бір элементті болатындай кез келген элементтен тұратын үйірді оңай құрастрып алуға болатынын көрсету.