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Constructive theory of boundary value problems for linear integral-differential equations

The necessary and sufficient conditions for solvability of boundary value problems of the linear integral-differential equations at phase and integral constraints are obtained. A method for constructing the solution of the boundary value problem with constraints by constructing minimizing sequences is developed. The basis of the proposed method for solving the boundary value problem is the principle of immersion. The principle of immersion is created by building the solution of a class of Fredholm integral equations of the first kind. The principal difference of the proposed method is that the origin value problem at the beginning immersed to the controllability problem with fictitious controls of functional spaces, followed by reduction to the initial problem of optimal control. Solvability and construction of a solution of the boundary value problems are solved together by solving an optimization problem. Creating a general theory of boundary value problems for linear integral-differential equations with complex boundary conditions in the presence of the phase and integral constraints is a topical problem with applications in the natural sciences, economics and ecology.

Key words: constructive theory, boundary value problems, linear integro-differential equations, the principle of immersion.

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Сызықты интегро-дифференцаилдық теңдеулер үшин шектік есептердің құрылымдық теориясы

Фазалық және интегралдық шектеулері бар сызықты интегро-дифференциалдық теңдеулер үшін шектік есептің шешілімділігінің қажетті және жеткілікті шарттары алынған. Минимумдаушы тізбектерді құру жолымен шектеулері бар шектік есептің шешімін құру әдісі жасалған. Шектік есепті шешуге ұсынылған әдістің негізі - батыру қағидасы болып табылады. Батыру қағидасы бір класстағы Фредгольмнің бірінші текті интегралдық теңдеулерінің жалпы шешімін түрғызу негізінде құрылған. ұсынылған әдістің түбегейлі өзгешелігі - берілген шектік есеп алғашында функционалдық кеңістіктерден алынған жалған басқарулары бар басқарымдылық есебіне, содан кейін тиімді басқарудың бастапқы есебіне келтірілуі. Шектік есептің шешімдерін тұрғызу және оның шешілімділігі тиімділік есебін шешу негізінде алынады. Фазалық және интегралдық шектеулері мен күрделі шектік шарттары бар сызықты интегро-дифференциалдық теңдеулер үшін шектік есептің жалпы теориясын құру жаратылыстану ғылымдарында, экономикада және экологияда көптеген қосымшалары бар актуалды мәселе болып табылады.

Түйін сөздер: құрылымдық теориясы, шектік есептер, сызықты интегро-дифференциалдық теңдеулер, батыру қағидасы.

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Конструктивная теория краевых задач для линейных интегро-дифференциальных уравнений

Получены необходимые и достаточные условия разрешимости краевых задач для линейных интегро-дифференциальных уравнений при наличии фазовых и интегральных ограничений. Разработан метод построения решения краевой задачи с ограничениями, путем построения минимизирующих последовательностей. Основой предлагаемого метода решения краевой задачи является принцип погружения. Принцип погружения был создан путем построения общего решения одного класса интегральных уравнений Фредгольма первого рода. Принципиальное отличие предлагаемого метода состоит в том, что исходная краевая задача в начале погружается в задачу управляемости с фиктивными управлениями из функциональных пространств, с последующим сведением к начальной задаче оптимального управления. Разрешимость и построение решения краевой задачи решаются воедино, путем решения оптимизационной задачи. Создание общей теории краевых задач для линейных интегро-дифференциальных уравнений со сложными краевыми условиями при наличии фазовых и интегральных ограничений является актуальной проблемой с многочисленными приложениями в естественных науках, экономики и экологии.
Ключевые слова: конструктивная теория, краевые задачи, линейные интегро-дифференциальные уравнения, принцип погружения.

1. Problem statement

We consider the following boundary value problem for linear integral-differential equations

$$\dot{x} = A_0(t)x + B_0(t)\int\limits_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau + \mu(t), \quad t \in I = [t_0, t_1], \tag{1}$$

with boundary conditions

$$(x(t_0) = x_0, \quad x(t_1) = x_1) \in S \subset R^{2n}, \tag{2}$$

at phase constraints

$$\begin{gathered} x(t) \in G(t) : G(t) = \{x \in R^n / \alpha(t) \le L(t)x(t) \le \beta(t), \quad t \in I; \\ \alpha_1(t) \le \int\limits_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau \le \beta_1(t), \quad t \in I\}, \end{gathered} \tag{3}$$

as well as the integral constraints

$$g_j(x) \le c_j, \quad j = \overline{1, m_1}, \quad g_j(x) = c_j, \quad j = \overline{m_1 + 1, m_2} \tag{4}$$

$$g_j(x) = \int\limits_{t_0}^{t_1} [< a_j(t), x(t) > + < b_j(t), \int\limits_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau >]dt, \quad j = \overline{1, m_2}. \tag{5}$$

Here $A_0(t)$, $B_0(t)$, $K(t,\tau)$, $L(t)$, $t \in I$, $\tau \in I$ are prescribed matrixes with piecewise continuous elements of the dimension $n \times n$, $n \times m$, $m \times n$, $s \times n$ respectively, $\mu(t)$, $t \in I$ is given $n-$ dimensional vector function with piecewise continuous components, S is given convex closed set, $a_j(t) = (a_{1j}(t), \dots, a_{nj}(t))$, $b_j(t) = (b_{1j}(t), \dots, b_{nj}(t))$, $t \in I$, $j = \overline{1, m_2}$ are known vector functions with piecewise continuous elements, $\alpha(t) = (\alpha_1(t), \dots, \alpha_s(t))$, $\beta(t) = (\beta_1(t), \dots, \beta_s(t))$, $\alpha_1(t) = (\alpha_{11}(t), \dots, \alpha_{m1}(t))$, $\beta_1(t) = (\beta_{11}(t), \dots, \beta_{m1}(t))$, $t \in I$ are given continuous function. The values c_j, $j = \overline{1, m_2}$ are prescribed constants.

The following problems are set:

Problem 1 *Find necessary and sufficient conditions for the existence of solutions of the boundary problem (1) – (5).*

Problem 2 *Construct a solution of the boundary value problem (1) – (5).*

As it follows from statements of the problems it is necessary to prove the existence of a pair $(x_0, x_1) \in S$ such that the solution of the system (1), coming from the point x_0 at the moment time t_0, passes through the point x_1 at the moment time t_1, in this case along the solutions of the system (1) for each time moment the phase constraint is performed (3) and integrals (5) satisfy the conditions (4). In particular, the set S is given by relation

$$S = \{(x_0, x_1) \in R^{2n} / H_j(x_0, x_1) \leq 0, \;\; j = \overline{1, p_1};$$

$$< \overline{a}_j, x_0 > + < \overline{b}_j, x_1 > - \overline{d}_j = 0, \;\; j = \overline{p_1 + 1, s_1}\},$$

where $H_j(x_0, x_1)$, $j = \overline{1, p}$ are convex functions in the variables (x_0, x_1), $x_0 = x(t_0)$, $x_1 = x(t_1)$, $\overline{a}_j \in R^n$, $\overline{b}_j \in R^n$, $\overline{d}_j \in R^1$, $j = \overline{p_1 + 1, s_1}$ are prescribed vectors and numbers, $< \cdot, \; \cdot >$ is scalar product.

Integral-differential equation connects together the present, future and past of the process. These mathematical models of phenomena more adequately describe its properties. One of the founders of quantum mechanics, V. Heisenberg, in his book "Physics and Philosophy" makes the following suggestion: "... the basic equation of matter regarded as a mathematical representation of the whole matter should take the form of a complex system of integral-differential equations."

Particular cases of the boundary value problem (1) – (6) in the absence of phase and integral restrictions with affine set S are studied in the works [1-3]. This work is a continuation of research of [4, 5].

The essence of the method is that the first stage of research the origin problem is immersed to the controllability problem by transformation and introduction of a fictitious control. Further elucidation of the existence of solutions of the original problem and the construction of its solution is carried out by solving the problem of optimal control of a special kind. With this approach, the necessary and sufficient conditions for the existence of a solution of the boundary value problem (1) – (5) can be obtained from the condition of the lower bound of the functional on a given set, and the solution of the original problem is determined by the limiting points of the minimizing sequences.

Constructive theory of boundary value problems with phase and integral constraints for ordinary differential equations, as well as for the parabolic equations are presented in [6-10].

2. Transformation

Introducing the additional variables $d = (d_1, \dots, d_{m_1}) \in R^{m_1}$, $d \geq 0$, the relations (4), (5) can be presented as

$$g_j(x) = \int_{t_0}^{t_1} [< a_j(t), x(t) > + < b_j(t), \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau >]dt = c_j - d_j, \quad j = \overline{1, m_1}$$

where

$$d \in D = \{d \in R^{m_1} / d \geq 0\}.$$

Let the vector $\overline{c} = (\overline{c}_1, \dots, \overline{c}_{m_2})$ has the components $\overline{c}_j = c_j - d_j$, $j = \overline{1, m_1}$, $\overline{c}_j = c_j$, $j = \overline{m_1 + 1, m_2}$. We introduce the vector functions $\eta(t) = (\eta_1(t), \dots, \eta_{m_2})$, $t \in I$ by equality

$$\eta_j(t) = \int_{t_0}^{t} [< a_j(\tau), x(\tau) > + < b_j(\tau), \int_{t_0}^{t_1} K(\tau,\rho)x(\rho)d\rho >]d\tau, \quad t \in I,$$

then

$$\dot{\eta}_j(t) =< a_j(t), x(t) > + < b_j(t), \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau >, \quad j = \overline{1, m_2},$$

$$\eta_j(t_0) = 0, \quad \eta_j(t_1) = \overline{c}_j, \quad j = \overline{1, m_2}, \quad d \in D.$$

It follows that

$$\dot{\eta}(t) = A_1(t)x(t) + B_1(t) \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau, \quad t \in I,$$

where

$$A_1(t) = \begin{pmatrix} a_1^*(t) \\ \cdots \\ a_{m_2}^*(t) \end{pmatrix}, \quad B_1(t) = \begin{pmatrix} b_1^*(t) \\ \cdots \\ b_{m_2}^*(t) \end{pmatrix}, \quad \eta(t) = \begin{pmatrix} \eta_1(t) \\ \cdots \\ \eta_{m_2}(t) \end{pmatrix},$$

$$\overline{c} \in C = \{\overline{c} \in R^{m_2} / \overline{c}_j = c_j - d_j, \quad j = \overline{1, m_1}, \quad \overline{c}_j = c_j, \quad j = \overline{m_1 + 1, m_2}\},$$

$$\eta(t_0) = O_{m_2,1}, \quad \eta(t_1) = \overline{c}, \quad d \in D.$$

Now the original boundary value problem (1) – (5) is written as

$$\dot{\xi} = A(t)\xi + B(t) \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau + \mu_1(t), \quad t \in I, \tag{6}$$

$$\xi(t_0) = \xi_0 = (x_0, O_{m_2,1}), \quad \xi(t_1) = \xi_1 = (x_1, \overline{c}), \tag{7}$$

$$(x_0, x_1) \in S, \quad d \in D, \quad P\xi(t) \in G(t), \quad t \in I, \tag{8}$$

where

$$\xi(t) = \begin{pmatrix} x(t) \\ \eta(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} A_0(t) & O_{nm_2} \\ A_1(t) & O_{m_2m_2} \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_0(t) \\ B_1(t) \end{pmatrix},$$

$$\mu_1(t) = \begin{pmatrix} \mu(t) \\ O_{m_21} \end{pmatrix}, \quad P = (I_n,\ O_{nm_2}), \quad P\xi = x,$$

$O_{jk} - j \times k$ is matrix with zero elements, $\xi = (\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+m_2})$, I_n is an identity matrix of order $n \times n$.

3. Linear control system

Along with the differential equation (6) with boundary conditions (7) we consider the linear controlled system

$$\dot{y} = A(t)y + B(t)w(t) + \mu_1(t), \quad t \in I, \tag{9}$$

$$y(t_0) = \xi_0 = (x_0, O_{m_2,1}), \quad y(t_1) = \xi_1 = (x_1, \bar{c}), \tag{10}$$

$$(x_0, x_1) \in S, \quad d \in D, \quad w(\cdot) \in L_2(I, R^m), \tag{11}$$

where $A(t)$, $B(t)$ are matrixes with piecewise continuous elements of the order $(n + m_2) \times (n+m_2)$, $(n+m_2)\times m$ respectively. It is easy to make sure that the control $w(\cdot) \in L_2(I, R^m)$ that transfers the trajectory of the system (9) from any initial state ξ_0 to any desired final state ξ_1, is a solution of the integral equation

$$\int_{t_0}^{t_1} \Phi(t_0, t)B(t)w(t)dt = a, \tag{12}$$

where $\Phi(t, \tau) = \theta(t)\theta^{-1}(\tau)$, $\theta(t)$ is a fundamental matrix of solutions of the linear homogeneous system $\dot{\omega} = A(t)\omega$, vector

$$a = a(\xi_0, \xi_1) = \Phi(t_0, t_1)\xi_1 - \xi_0 - \int_{t_0}^{t_1} \Phi(t_0, t)\mu_1(t)dt.$$

Theorem 1 *The integral equation (12) at any fixed $a \in R^{n+m_2}$ has a solution if and only if $(n + m_2) \times (n + m_2)$ matrix*

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B^*(t)\Phi^*(t_0, t)dt$$

is positive definited, where $(*)$ denotes transposition.

The proof of the theorem is given in [11].

Theorem 2 *Let the matrix $W(t_0,t_1)>0$. Control $w(\cdot)\in L_2(I,R^m)$ transforms the trajectory of system (9) from any starting point $\xi_0\in R^{n+m_2}$ to any final state $\xi_1\in R^{n+m_2}$ if and only if*

$$\begin{gathered} w(t)\in W=\{w(\cdot)\in L_2(I,R^m)/w(t)=v(t)+\lambda_1(t,\xi_0,\xi_1)+N_1(t)z(t_1,v),\\ t\in I,\ \ v(\cdot)\in L_2(I,R^m)\}, \end{gathered} \tag{13}$$

where the function $z(t)=z(t,v)$, $t\in I$, is the solution of the differential equation

$$\dot z=A(t)z+B(t)v(t),\ \ z(t_0)=0,\ \ v(\cdot)\in L_2(I,R^m). \tag{14}$$

Moreover the solution of the differential equation (9), corresponding to the control $w(t)\in W$ defined by equality

$$y(t)=z(t)+\lambda_2(t,\xi_0,\xi_1)+N_2(t)z(t_1,v),\ \ t\in I. \tag{15}$$

Here $v(\cdot)\in L_2(I,R^m)$ is any function, $\lambda_1(t,\xi_0,\xi_1)$, $N_1(t)$, $\lambda_2(t,\xi_0,\xi_1)$, $N_2(t)$ are defined by formulas

$$\lambda_1(t,\xi_0,\xi_1)=B^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1)a,\ \ N_1(t)=-B^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1)\Phi(t_0,t_1),$$

$$\lambda_2(t)=\Phi(t,t_0)W(t,t_1)W^{-1}(t_0,t_1)\xi_0+\Phi(t,t_0)W(t_0,t)W^{-1}(t_0,t_1)\Phi(t_0,t_1)\xi_1+$$

$$+\int_{t_0}^{t}\Phi(t,\tau)\mu_1(\tau)d\tau-\Phi(t,t_0)W(t_0,t)W^{-1}(t_0,t_1)\int_{t_0}^{t_1}\Phi(t_0,\tau)\mu_1(\tau)d\tau,$$

$$N_2(t)=-\Phi(t_1,t_0)W(t_0,t)W^{-1}(t_0,t_1)\Phi(t_0,t_1),\ \ W(t,t_1)=\int_{t}^{t_1}\Phi(t_0,\tau)B(\tau)\times$$

$$\times B^*(\tau)\Phi^*(t_0,\tau)d\tau,\ \ W(t_0,t)=\int_{t_0}^{t}\Phi(t_0,\tau)B(\tau)B^*(\tau)\Phi^*(t_0,\tau)d\tau.$$

The proof of the analogous theorem can be found in [6].

4. Optimization problem

We consider the following optimization problem: minimize the functional

$$\begin{gathered} J(v,u,p,x_0,x_1,d)=\int_{t_0}^{t_1}[|w(t)-u(t)|^2+|p(t)-L(t)Py(t)|^2+\\ +|w(t)-\int_{t_0}^{t_1}K(t,\tau)Py(\tau)d\tau|^2]dt\to\inf \end{gathered} \tag{16}$$

at conditions

$$\dot z=A(t)z+B(t)v(t),\ \ z(t_0)=0,\ \ v(\cdot)\in L_2(I,R^m), \tag{17}$$

$$u(t) \in U(t) = \{u(\cdot) \in L_2(I, R^m)/\alpha_1(t) \le u(t) \le \beta_1(t), \ \ t \in I\}, \tag{18}$$

$$p(t) \in V(t) = \{p(\cdot) \in L_2(I, R^s)/\alpha(t) \le p(t) \le \beta(t), \ \ t \in I\}, \tag{19}$$

$$(x_0, x_1) \in S, \ \ d \in D, \tag{20}$$

where the functions $w(t)$, $y(t)$, $t \in I$ are defined by the formulas (13) – (15) respectively.

We denote

$$X = L_2(I, R^m) \times U(t) \times V(t) \times S \times D \subset H = L_2(I, R^m) \times L_2(I, R^m) \times$$

$$\times L_2(I, R^s) \times R^{2n} \times R^{m_1}, \ \ J_* = \inf_{\theta \in X} J(\theta),$$

$$\theta = (v(t), u(t), p(t), x_0, x_1, d) \in X, \ \ X_* = \{\theta_* \in X/J(\theta_*) = \inf_{\theta \in X} J(\theta) = \min_{\theta \in X} J(\theta)\}.$$

We introduce the following notations

$$F_0(q(t), t) = |w(t) - u(t)|^2 + |p(t) - L(t)Py(t)|^2 = |v(t) + \lambda_1(t, \xi_0, \xi_1) +$$

$$+N_1(t)z(t_1, v) - u(t)|^2 + |p(t) - L(t)P[z(t, v) + \lambda_2(t, \xi_0, \xi_1) + N_2(t)z(t_1, v)]|^2 =$$

$$= |v(t) + T_1(t)x_0 + T_2(t)x_1 + T_3(t)d + \mu_2(t) + N_1(t)z(t_1, v) - u(t)|^2 +$$

$$+|p(t) - L(t)P[z(t, v) + E_1(t)x_0 + E_2(t)x_1 + E_3(t)d + \mu_3(t) + N_2(t)z(t_1, v)]|^2 =$$

$$= q^*(t)Q(t)q(t) + 2q^*(t)\overline{a}(t) + \overline{b}(t) \ge 0,$$

where

$$Q(t) = Q^*(t) \ge 0, \ \ t \in I, \ \ q(t) = (\theta(t), z(t, v), z(t_1, v));$$

$$F_1(q(t), t) = |w(t) - \int_{t_0}^{t_1} K(t, \tau)Py(\tau)d\tau|^2 = |v(t) + T_1(t)x_0 + T_2(t)x_1 +$$

$$+T_3(t)d + \mu_2(t) + N_1(t)z(t_1, v) - \int_{t_0}^{t_1} K(t, \tau)P[z(\tau, v) + E_1(\tau)x_0 + E_2(\tau)x_1 +$$

$$+E_3(\tau)d + \mu_3(\tau) + N_2(\tau)z(t_1, v)]d\tau|^2 = |v(t) + [T_1(t) - \int_{t_0}^{t_1} K(t, \tau)PE_1(\tau)d\tau]x_0 +$$

$$+[T_2(t) - \int_{t_0}^{t_1} K(t, \tau)PE_2(\tau)d\tau]x_1 + [T_3(t) - \int_{t_0}^{t_1} K(t, \tau)PE_3(\tau)d\tau]d +$$

$$+[\mu_2(t) - \int_{t_0}^{t_1} K(t, \tau)P\mu_3(\tau)d\tau] + [N_1(\tau) - \int_{t_0}^{t_1} K(t, \tau)PN_2(\tau)d\tau]z(t_1, v) -$$

$$- \int_{t_0}^{t_1} K(t, \tau)Pz(\tau, v)d\tau|^2 = |v(t) + \overline{T}_1(t)x_0 + \overline{T}_2(t)x_1 + \overline{T}_3(t)d +$$

$$+\overline{\mu}_2(t)+\overline{N}_1(t)z(t_1,v)-\int\limits_{t_0}^{t_1}K(t,\tau)Pz(\tau,v)d\tau|^2=$$

$$=\int\limits_{t_0}^{t_1}|\overline{w}(t)-\int\limits_{t_0}^{t_1}K(t,\tau)Pz(\tau,v)d\tau|^2dt,$$

where $\overline{w}(t)=v(t)+\overline{T}_1(t)x_0+\overline{T}_2(t)x_1+\overline{T}_3(t)d+\overline{\mu}_2(t)+\overline{N}_1(t)z(t_1,v)$, $q(t)=(v(t),x_0,x_1,d,z(t,v),$ $z(t_1,v))$, $t\in I$, $\tau\in I$.

Now the problem (16) – (20) can be written as

$$J(v,u,p,x_0,x_1,d)=\int\limits_{t_0}^{t_1}F_0(q(t),t)dt+\int\limits_{t_0}^{t_1}F_1(q(t),t)dt\to\inf \qquad (21)$$

at conditions (17) – (20).

Lemma 1 *Let $S\subset R^{2n}$ be a convex set. Then:*

1) functional (21) at conditions (17) – (20) is convex;

2) the partial derivatives of the function $F_0(q,t)$, $F_1(q,t)$ in the variable $q=(v,u,p,x_0,x_1,d,$ $z,z(t_1))\in R^N$, $N=2m+S+2n+m_1+2(n+m_2)$ satisfy Lipschitz conditions.

Proof. Since $F_0(q,t)=q^*Q(t)q+2q^*\overline{a}(t)+\overline{b}(t)$, $t\in I$, $q\in R^N$, that $\partial^2F_0(t,q)/\partial^2q=2Q(t)\geq 0$, the function $F_0(t,q)$ with respect to variable $q\in R^N$ is convex. The function $F_1(q_1,t)$ is convex in the variables $(v,x_0,x_1,d,z(t_1))=\overline{q}_1$, due to the fact that $\overline{w}^*\overline{w}=\overline{q}_1^*Q_1\overline{q}_1$, where $Q_1=Q_1^*\geq 0$.

We notice, that

$$\int\limits_{t_0}^{t_1}|\overline{w}(t)-\int\limits_{t_0}^{t_1}K(t,\tau)Pz(\tau,v)d\tau|^2dt=\int\limits_{t_0}^{t_1}[\overline{w}^*(t)\overline{w}(t)-2\overline{w}^*(t)\times$$

$$\times\int\limits_{t_0}^{t_1}K(t,\tau)Pz(\tau)d\tau+\int\limits_{t_0}^{t_1}\int\limits_{t_0}^{t_1}z^*(\tau)P^*K^*(t,\tau)K(t,\sigma)Pz(\sigma)d\sigma d\tau]dt,$$

where

$$F_1(q(t),t)=\overline{w}^*(t)\overline{w}(t)-2\overline{w}^*(t)\times\int\limits_{t_0}^{t_1}K(t,\tau)Pz(\tau)d\tau+$$

$$+\int\limits_{t_0}^{t_1}\int\limits_{t_0}^{t_1}z^*(\tau)P^*K^*(t,\tau)K(t,\sigma)Pz(\sigma)d\sigma d\tau,$$

$$\int\limits_{t_0}^{t_1}\int\limits_{t_0}^{t_1}z^*(\tau)P^*K^*(t,\tau)K(t,\sigma)Pz(\sigma)d\sigma d\tau=\left[\int\limits_{t_0}^{t_1}K(t,\tau)Pz(\tau)d\tau\right]^2\geq 0.$$

It follows that the function $F_1(q,t)$ is convex in the variables q_1. From the convexity of the function $F_0(q,t)$, $F_1(q,t)$, with taking into account, that

$$z(t, \alpha v_1 + (1-\alpha)v_2) = \alpha z(t, v_1) + (1-\alpha) z(t, v_2), \;\; t \in I, \; \forall v_1, \; v_2 \in L_2(I, R^m),$$

we obtain

$$J(\alpha\theta_1 + (1-\alpha)\theta_2) = \int_{t_0}^{t_1} F_0(\alpha\overline{q} + (1-\alpha)\overline{\overline{q}}, t)dt +$$

$$+ \int_{t_0}^{t_1} F_1(\alpha\overline{q}_1 + (1-\alpha)\overline{\overline{q}}, t)dt \leq \alpha J(\theta_1) + (1-\alpha)J(\theta_2), \;\; \forall \theta_1, \theta_2 \in X.$$

Consequently, the function (21) at conditions (17) – (20) is convex.

The partial derivatives of the function $F(v, u, p, x_0, x_1, d, z, z(t_1)) = F_0(q,t) + F(q_1, t)$ equal:

$$F_v(q,t) = \frac{\partial F(q,t)}{\partial v} = 2(w-u) + 2[\overline{w} - \int_{t_0}^{t_1} K(t,\tau)Pz(\tau)d\tau];$$

$$F_u(q,t) = \frac{\partial F(q,t)}{\partial u} = -2(w-u); \quad F_p(q,t) = \frac{\partial F(q,t)}{\partial p} = 2[p - L(t)Py(t)];$$

$$F_{x_0}(q,t) = \frac{\partial F(q,t)}{\partial x_0} = 2T_1^*(w-u) - 2E_1^*P^*L^*(p - L(t)Py) + 2\overline{T}_1^*[\overline{w} - \int_{t_0}^{t_1} K(t,\tau)Pz(\tau)d\tau];$$

$$F_{x_1}(q,t) = \frac{\partial F(q,t)}{\partial x_1} = 2T_2^*(w-u) - 2E_2^*P^*L^*(p - L(t)Py) + 2\overline{T}_2^*[\overline{w} - \int_{t_0}^{t_1} K(t,\tau)Pz(\tau)d\tau];$$

$$F_d(q,t) = \frac{\partial F(q,t)}{\partial d} = 2T_3^*(w-u) - 2E_3^*P^*L^*(p - L(t)Py) + 2\overline{T}_3^*[\overline{w} - \int_{t_0}^{t_1} K(t,\tau)Pz(\tau)d\tau];$$

$$\begin{aligned} F_z(q,t) = \frac{\partial F(q,t)}{\partial z} &= -2P^*L^*(p - LPy) - 2\int_{t_0}^{t_1} P^*K^*(\sigma,t)\overline{w}(\sigma)d\sigma + \\ &+2\int_{t_0}^{t_1}\int_{t_0}^{t_1} P^*K^*(\sigma,t)K(t,\xi)Pz(\xi)d\xi d\sigma; \end{aligned} \tag{22}$$

$$F_{z(t_1)}(q,t) = \frac{\partial F(q,t)}{\partial z(t_1)} = 2N_1^*(w-u) - 2N_2^*P^*L^*(p - LPy) + 2\overline{N}_1^*[\overline{w} - \int_{t_0}^{t_1} K(t,\tau)Pz(\tau)d\tau];$$

As it follows from (22), the partial derivatives $F(q,t) = F_0(q,t) + F_1(q,t)$ satisfy Lipschitz conditions. Lemma is proved.

Theorem 3 *Let the matrix* $W(t_0, t_1) > 0$. *Then the functional (21) at conditions (17) – (20) is continuously Frechet differentiable, the gradient of the functional*

$$J'(\theta) = (J'_v(\theta), J'_u(\theta), J'_p(\theta), J'_{x_0}(\theta), J'_{x_1}(\theta), J'_d(\theta)) \in H$$

at any point $\theta \in X$ *is calculated by the formula*

$$J'_v(\theta) = \frac{\partial F(q,t)}{\partial v} - B^*(t)\psi(t), \quad J'_u(\theta) = \frac{\partial F(q,t)}{\partial u}, \quad J'_p(\theta) = \frac{\partial F(q,t)}{\partial p},$$

$$J'_{x_0}(\theta) = \int_{t_0}^{t_1} \frac{\partial F(q,t)}{\partial x_0} dt, \quad J'_{x_1}(\theta) = \int_{t_0}^{t_1} \frac{\partial F(q,t)}{\partial x_1} dt, \quad J'_d(\theta) = \int_{t_0}^{t_1} \frac{\partial F(q,t)}{\partial d} dt, \tag{23}$$

where $z(t) = z(t, v)$, $t \in I$ *is a solution of differential equation (17), and function* $\psi(t)$, $t \in I$ *is a solution of the adjoint system*

$$\dot{\psi} = \frac{\partial F(q,t)}{\partial z} - A^*(t)\psi(t), \quad \psi(t_1) = -\int_{t_0}^{t_1} \frac{\partial F(q,t)}{\partial z(t_1)} dt. \tag{24}$$

In addition, the gradient $J'(\theta) \in H$ *satisfies Lipschitz condition*

$$\|J'(\theta_1) - J'(\theta_2)\|_H \le K\|\theta_1 - \theta_2\|_H, \quad \forall \theta_1, \ \theta_2 \in X. \tag{25}$$

Proof. Let $\theta(t) \in X$, $\theta(t) + \Delta\theta(t) \in X$. Then the increment of the functional

$$\begin{gathered} \Delta J = J(\theta + \Delta\theta) - J(\theta) = \int_{t_0}^{t_1} [F(q(t) + \Delta q(t), t) - F(q(t), t)]dt = \int_{t_0}^{t_1} [h^*(t)F_v(q(t), t) + \\ + \Delta u^*(t)F_u(q(t), t) + \Delta p^*(t)F_p(q,t) + \Delta x_0^* F_{x_0}(q,t) + \Delta x_1^* F_{x_1}(q,t) + \\ + \Delta d^* F_d(q,t) + \Delta z^*(t)F_z(q,t) + \Delta z^*(t_1)F_{z(t_1)}(q,t)]dt + \sum_{i=1}^{8} R_i, \end{gathered} \tag{26}$$

where $\Delta q(t) = (h(t), \Delta u(t), \Delta p(t), \Delta x_0, \Delta x_1, \Delta d, \Delta z, \Delta z(t_1))$.

Hence, by the fact that

$$|\Delta z(t)| \le \int_{t_0}^{t_1} \|\Phi(t,\tau)B(\tau)\| |h(\tau)| d\tau \le c_1 \|h\|_{L_2}$$

$$\int_{t_0}^{t_1} \Delta z^*(t_1) F_{z(t_1)}(q(t), t)dt = \Delta z^*(t_1) \int_{t_0}^{t_1} F_{z(t_1)}(q(t), t)dt = -\Delta z^*(t_1)\psi(t_1) =$$

$$= -\int_{t_0}^{t_1} \frac{\partial}{\partial t}[\Delta z^*(t)\psi(t)]dt = -\int_{t_0}^{t_1} [\Delta \dot{z}^*(t)\psi(t) + \Delta z^*(t)\dot{\psi}(t)]dt =$$

$$= -\int_{t_0}^{t_1} [\Delta z^*(t)A^*(t) + h^*(t)B^*(t)]\psi(t)dt - \int_{t_0}^{t_1} \Delta z^*(t)[F_z(q(t),t) - A^*(t)\psi(t)]dt =$$

$$= -\int_{t_0}^{t_1} h^*(t)B^*(t)\psi(t)dt - \int_{t_0}^{t_1} \Delta z^*(t)F_z(q(t),t)dt,$$

$$\int_{t_0}^{t_1} \Delta z^*(t)F_z(q(t),t)dt + \int_{t_0}^{t_1} \Delta z^*(t_1)F_{z(t_1)}(q(t),t)dt = -\int_{t_0}^{t_1} h^*(t)B^*(t)\psi(t)dt,$$

The increment of the functional (26) can be represented as

$$\Delta J = \int_{t_0}^{t_1} \{h^*(t)[F_v(q(t),t) - B^*(t)\psi(t)] + \Delta u^*(t)F_u(q(t),t) + \Delta p^*(t)F_p(q,t)+$$
$$+\Delta x_0^* F_{x_0}(q(t),t) + \Delta x_1^* F_{x_1}(q(t),t) + \Delta d^* F_d(q(t),t)\}dt + \sum_{i=1}^{8} R_i. \qquad (27)$$

Further, taking into consideration that the partial derivatives $F(q,t)$ satisfy LIpschitz condition we obtain $\sum_{i=1}^{8} |R_i| \le c_2\|\Delta\theta\|^2$, $\Delta\theta = (h, \Delta u, \Delta p, \Delta x_0, \Delta x_1, \Delta d)$. Then from (27) follows that the gradient $J'(\theta)$ is defined by formula (23), where $\psi(t)$, $t \in I$ is solution (24). Let $\theta_2 = \theta$, $\theta_1 = \theta + \Delta\theta$. Then from (23) follows

$$|J'(\theta_1) - J'(\theta_2)| \le L_1|\Delta q(t)| + L_2|\Delta\psi(t)| + L_3\|\Delta q\|,$$
$$\|J'(\theta_1) - J'(\theta_2)\|^2 = \int_{t_0}^{t_1} |J'(\theta_1) - J'(\theta_2)|^2 dt \le L_4\|\Delta q\|^2 + L_5 \int_{t_0}^{t_1} |\Delta\psi(t)|^2 dt. \qquad (28)$$

Since

$$\Delta\dot{\psi}(t) = [F_z(q(t) + \Delta q(t),t) - F_z(q(t),t)] - A^*(t)\Delta\psi(t), \quad t \in I,$$

$$\Delta\psi(t_1) = -\int_{t_0}^{t_1} [F_{z(t_1)}(q(t) + \Delta q(t),t) - F_{z(t_1)}(q(t),t)]dt,$$

that by using Granuolla's lemma we obtain

$$|\Delta\psi(t)| \le L_6\|\Delta q\|, \quad t \in I. \qquad (29)$$

(25) follows from estimations (28), (29). Theorem is proved.

For solving of the applied problems can assume, that

$$v(t) \in V_1 = \{v(\cdot) \in L_2(I, R^m)/|v(t)| \le \gamma_0, \ \ \gamma_0 < \infty, \ \ \text{п.в.} \ \ t \in I\},$$

$$d \in D_1 = \{d \in R^m/|d_1| \le \gamma_1 < \infty\},$$

where $\gamma_0 > 0$, $\gamma_1 > 0$ are sufficiency large numbers.

Lemma 2 *Let $v(t) \in V_1$, $d \in D_1$, S be bounded convex closed set. Then the functional (21) at conditions (17) – (20) gets a lower bound on the set*

$$X_1 = V_1 \times U(t) \times V(t) \times S \times D_1 \subset H,$$

$$J_* = \inf_{\theta \in X} J(\theta) = \inf_{\theta \in X_1} J(\theta) = \min_{\theta \in X_1} J(\theta) = J(\theta_*), \quad \theta_* \in X_1.$$

Proof. Since the set X_1 is bounded convex closed set in reflexible Banach space H, that X_1 is weakly bicompactly [12]. Continuous and convex functional (21) on the convex set X_1 is weakly semicontinuous below. Then acoording to Weierstrass' theorem the weakly semicontinuous functional gets a lower bound on the weakly bicompact set. Lemma is proved.

We construct the sequences $\{\theta_n\} \subset X_1$ by the rules:

$$\begin{array}{c} v_{n+1} = P_{V_1}[v_n - \alpha_n J'_v(\theta_n)], \quad u_{n+1} = P_{U_1}[u_n - \alpha_n J'_u(\theta_n)], \\ p_{n+1} = P_V[p_n - \alpha_n J'_p(\theta_n)], \quad x_{0n+1} = P_{S_1}[x_{0n} - \alpha_n J'_{x_0}(\theta_n)], \\ x_{1n+1} = P_{S_1}[x_{1n} - \alpha_n J'_{x_1}(\theta_n)], \quad d_{n+1} = P_{D_1}[d_n - \alpha_n J'_d(\theta_n)], \\ 0 < \varepsilon_0 \le \alpha_n \le \dfrac{2}{K + 2\varepsilon_1}, \quad \varepsilon_1 > 0, \quad n = 0, 1, 2, \ldots, \end{array} \tag{30}$$

where $P_{W_1}[\cdot]$ is a projection of the point on the set W_1, $K = const$ is a Lipschitz constant (25). In particular, at $\varepsilon_0 = \dfrac{1}{K}$, $\varepsilon_1 = \dfrac{K}{2}$, the value $\alpha_n - \dfrac{1}{K} = const > 0$.

Theorem 4 *Let the matrix be $W(t_0, t_1) > 0$, the sequence $\{\theta_n\} \subset X_1$ is defined by formula (30). Then*

1) the sequence $\{\theta_n\} = \{v_n, u_n, p_n, x_{0n}, x_{1n}, d_n\} \subset X_1$ is minimizing, i.e.

$$\lim_{n \to \infty} J(\theta_n) = J_* = \inf_{\theta \in X_1} J(\theta);$$

2) the sequence $\{\theta_n\} \subset X_1$ weakly converges to the set $X_* \subset X_1 \subset X$, where

$$X_* = \{\theta_* = (v_*, u_*, p_*, x_{0*}, x_{1*}, d_*) \in X_1 / J(\theta_*) = J_* = \inf_{\theta \in X_1} J(\theta) = \min_{\theta \in X_1} J(\theta)\}$$

$$v_n \stackrel{сл}{\to} v_*, \quad u_n \stackrel{сл}{\to} u_*, \quad p_n \stackrel{сл}{\to} p_*, \quad x_{0n} \to x_{0*}, \quad x_{1n} \to x_{1*}, \quad d_n \to d_* \ \text{при} \ n \to \infty;$$

3) the following estimation of the convergence rate is valid

$$J(\theta_n) - J_* \le \frac{c}{n}, \quad c = const > 0, \quad n = 1, 2, \ldots .$$

Proof. From (30), with taking into account the property of the point projection on the set we obtain

$$< v_{n+1} - v_n + \alpha_n J'_v(\theta_n), \ v - v_{n+1} >_{L_2} \ge 0, \quad < u_{n+1} - u_n + \alpha_n J'_u(\theta_n), \ u - u_{n+1} >_{L_2} \ge 0,$$

$$< p_{n+1} - p_n + \alpha_n J'_p(\theta_n), \ p - p_{n+1} >_{L_2} \ge 0, \quad < x_{0n+1} - x_{0n} + \alpha_n J'_{x_0}(\theta_n), \ x_0 - x_{0n+1} >_{R^n} \ge 0,$$

$$< x_{1n+1} - x_{1n} + \alpha_n J'_{x_1}(\theta_n), \ x_1 - x_{1n+1} >_{R^n} \ge 0, \quad < d_{n+1} - d_n + \alpha_n J'_d(\theta_n), \ d - d_{n+1} >_{R^{m_1}} \ge 0,$$

$$\forall v, \ v \in V_1, \ \ \forall u, \ u \in U, \ \ \forall p, \ p \in V, \ \ \forall x_0, \ \forall x_1, \ (x_0, x_1) \in S, \ \ \forall d, \ d \in D_1.$$

Hence, in particular, when $\theta = (v, u, p, x_0, x_1, d) = \theta_n$, we get

$$< J'_v(\theta_n), v_n - v_{n+1} >_{L_2} \geq \frac{1}{\alpha_n}\|v_n - v_{n+1}\|^2, \;\; < J'_u(\theta_n), u_n - u_{n+1} >_{L_2} \geq \frac{1}{\alpha_n}\|u_n - u_{n+1}\|^2,$$

$$< J'_p(\theta_n), p_n - p_{n+1} >_{L_2} \geq \frac{1}{\alpha_n}\|p_n - p_{n+1}\|^2, \;\; < J'_{x_0}(\theta_n), x_{0n} - x_{0n+1} >_{R^n} \geq \frac{1}{\alpha_n}\|x_{0n} - x_{0n+1}\|^2,$$

$$< J'_{x_1}(\theta_n), x_{1n} - x_{1n+1} >_{R^n} \geq \frac{1}{\alpha_n}\|x_{1n} - x_{1n+1}\|^2, \;\; < J'_d(\theta_n), d_n - d_{n+1} >_{L_2} \geq \frac{1}{\alpha_n}\|d_n - d_{n+1}\|^2. \tag{31}$$

Since the functional $J(\theta) \in C^{1,1}(X_1)$, that the inequality is valid

$$J(\theta_n) - J(\theta_{n+1}) \geq < J'(\theta_n), \theta_n - \theta_{n+1} > -\frac{K}{2}\|\theta_n - \theta_{n+1}\|^2. \tag{32}$$

Then of (31), (32) follows, that

$$J(\theta_n) - J(\theta_{n+1}) \geq \left(\frac{1}{\alpha_n} - \frac{K}{2}\right)\|\theta_n - \theta_{n+1}\|^2 \geq \varepsilon_1\|\theta_n - \theta_{n+1}\|^2, \tag{33}$$

where $\dfrac{1}{\alpha_n} \geq \dfrac{K + 2\varepsilon_1}{2}$, $\dfrac{1}{\alpha_n} - \dfrac{K}{2} \geq \varepsilon_1$. From (33) follows, that the numeric sequence $\{J(\theta_n)\}$ decreases strictly. Since the value of the functional $J(\theta_n)$ is bounded from below, i.e. $J(\theta_n) \geq 0$, $\forall\theta$, $\theta \in X_1$, that the numeric sequence $\{J(\theta_n)\}$ is converged. Consequently, $\lim\limits_{n\to\infty}[J(\theta_n) - J(\theta_{n+1})] = 0$. Then by transferring to the limit from (33) we get $\|\theta_n - \theta_{n+1}\| \to 0$, at $n \to \infty$.

We show, that the sequence $\{\theta_n\} \subset X_1$ is minimizing. As it follows from the lemma 1, the functional $J(\theta) \in C^{1,1}(X_1)$ is convex. Then necessarily and sufficiently the inequality is satisfied

$$J(\theta_2) - J(\theta_1) \leq < J'(\theta_2), \theta_2 - \theta_1 >_{L_2}, \;\; \forall\theta_1, \; \theta_2 \in X_1.$$

From the inequality at $\theta_1 = \theta_* \in X_* \subset X_1$, $\theta_2 = \theta_n \in X$, we get

$$J(\theta_n) - J(\theta_*) \leq < J'(\theta_n), \theta_n - \theta_* >_{L_2} = < J'(\theta_n), \theta_n - \theta_{n+1} > - < J'(\theta_n), \theta_* - \theta_{n+1} > . \tag{34}$$

From (31) at $\theta = \theta_n$, we obtain

$$< J'(\theta_n), \theta_* - \theta_{n+1} > \geq \frac{1}{\alpha_n} < \theta_n - \theta_{n+1}, \theta_* - \theta_{n+1} > . \tag{35}$$

From (34), (35) we get

$$\begin{gathered} J(\theta_n) - J(\theta_*) \leq < J'(\theta_n) - \frac{1}{\alpha_n}(\theta_* - \theta_{n+1}), \theta_n - \theta_{n+1} > \leq \\ \leq \|J'(\theta_n) - \frac{1}{\alpha_n}(\theta_* - \theta_{n+1})\|\|\theta_n - \theta_{n+1}\| \leq \left(\sup\|J'(\theta_n)\| + \frac{r}{\varepsilon_0}\right)\|\theta_n - \theta_{n+1}\| = \\ = l\|\theta_n - \theta_{n+1}\|, \;\; l = const > 0, \end{gathered} \tag{36}$$

where r is a diameter of the set X_1, $\theta_* - \theta_{n+1} \leq r$, $\dfrac{1}{\alpha_n} \leq \dfrac{1}{\varepsilon_0}$, $0 \leq \varepsilon_0 \leq \alpha_n$.

Since $\|\theta_n - \theta_{n+1}\| \to 0$ at $n \to \infty$, that from (36) follows $\lim\limits_{n\to\infty} J(\theta_n) = J(\theta_*) = J_* = \inf\limits_{\theta \in X_1} J(\theta)$. This means, that the sequence $\{\theta_n\} \subset X_1$ is minimizing.

We show, that the sequence $\{\theta_n\} \subset X_1$ weakly converges to the point $\theta_* \in X_*$. In fact, the set X_1 is weakly bicompactly, the sequence $\{\theta_n\} \subset X_1$. Consequently, the sequence $\{\theta_n\} \subset X_1$ has at least one subsequence $\{\theta_{k_m}\} \subset X_1$ such, that $\theta_{k_m} \overset{\text{сл}}{\to} \theta_*$ at $m \to \infty$, moreover $\theta_* \in X_1$. Since the sequence $\{J(\theta_n)\}$ converges to $J(\theta_*)$, that the numeric sequence $J(\theta_{k_m})$ as well as converges to the number $J(\theta_*)$ i.e. $\lim\limits_{m\to\infty} J(\theta_{k_m}) = J(\theta_*)$.

From the inequality (33), (36) follows, that

$$a_n \leq l\|\theta_n - \theta_{n+1}\|, \;\; a_n - a_{n+1} \geq \varepsilon_1 \|\theta_n - \theta_{n+1}\|, \;\; a_n = J(\theta_n) - J(\theta_*).$$

Then $a_n \leq \dfrac{1}{A_n}$, $n = 1, 2, \ldots$, $A = \dfrac{\varepsilon_1}{l^2} > 0$ (see. [5]). Hence it follows the third statement of the lemma. Theorem is proved.

5. Solution existence

Let $\theta_* = (v_*(t), u_*(t), p_*(t), x_{0*}, x_{1*}, d_*) \in X_1$ be a solution of the optimization problem (16) – (20). Then

$$w_*(t) = v_*(t) + \lambda_1(t, x_{0*}, x_{1*}) + N_1(t) z(t_1, v_*), \;\; t \in I, \;\; v_*(t) \in V_1, \;\; t \in I,$$

$$y_*(t) = z(t, v_*) + \lambda_2(t, \xi_{0*}, \xi_{1*}) + N_2(t) z(t_1, v_*), \;\; p_*(t) \in V(t), \;\; t \in I,$$

where $\xi_{0*} = (x_{0*}, O_{m_2,1})$, $\xi_{1*} = (x_1^*, \bar{c}_*)$, $\bar{c}_* = (c_1 - d_{1*}, \ldots, c_{m_1} - d_{m_1*}, c_{m_1+1}, \ldots, c_{m_2})$, $d_* = (d_{1*}, \ldots, d_{m_1*}) \in D_1$, $(x_{0*}, x_{1*}) \in S$.

We notice, that the value $J(\theta) \geq 0$, $\forall \theta$, $\theta \in X_1$. In particular, the value $J(\theta_*) = 0$.

Theorem 5 *Let the matrix be $W(t_0, t_1) > 0$. For existence of a solution of the boundary value problem (1) – (5) necessarily and sufficiently, that the value $J(\theta_*) = 0$, where $\theta_* = \theta_*(t) \in X_1$ is the solution of the optimization problem (16) – (20).*

Proof. Let the value be $J(\theta_*) = 0$. We show, that the boundary value problem (1) – (5) has a solution $x_*(t) = P y_*(t)$, $t \in I$. As it follows from optimization problem (16) – (20) the value $J(\theta_*) = 0$ if and only if

$$w_*(t) = v_*(t) + \lambda_1(t, x_{0*}, x_{1*}) + N_1(t) z(t_1, v_*) = u_*(t), \;\; t \in I,$$

$$p_*(t) = L(t) P y_*(t), \;\; t \in I, \;\; w_*(t) = \int_{t_0}^{t_1} K(t, \tau) P y_*(\tau) d\tau,$$

where $u_*(t) \in U(t)$, $p_*(t) \in V(t)$, $(x_{0*}, x_{1*}) \in S$, $d_* \in D_1$. The function $y_*(t)$, $t \in I$ is a solution of the linear controllable system (9) – (11). Consequently, the equality is valid

$$\dot{y}_*(t) = A(t) y_*(t) + B(t) u_*(t) + \mu_1(t), \;\; t \in I, \tag{37}$$

$$y_*(t_0) = \xi_{0*} = (x_{0*}, O_{m_2,1}), \;\; y_*(t_1) = \xi_{1*} = (x_{1*}, \bar{c}_*), \tag{38}$$

$$(x_{0*}, x_{1*}) \in S, \;\; d_* \in D_1, \;\; u_*(t) \in U(t), \;\; t \in I, \tag{39}$$

From the equality

$$w_*(t) = u_*(t) = \int_{t_0}^{t_1} K(t,\tau) P y_*(\tau) d\tau \in U(t), \;\; t \in I$$

and relations (37) – (39) follow that $\xi_*(t) = y_*(t)$, $Py_*(t) = x_*(t)$, $t \in I$, where $\xi_*(t) = (x_*(t), \eta_*(t))$, $x_*(t_0) = x_{0*}$, $x_*(t_1) = x_{1*}$, $\eta_*(t_0) = 0$, $\eta_*(t_1) = \overline{c}_*$. Since $P\xi_*(t) = Py_*(t) = x_*(t)$, $t \in I$, that the function

$$x_*(t) = Py_*(t) = P[z(t, v_*) + \lambda_2(t, \xi_{0*}, \xi_{1*}) + N_2(t) z(t_1, v_*)], \;\; t \in I$$

is the solution of the boundary value problem (1) – (5). We notice, that from inclusion $p_*(t) \in V(t)$, $t \in I$, $u_*(t) \in U(t)$ follows the function $x_*(t) \in G(t)$, $t \in I$. From the conditions that the function $\eta_*(t)$, $t \in I$ satisfies to the conditions $\eta_*(t_0) = 0$, $\eta_*(t_1) = \overline{c}_*$ follows satisfication of the integral constraints (4), (5). Theorem is proved.

6. Construction of a solution of the boundary value problem (1) – (5)

From the theorem 5 follows the algorithm for solving of the boundary value problem (1) – (5):

1) By introducing the auxiliary variables $\eta(t)$, $t \in I$, the origin boundary value problem (1) – (5) is transformed to the form (6) – (8).

2) To make sure, that the matrix $W(t_0, t_1)$ is positive defined. The condition $W(t_0, t_1) > 0$ is necessary and sufficient condition for solvability of the integral equation (12) at any $a \in R^{n+m_2}$. However this condition is necessry condition for solvability of the boundary value problem (1) – (5). If the matrix $W(t_0, t_1)$ is not positive defined, than the boundary value problem (1) – (5) has not a solution.

Fundamental matrix of the solutions $\theta(t)$, $t \in I$ of the linear system $\dot{\omega} = A(t)\omega$ is a solution of the initial problem $\dot{\theta}(t) = A(t)\theta(t)$, $\theta(t_0) = I_{n+m_2}$, $t \in I$, where I_{n+m_2} is an unitary matrix of the order $(n+m_2) \times (n+m_2)$. The matrix $\theta(t)$, $t \in I$ can be calculated by an accuracy.

3) To solve an optimization problem (16) – (20) by constructing a minimizing sequence $\{\theta_n\} \subset X_1$. As result, we find $\theta_* \in X_1$, $J(\theta_*) = \inf_{\theta \in X_1} J(\theta) = \min_{\theta \in X_1} J(\theta)$. The point $\theta_* \in X_1$ always exists at $W(t_0, t_1) > 0$. For existing of a solution for the boundary value problem (1) – (5) necessarily and sufficiently, that $J(\theta_*) = 0$. If $J(\theta_*) > 0$, then the boundary value problem (1) – (5) has not any solution.

In the case $J(\theta_*) = 0$ we find: $w_*(t) = u_*(t)$, $y_*(t)$, $t \in I$. We define the function $x_*(t) = Py_*(t)$, where $P = (I_n, O_{nm_2})$ is known matrix.

The function $x_*(t)$, $t \in I$ is the solution of the boundary value problem (1) – (5). We consider the following particular cases:

1) In the absence of the integral constraints (4), (5). In this case, $A(t) = A_0(t)$, $B(t) = B_0(t)$, $\mu_1(t) = \mu(t)$, $t \in I$. Linear controllable system (9) – (11) is written as

$$\dot{y} = A_0(t) y + B_0(t) w(t) + \mu(t), \;\; t \in I, \;\; w(\cdot) \in L_2(I, R^m)$$

$$y(t_0) = x_0, \quad y(t_1) = x_1, \quad (x_0, x_1) \in S,$$

function $\xi(t) = x(t)$, $t \in I$, and function $z(t)$, $t \in I$ is a solution of the differential equation

$$\dot{z} = A_0(t)z + B_0(t)v(t), \quad z(t_0) = 0, \quad v(\cdot) \in L_2(I, R^m).$$

2) In the absence of the integral constraints (4), (5) and phase constraints (3). In this case, the optimization problem (16) – (20) is written as: minimize the functional

$$J(v, x_0, x_1) = \int_{t_0}^{t_1} |w(t) - \int_{t_0}^{t_1} K(t, \tau)y(\tau)d\tau|^2 \to \inf$$

at conditions

$$\dot{z} = A_0(t)z + B_0(t)v(t), \quad z(t_0) = 0, \quad v(\cdot) \in L_2(I, R^m), \quad (x_0, x_1) \in S.$$

We show the example of the basic results considered above.

Example. We consider the following boundary value problem:

$$\begin{aligned} \dot{x}_1 = \frac{1}{t}x_1 - \int_1^2 e^{t\tau}x_2(\tau)d\tau + \mu_1(t), \quad t \in [1, 2], \quad \tau \in [1, 2,]; \\ \dot{x}_2 = x_1 - \int_1^2 e^{t^2\tau}x_1(\tau)d\tau + \mu_2(t), \end{aligned} \tag{40}$$

where

$$\mu_1(t) = \frac{1}{2}\frac{e^t}{t}\left[\frac{e^t}{t^2}(3t^2 - 4t + 2) - \frac{1}{t^2}(-2t + 2)\right], \quad t \in [1, 2]$$

$$\mu_2(t) = \frac{e^{t^2}}{t^4}\left[(2t^2 - 1)e^{t^2} - t^2 + 1\right], \quad t \in [1, 2].$$

The boundary conditions have the form

$$\begin{aligned} x_1(1) + x_2(1) - x_1(2) - x_2(2) = -5/2, \\ -x_1(1) + 2x_2(1) + x_1(2) + 3x_2(2) = 11/2. \end{aligned} \tag{41}$$

The phase constraints are defined by relations:

$$\begin{aligned} x(t) = (x_1(t), x_2(t)) \in G(t) : G(t) = \Big\{x \in R^2 / 1 \le x_1(t) \le 2, \\ 0 \le x_2(t) \le \frac{3}{2}; \quad \frac{e^2}{2} \le \int_1^2 e^{t\tau}x_2(\tau)d\tau \le \frac{1}{8}(3e^4 + e^2); \\ e^2 \le \int_1^2 e^{t^2\tau}x_1(\tau)d\tau \le \frac{e^4}{16}(7e^4 - 3)\Big\}. \end{aligned} \tag{42}$$

The integral constraints have the form

$$g_1(x) = \int_1^2 x_1(t)dt \le 2, \;\; g_2(x) = \int_1^2 x_2(t)dt = 2/3. \tag{43}$$

As it follows from (40) – (43):

$$A_0(t) = \begin{pmatrix} \frac{1}{t} & 0 \\ 1 & 0 \end{pmatrix}, \;\; B_0(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; K(t,\tau) = \begin{pmatrix} 0 & -e^{t\tau} \\ -e^{t^2\tau} & 0 \end{pmatrix},$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \;\; x_0(t) = \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix}, \;\; x_1 = \begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix}, \;\; \mu(t) = \begin{pmatrix} \mu_1(t) \\ \mu_2(t) \end{pmatrix}.$$

Let

$$E = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \;\; F = \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix}, \;\; e = \begin{pmatrix} -5/11 \\ 11/2 \end{pmatrix}.$$

Then the boundary conditions (41) are written in the form

$$Ex_0 + Fx_1 = e.$$

A. Transformation. Being introduced the functions

$$\eta_1(t) = \int_{t_0}^{t} x_1(\tau)d\tau, \;\; \eta_2(t) = \int_{t_0}^{t} x_2(\tau)d\tau, \;\; t_0 = 1, \;\; t \in [1,2]$$

the integral constrains (43) we write in the form

$$\dot\eta_1 = x_1, \;\; \dot\eta_2 = x_2, \;\; \eta_1(1) = 0, \;\; \eta_1(2) = 2 - d, \;\; \eta_2(2) = 2/3,$$

$$d \in D = \{d \in R^1 / d \ge 0\}.$$

Then the boundary value problem (40) – (43) is written as

$$\dot\xi = A(t)\xi + B(t)\int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau + \overline{\mu}_1(t), \;\; t \in [1,2] = I,$$

$$\xi(1) = \xi_0 = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix}, \;\; \xi(t_1) = \xi_1 = \begin{pmatrix} x_1 \\ 2-d \\ 2/3 \end{pmatrix},$$

$$(x_0, x_1) \in S = \{(x_0, x_1) \in R^4 / Ex_0 + Fx_1 = e\}, \;\; d \in D, \;\; P\xi(t) \in G(t),$$

where

$$A(t) = \begin{pmatrix} \frac{1}{t} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \;\; B(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \;\; \overline{\mu}(t) = \begin{pmatrix} \mu(t) \\ 0 \\ 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P\xi = x, \quad \xi = \xi(t) = \begin{pmatrix} x(t) \\ \eta(t) \end{pmatrix}, \quad \eta(t) = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix}.$$

B. Linear controllable system. For this example the relations (9) – (11) are written as

$$\dot{y} = A(t)y + Bw(t) + \overline{\mu}(t), \quad t \in [1,2] = I,$$

$$y(1) = \xi_0 = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix}, \quad y(2) = \xi_1 = \begin{pmatrix} x_1 \\ 2-d \\ 2/3 \end{pmatrix}, \quad \overline{c} = \begin{pmatrix} 2-d \\ 2/3 \end{pmatrix},$$

$$(x_0, x_1) \in S, \quad d \in D, \quad w(\cdot) \in L_2(I, R^2), \quad n = 2, \; m_2 = 2, \; m = 2.$$

Fundamental matrix of the solutions $\theta(t)$ of the lineat homogeneous system $\dot{\omega} = A(t)\omega$ is defined by solution of the equation

$$\dot{\theta}(t) = A(t)\theta, \quad \theta(1) = I_4, \quad t \in I,$$

where I_4 is unitary matrix of the order 4×4. Solution of the equation is the matrix

$$\theta(t) = \begin{pmatrix} t & 0 & 0 & 0 \\ \frac{t^2}{2} - \frac{1}{2} & 1 & 0 & 0 \\ \frac{t^2}{2} - \frac{1}{2} & 0 & 1 & 0 \\ \frac{t^3}{6} - \frac{t}{2} + \frac{1}{3} & t-1 & 0 & 1 \end{pmatrix}, \quad t \in [1;2].$$

Inverse matrix $\theta^{-1}(t)$ equals

$$\theta^{-1}(t) = \begin{pmatrix} \frac{1}{t} & 0 & 0 & 0 \\ -\frac{t}{2} + \frac{1}{2t} & 1 & 0 & 0 \\ -\frac{t}{2} + \frac{1}{2t} & 0 & 1 & 0 \\ \frac{t^2}{3} - \frac{t}{2} + \frac{1}{6t} & -(t-1) & 0 & 1 \end{pmatrix}.$$

The matrixes

$$\Phi(t_0, t) = \Phi(1, t) = \theta(1)\theta^{-1}(t) = \theta^{-1}(t), \quad \Phi(t_0, t_1) = \Phi(1, 2) = \theta(1)\theta^{-1}(2) = \theta^{-1}(2),$$

vector

$$a = \Phi(1,2)\xi_1 - \xi_0 - \int_1^2 \Phi(1,t)\overline{\mu}(t)dt,$$

and integral equation (12) for this example has the form

$$\int_1^2 \Phi(1,t)B(t)w(t)dt = a.$$

As it follows from theorem 1, the matrix

$$W(1,2) = \int_1^2 \Phi(1,t)B(t)B^*(t)\Phi^*(1,t)dt = \int_1^2 \theta^{-1}(t)B(t)B^*(t)\theta^{*-1}(t)dt =$$

$$= \int_1^2 \begin{bmatrix} \frac{1}{t^2} & -\frac{1}{2}+\frac{1}{2t^2} & -\frac{1}{2}+\frac{1}{2t^2} \\ -\frac{1}{2}+\frac{1}{2t^2} & (\frac{t}{2}-\frac{1}{2t})^2+1 & (\frac{t}{2}-\frac{1}{2t})^2 \\ -\frac{1}{2}+\frac{1}{2t^2} & (\frac{t}{2}-\frac{1}{2t})^2 & (\frac{t}{2}-\frac{1}{2t})^2 \\ \frac{t}{3}-\frac{1}{2}+\frac{1}{6t^2} & (-\frac{t^2}{3}+\frac{t}{2}-\frac{1}{6t})(\frac{t}{2}-\frac{1}{2t})-(t-1) & (-\frac{t^2}{3}+\frac{t}{2}-\frac{1}{6t})(\frac{t}{2}-\frac{1}{2t}) \end{bmatrix}$$

$$\begin{bmatrix} \frac{t}{3}-\frac{1}{2}+\frac{1}{6t^2} \\ (\frac{t}{2}-\frac{1}{2t})(-\frac{t^2}{3}+\frac{t}{2}-\frac{1}{6t})-(t-1) \\ (\frac{t}{2}-\frac{1}{2t})(-\frac{t^2}{3}+\frac{t}{2}-\frac{1}{6t}) \\ (-\frac{t^2}{3}+\frac{t}{2}-\frac{1}{6t})^2+(t-1)^2 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{12} \\ -\frac{1}{4} & \frac{29}{24} & \frac{5}{24} & -\frac{7}{12} \\ -\frac{1}{4} & \frac{5}{24} & \frac{5}{24} & -\frac{1}{12} \\ \frac{1}{12} & -\frac{7}{12} & -\frac{1}{12} & \frac{133}{360} \end{bmatrix}.$$

The main minors of the matrix $W(1,2)$ equal

$$\Delta_1 = \frac{1}{2} > 0, \quad \Delta_2 = \begin{vmatrix} 1/2 & -1/4 \\ -1/4 & 29/24 \end{vmatrix} = \frac{13}{24} > 0,$$

$$\Delta_3 = \begin{vmatrix} 1/2 & -1/4 & -1/4 \\ -1/4 & 29/24 & 5/24 \\ -1/4 & 5/24 & 5/24 \end{vmatrix} = \frac{1}{24} > 0, \quad \Delta_4 = |W(1,2)| = \frac{61}{17280} > 0.$$

Consequently, the matrix $W(1,2)$ is positive defined.

The inverse matrix

$$W^{-1}(1,2) = \frac{1}{\Delta} \begin{pmatrix} \frac{93}{5184} & \frac{1}{576} & \frac{61}{2880} & \frac{1}{288} \\ \frac{1}{576} & \frac{241}{17280} & -\frac{61}{17280} & \frac{1}{48} \\ \frac{61}{2880} & -\frac{61}{17280} & \frac{793}{17280} & 0 \\ \frac{1}{288} & \frac{1}{48} & 0 & \frac{1}{24} \end{pmatrix} = \begin{bmatrix} \frac{310}{61} & \frac{310}{61} & 6 & \frac{60}{61} \\ \frac{30}{61} & \frac{240}{61} & -1 & \frac{360}{61} \\ 6 & -1 & \frac{793}{61} & 0 \\ \frac{60}{61} & \frac{360}{61} & 0 & \frac{720}{61} \end{bmatrix},$$

where $\Delta = \Delta_4 = \dfrac{61}{17280}$. Matrixes

$W(1,t) = \int_1^t \theta^{-1}(\tau)B(\tau)B^*(\tau)\theta^{*-1}(\tau)d\tau =$

$$= \begin{bmatrix} -\frac{1}{t}+1 & -\frac{t}{2}-\frac{1}{2t}+1 \\ -\frac{t}{2}-\frac{1}{2t}+1 & \frac{t^3}{12}+\frac{t}{2}-\frac{1}{4t}-\frac{1}{3} \\ -\frac{t}{2}-\frac{1}{2t}+1 & \frac{t^3}{12}-\frac{t}{2}-\frac{1}{4t}+\frac{2}{3} \\ t^2-\frac{t}{2}-\frac{1}{6t}+\frac{1}{2} & -\frac{t^4}{24}+\frac{t^3}{12}-\frac{5t^2}{12}-\frac{1}{12t}+\frac{2}{3}t-\frac{5}{24} \end{bmatrix}$$

$$\left.\begin{matrix} -\frac{t}{2}-\frac{1}{2t}+1 & \frac{t^2}{6}-\frac{t}{2}-\frac{1}{6t}+\frac{1}{2} \\ \frac{t^3}{12}-\frac{t}{2}-\frac{1}{4t}+\frac{2}{3} & -\frac{t^4}{24}+\frac{t^3}{12}-\frac{5t^2}{12}-\frac{1}{12t}+\frac{2}{3}t-\frac{5}{24} \\ \frac{t^3}{12}-\frac{t}{2}-\frac{1}{4t}+\frac{2}{3} & -\frac{t^4}{24}+\frac{t^3}{12}-\frac{2t}{3}+\frac{5t^2}{12}-\frac{1}{12t}+\frac{7}{24} \\ -\frac{t^4}{24}+\frac{t^3}{12}-\frac{t^2}{12}-\frac{2t}{3}-\frac{1}{12}t+\frac{7}{24} & \frac{t^5}{45}-\frac{t^4}{12}+\frac{5t^3}{12}-\frac{1}{36t}+\frac{5}{6}t-\frac{17t^2}{18}-\frac{78}{360} \end{matrix}\right]$$

The matrix $W(t,2) = W(1,2) - W(1,t) =$

$$= \left[\begin{matrix} -\frac{1}{2}+\frac{1}{t} & -\frac{5}{4}+\frac{t}{2}+\frac{1}{2t} \\ -\frac{5}{4}+\frac{t}{2}+\frac{1}{2t} & \frac{37}{24}-\frac{t^3}{12}-\frac{t}{2}+\frac{1}{4t} \\ -\frac{5}{4}+\frac{t}{2}+\frac{1}{2t} & -\frac{11}{24}-\frac{t^3}{12}+\frac{t}{2}+\frac{1}{4t} \\ -\frac{5}{12}-t^2+\frac{t}{2}+\frac{1}{6t} & -\frac{9}{24}+\frac{t^4}{24}-\frac{t^3}{12}+\frac{5t^2}{12}+\frac{1}{12t}-\frac{2}{3}t \end{matrix}\right.$$

$$\left.\begin{matrix} -\frac{5}{4}+\frac{t}{2}+\frac{1}{2t} & -\frac{5}{12}-\frac{t^2}{6}+\frac{t}{2}+\frac{1}{6t} \\ -\frac{11}{24}-\frac{t^3}{12}+\frac{t}{2}+\frac{1}{4t} & -\frac{9}{24}+\frac{t^4}{24}-\frac{t^3}{12}+\frac{5t^2}{12}+\frac{1}{12t}-\frac{2}{3}t \\ -\frac{11}{24}-\frac{t^3}{12}+\frac{t}{2}+\frac{1}{4t} & -\frac{9}{24}+\frac{t^4}{24}-\frac{t^3}{12}+\frac{2}{3}t-\frac{5t^2}{12}+\frac{1}{12t} \\ -\frac{9}{24}+\frac{t^4}{24}-\frac{t^3}{12}-\frac{t^2}{12}+\frac{2}{3}t+\frac{1}{12t} & \frac{211}{360}-\frac{t^5}{45}+\frac{t^4}{12}-\frac{5t^3}{12}+\frac{1}{36t}-\frac{5}{6}t+\frac{17t^2}{18} \end{matrix}\right]$$

By known matrixes $\theta(t)$, $\theta(1) = \theta^{-1}(1) = I_4$, $\theta^{-1}(t)$, $\theta^{-1}(\tau)$, $W(1,2)$, $W^{-1}(1,2)$, $W(1,t)$, $W(t,2)$ we define

$$a = \theta^{-1}(2)\xi_1 - \xi_0 - \int_1^2 \theta^{-1}(t)\overline{\mu}(t)dt, \quad \lambda_1(t,\xi_0,\xi_1) = B^*(t)\theta^{*-1}(t)\times$$

$$\times W^{-1}(1,2)a, \quad N_1(t) = -B^*(t)\theta^{*-1}(t)W^{-1}(1,2)\theta^{-1}(2),$$

$$\lambda_2(t,\xi_0,\xi_1) = \theta(t)W(t,2)W^{-1}(1,2)\xi_0 + \theta(t)W(1,t)W^{-1}(1,2)\theta^{-1}(t_1)\xi_1 +$$

$$+ \int_1^t \theta(t)\theta^{-1}(\tau)\overline{\mu}(\tau)d\tau - \theta(t)W(1,t)W^{-1}(1,2)\int_1^2 \theta^{-1}(\tau)\overline{\mu}(\tau)d\tau,$$

$$N_2(t) = -\theta(t_1)W(1,t)W^{-1}(1,2)\theta^{-1}(t_1),$$

as well as the functions,

$$w(t) = v(t) + \lambda_1(t,\xi_0,\xi_1) + N_1(t)z(2,v),$$

$$y(t) = z(t) + \lambda_2(t,\xi_0,\xi_1) + N_2(t)z(2,v), \quad t \in I = [1;2],$$

where $z(t) = z(t, v)$, $t \in I$ is a solution of the differential equation

$$\dot{z} = A(t)z + B(t)v(t), \ \ z(1) = 0, \ \ v(\cdot) \in L_2(I, R^2).$$

We notice, that

$$z(t) = \int_1^t \theta(t)\theta^{-1}(\tau)B(\tau)v(\tau)d\tau, \ \ z(\tau) = \int_1^\tau \theta(\tau)\theta^{-1}(\eta)B(\eta)v(\eta)d\eta.$$

Optimization problem. For this example, an optimization problem (16) – (20) has the form

$$J(v, u, p, x_0, x_1, d) = \int_1^2 \{|w_1(t) - u_1(t)|^2 + |w_2(t) - u_2(t)|^2+$$

$$+|p_1(t) - y_1(t)|^2 + |p_2(t) - y_2(t)|^2 + |w_1(t) - \int_1^2 e^{t\tau} y_2(\tau)d\tau|^2+$$

$$+|w_2(t) - \int_1^2 e^{t^2\tau} y_1(\tau)d\tau|^2\}dt \to \inf$$

at conditions

$$\dot{z} = A(t)z + B(t)v(t), \ \ z(1) = 0, \ \ v(\cdot) = (v_1(\cdot), v_2(\cdot)) \in L_2(I, R^2),$$

$$u_1(t) \in U_1 = \{u_1(\cdot) \in L_2(I, R^1)/e^2/2 \le u_1(t) \le \frac{1}{8}(3e^4 + e^2), \ \ t \in I\},$$

$$u_2(t) \in U_2 = \{u_2(\cdot) \in L_2(I, R^1)/e^2 \le u_2(t) \le \frac{e^2}{16}(7e^4 - 3)\},$$

$$p_1(t) \in \overline{V}_1 = \{p_1(\cdot) \in L_2(I, R^1)/1 \le p_1(t) \le 2, \ \ t \in I\},$$

$$p_2(t) \in \overline{V}_2 = \{p_2(\cdot) \in L_2(I, R^1)/0 \le p_2(t) \le \frac{3}{2}, \ \ t \in I\},$$

$$Ex_0 + Fx_1 = e, \ \ d \in \{d \in R^1/d \ge 0\} = D,$$

where $w(t) = (w_1, w_2(t))$, $w(\cdot) \in L_2(I, R^2)$, $v(t) = (v_1(t), v_2(t))$, $v(\cdot) \in L_2(I, R^2)$, $S = \{(x_0, x_1) \in R^4/Ex_0 + Fx_1 = e\}$.

The sets $X = L_2(I, R^1) \times L_2(I, R^1) \times U_1 \times U_2 \times V_1 \times V_2 \times S \times D$, $H = L_2(I, R^1) \times L_2(I, R^1) \times L_2(I, R^1) \times L_2(I, R^1) \times L_2(I, R^1) \times L_2(I, R^1) \times R^4 \times R^1$, $n = 2$, $m_1 = 1$.

Since the function

$$w_1(t) = v_1(t) + T_{11}(t)x_0 + T_{21}x_1 + T_{31}(t)d + \mu_{11}(t) + N_{11}(t)z(2, v_1, v_2), \ \ t \in I,$$

$$w_2(t) = v_2(t) + T_{12}(t)x_0 + T_{22}x_1 + T_{32}(t)d + \mu_{12}(t) + N_{12}(t)z(2, v_1, v_2),$$

$$y_1(t) = z_1(t, v_1, v_2) + E_{11}(t)x_0 + E_{21}(t)x_1 + E_{31}(t)d + \mu_{31}(t) + N_{21}(t)z(2, v), \ \ t \in I,$$

$$y_2(t) = z_2(t, v_1, v_2) + E_{12}(t)x_0 + E_{22}(t)x_1 + E_{32}(t)d + \mu_{32}(t) + N_{22}(t)z(2, v), \ \ t \in I,$$

$$y_3(t) = z_3(t,v) + E_{13}(t)x_0 + E_{23}(t)x_1 + E_{33}(t)d + \mu_{33}(t) + N_{22}(t)z(2,v), \quad t \in I,$$
$$y_4(t) = z_4(t,v) + E_{14}(t)x_0 + E_{24}(t)x_1 + E_{34}(t)d + \mu_{34}(t) + N_{24}(t)z(2,v), \quad t \in I,$$

that $F_0(q(t),t) = |w_1-u_1|^2+|w_2-u_2|^2+|p_1-y_1|^2+|p_2-y_2|^2$, $F_1(q,t) = |w_1-\int\limits_1^2 e^{t\tau}y_2(\tau)d\tau|^2+$ $|w_2-\int\limits_1^2 e^{t^2\tau}y_1(\tau)d\tau|^2$, where

$$\overline{T}_{11}(t) = T_{11}(t) - \int\limits_1^2 e^{t\tau}E_{12}(\tau)d\tau, \quad \overline{T}_{21}(t) = T_{21}(t) - \int\limits_1^2 e^{t\tau}E_{21}(\tau)d\tau,$$

$$\overline{T}_{31}(t) = T_{31}(t) - \int\limits_1^2 e^{t\tau}E_{32}(\tau)d\tau, \quad \overline{\mu}_{21}(t) = \mu_{11}(t) - \int\limits_1^2 e^{t\tau}\mu_{32}(\tau)d\tau,$$

$$\overline{N}_{11}(t) = N_{11}(t) - \int\limits_1^2 e^{t\tau}N_{22}(\tau)d\tau, \quad \overline{w}_1(t) = v_1(t) + \overline{T}_{11}(t)x_0 + \overline{T}_{21}(t)x_1 +$$

$$+\overline{T}_{31}(t)d + \overline{N}_{11}(t)z(2,v), \quad w_1(t) - \int\limits_1^2 e^{t\tau}y_2(\tau)d\tau = \overline{w}_1(t) -$$

$$-\int\limits_1^2 e^{t\tau}z_2(\tau,v)d\tau, \quad w_2(t) - \int\limits_1^2 e^{t^2\tau}y_1(\tau)d\tau = \overline{w}_2(t) - \int\limits_1^2 e^{t^2\tau}z_1(\tau,v)d\tau.$$

Partial derivatives of the functions $F(q,t) = F_0(q,t) + F_1(q,t)$ are calculated by formula (22) and sequences $\{\theta_n\} \subset X$ are defined by algorithm (30):

$$v^1_{n+1} = P_{V_1}[v^1_n - \alpha_n J'_{v_1}(\theta_n)], \quad v^2_{n+1} = P_{V_1}[v^2_n - \alpha_n J'_{v_2}(\theta_n)],$$
$$u^1_{n+1} = P_{U_1}[u^1_n - \alpha_n J'_{u_1}(\theta_n)], \quad u^2_{n+1} = P_{U_2}[u^2_n - \alpha_n J'_{u_2}(\theta_n)],$$
$$p^1_{n+1} = P_{\overline{V}_1}[p^1_n - \alpha_n J'_{p_1}(\theta_n)], \quad p^2_{n+1} = P_{\overline{V}_2}[p^2_n - \alpha_n J'_{p_2}(\theta_n)],$$
$$x^{n+1}_0 = P_S[x^n_0 - \alpha_n J'_{x_0}(\theta_n)], \quad x^{n+1}_1 = P_S[x^n_1 - \alpha_n J'_{x_1}(\theta_n)],$$
$$d_{n+1} = P_{D_1}[d_n - \alpha_n J'_d(\theta_n)], \quad n = 0,1,2,\dots, \quad \alpha_n = \frac{1}{K} = const > 0.$$

We notice, that

$$(x^{n+1}_0, x^{n+1}_1) = (x^n_0 - \alpha_n J'_{x_0}(\theta_n), x^n_1 - \alpha_n J'_{x_1}(\theta_n)) -$$

$$-\begin{pmatrix} E^* \\ F^* \end{pmatrix} \left\{ (E,\ F) - \begin{pmatrix} E^* \\ F^* \end{pmatrix} \right\}^{-1} \left\{ (E,\ F) - \begin{pmatrix} x^n_0 - \alpha_n J'_{x_0}(\theta_n) \\ x^n_1 - \alpha_n J'_{x_1}(\theta_n) \end{pmatrix} - e \right\},$$

where

$$(E,\ F)\begin{pmatrix} E^* \\ F^* \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -3 & 15 \end{pmatrix}, \quad \left\{ (E,\ F)\begin{pmatrix} E^* \\ F^* \end{pmatrix} \right\}^{-1} = \begin{pmatrix} 15/51 & 3/51 \\ 3/51 & 4/51 \end{pmatrix}.$$

Construction of the minimizing sequence.

1. The initial point is choosen $\theta_0 = (v_0^1, v_0^2, u_0^1, u_0^2, p_0^1, p_0^2, x_0^0, x_1^0, d_0) \in X$. In particular, $v_0^1(t) = \sin t$, $v_0^2(t) = \cos t$, $u_0^2(t) = [e^2 + \frac{e^2}{16}(7e^4 - 3)]/2$, $u_0^1(t) = [\frac{e^2}{2} + \frac{1}{8}(3e^4 + e^2)]/2$, $p_0^1(t) = 3/2$, $p_0^2(t) = 3/4$, $x_0^0 = (x_1(1) = -3,\ x_2(1) = 1)$, $x_1^0 = (x_1(2) = 1/2,\ x_2(2) = 0)$, $(x_0^0, x_1^0) \in S$.

2. To find a solution of the differential equation $\dot z_0 = A(t)z_0 + B(t)v_0(t)$, $z_0(1) = 0$, $v_0(t) = (v_0^1(t), v_0^2(t))$. As result we get $z_0(t) = z_0(t, v_0^0, v_0^1)$, $t \in I = [1;2]$.

3. To find a value $\psi_0(2) = -\int\limits_1^2 \frac{\partial F(q_0(t),t)}{\partial z_0(t_1)}dt$, $q_0(t) = (v_0^1, v_0^2, u_0^1, u_0^2, p_0^1, p_0^2, x_0^0, x_1^0, d_0, z_0(t), z_0(t_1))$. Solve the differential equation

$$\dot\psi_0(t) = \frac{\partial F(q_0(t),t)}{\partial z_0} - A^*(t)\psi_0(t), \quad \psi_0(2) = -\int\limits_1^2 \frac{\partial F(q_0(t),t)}{\partial z_0(t_1)}dt$$

and determine a function $\psi_0(t)$, $t \in [1,2]$.

4. To compute the partial derivatives in the initial point $\theta_0 \in X$. Finally, it is known $\frac{\partial F(q_0(t),t)}{\partial v}$, $\frac{\partial F(q_0(t),t)}{\partial u}$, $\frac{\partial F(q_0(t),t)}{\partial p}$, $\int\limits_1^2 \frac{\partial F(q_0(t),t)}{\partial x_0}dt$, $\int\limits_1^2 \frac{\partial F(q_0(t),t)}{\partial x_1}dt$, $\int\limits_{t_0}^{t_1} \frac{\partial F(q_0(t),t)}{\partial d}dt$.

5. To find

$$v_1^1 = P_{V_1}[v_0^1 - \alpha_0 J'_{v_1}(\theta_0)], \quad v_1^2 = P_{V_1}[v_0^2 - \alpha_0 J'_{v_2}(\theta_0)],$$
$$u_1^1 = P_{U_1}[u_0^1 - \alpha_0 J'_{u_1}(\theta_0)], \quad u_1^2 = P_{U_2}[u_0^2 - \alpha_0 J'_{u_2}(\theta_0)],$$
$$p_1^1 = P_{\overline{V}_1}[p_0^1 - \alpha_0 J'_{p_1}(\theta_0)], \quad p_1^2 = P_{\overline{V}_2}[p_0^2 - \alpha_0 J'_{p_2}(\theta_0)],$$
$$x_0^1 = P_S[x_0^0 - \alpha_0 J'_{x_0}(\theta_0)], \quad x_1^1 = P_S[x_1^0 - \alpha_0 J'_{x_1}(\theta_0)],$$
$$d_1 = P_{D_1}[d_0 - \alpha_0 J'_d(\theta_0)], \quad n = 0,1,2,\dots, \quad \alpha_0 = const > 0.$$

6. To repeat the items 1 – 5.

As it follows from theorem 4, the constructed sequences is minimizing i.e. $\lim\limits_{n\to\infty} J(\theta_n) = J_* = \inf\limits_{\theta \in X_1} J(\theta) = J(\theta_*)$, where $\theta_* = (v_{1*}(t), v_{2*}(t), u_{1*}(t), u_{2*}(t), p_{1*}(t), p_{2*}(t), x_{0*}, x_{1*}, d_*) \in X$ is solution of the optimization problem. If $J(\theta_*) = 0$, then $y_{1*}(t) = x_{1*}(t)$, $y_{2*}(t) = x_{2*}(t)$, $t \in [1;2]$ is solution of the boundary value problem (40) – (43). For this example the following results are obtained: $x_{1*}(t) = t$, $x_{2*}(t) = \frac{t^2}{2} - \frac{1}{2}$, $t \in [1;2]$, $x_{0*} = (1;0)$, $x_{1*} = (2;3/2)$, $d_* = 1/2$, $u_{1*}(t) = \frac{1}{2}\frac{e^t}{t}[\frac{e^t}{t^2}(3t^2 - 4t + 2) - \frac{1}{t^2}(-2t + 2)]$, $u_{2*}(t) = \frac{e^{t^2}}{t^4}[(2t^2 - 1)e^{t^2} - t^2 + 1]$, $t \in [1;2]$.

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References

[1] *Bykov Ja.V.* Some problems of integral-differential equations. Frunze, 1957. – 400 p.

[2] *Samoilenko A.M., Bojchuk O.A., Krivosheja S.A.* Boundary value problems for systems of the linear integral-differential equations with degenerate kernel // Ucr. math. journ. 1996. v. 48. No 11. P. 1576–1579.

[3] *Dzhumabaev D.S., Bakirova Je.A.* About properties of solvability of the linear twopoint boundary value problem for systems of the integral-differential equations // Diff.equat. 2013. v. 49, No 9. P. 1125-1-140.

[4] *Aisagaliev S.A.* Constructive theory of the boundary value problems of integral-differential equations with phase constraints // Proceedings of the International scientific conference "Problems of mathematics and informatics in XXI century". Bulletin of Kyrgyz state national university. Ser. 3, No. 4, 2000. P. 127–133.

[5] *Aisagaliev S.A., Aisagaliev T.S.* Methods of solution of the boundary value problems. – Almaty, Kazakh university, 2002. – 348 p.

[6] *Aisagaliev S.A.* Control by some system of differential equations // Diff. equat. 1991, v. 27, No 9. p. 1475–1486.

[7] *Aisagaliev S.A., Belogurov A.P.* Controllability and speed of the process described by parabolic equation with limited control // Siberian mathematical journal. – 2012. – v. 53, № 1. – P. 20-37.

[8] *Ajsagaliev S.A.* General solution of a class integral equations // Mathematical journal. Institute of Mathematics MES RK. - 2005. - v. 5, № 4. - P. 7-13.

[9] *Aisagaliev S.A., Kabidoldanova A.A.* Optimal control by linear systems with linear quality criteria and restrictions // Differential equations. – 2012. – v. 48, № 6. – P. 826-838.

[10] *Aisagaliev S.A., Kalimoldaev M.N.* Constructive method for solution of the boundary value problems for ordinary differential equations // Diff. equat. 2015. v. 51. No 2. P. 147–160.

[11] *Kalman R.* On general theory of control systems. Proceedings of the i-st congress IFAK, v. 2. Moscow, 1961, pp. 521-547.

[12] *Kolmogorov A.N., Fomin S.V.* Elements of the function theory and functional analysis. – M.: Nauka, 1989. – 623 p.