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On the general net spaces

The theory of the net spaces has a lot of applications in harmonic analysis and approximation theory. By the interpolation theorems of the net spaces it was obtained new results in many classical problems of analysis. In particular, new estimates of norms of the convolution operator were proved in the terms of the net spaces. Also, in the context of the theory of net spaces were considered questions related to the Fourier multipliers, estimates of norms of the Fourier transforms (Hardy and Littlewoods's theorems), inequalities in various metrics and other problems of analysis. In this paper are introduced general net spaces $N_{p,q,\alpha}$ depending on three positive parameters. Introduced spaces is close to the known net spaces. In paper we study properties of $N_{p,q,\alpha}$ spaces. In particular, we study their interpolation properties. Besides that we give equivalent norms in these spaces. Also we show that these spaces increase by the second parameter q. In paper we give some applications of general net spaces. We obtain estimate of norm of the Fourier transform of function from the Lorentz space. Analogous estimates were proved earlier for the net spaces and were used for proving criteria of integrability of Fourier transforms.
Key words: net spaces, interpolation, Lorentz spaces, Fourier transform.

Муканов А.Б.
Жалпыланған торлы кеңістіктер жайлы

Торлы кеңістіктер теориясы гармоникалық анализде және жуықтау теориясында көп қолданылады. Торлы кеңістіктердің интерполяциондық теоремалары арқылы анализдің көптеген мәселелерінде жаңа нәтижелер алынды. Соның ішінде торлы кеңістіктер терминінде орама операторының нормасының жаңа бағалаулары дәлелденген. Сонымен қатар торлы кеңістіктер теориясының мәнмәтінінде Фурье мультипликаторларына, Фурье түрлендіруінің нормаларын бағалауына (Харди және Литтлвуд теоремалары), әр түрлі метрикалар теңсіздіктеріне және анализдің басқа мәселелеріне байланысты сұрақтар қарастырылған. Бұл мақалада үш оң параметрлерге тәуелді жалпыланған торлы кеңістіктер енгізілді. Енгізілген кеңістіктер белгілі торлы кеңістіктерге жақын. Мақалада біз $N_{p,q,\alpha}$ кеңістіктерінің қасиеттерін зерттейміз. Соның ішінде олардың интерполяциондық қасиеттері зерттелген. Сонымен қатар берілген кеңістіктерде эквивалентті нормалар келтірілген. Және де кеңістіктердің екінші параметрі q бойынша үлкейетіні көрсетілген. Оған қоса мақалада жалпыланған торлы кеңістіктердің қолданыстары келтірілген. Лорентц кеңістіктегі функцияның Фурье түрлендіруінің нормасының бағалауы алынған. Мұндай бағалаулар бұрын торлы кеңістіктер үшін алынған және Фурье түрлендіруінің интегралдану критерийінде пайдаланылған.
Түйін сөздер: торлы кеңістіктер, интерполяция, Лорентц кеңістіктері, Фурье түрлендіруі.

Муканов А.Б.
Об обобщенных сетевых пространствах

Теория сетевых пространств имеет много приложений в гармоническом анализе и теории приближений. Благодаря интерполяционным теоремам сетевых пространств были получены новые результаты во многих классических вопросах анализа. В частности, в терминах сетевых пространств были доказаны новые оценки для нормы оператора свертки. Также в контексте теории сетевых пространств были рассмотрены вопросы, касающиеся мультипликаторов Фурье, оценок для норм преобразований Фурье (теоремы Харди и Литтлвуда), неравенств разных метрик и других проблем анализа.

В настоящей статье вводятся обобщенные сетевые пространства $N_{p,q,\alpha}$, зависящие от трех положительных параметров. Введенные пространства близки к известным сетевым пространствам. В статье мы изучаем свойства пространств $N_{p,q,\alpha}$. В частности, исследованы их интерполяционные свойства. Помимо этого приведены эквивалентные нормировки в данных пространствах. Также показано, что пространства расширяются по второму параметру q. В статье также приводятся приложения обобщенных сетевых пространств. Получена оценка для нормы преобразования Фурье функции из пространства Лоренца. Подобные оценки были ранее получены для сетевых пространств и были использованы для критерия интегрируемости преобразований Фурье.
Ключевые слова: сетевые пространства, интерполяция, пространства Лоренца, преобразование Фурье.

1. Introduction

Let μ be Lebesgue measure on $\mathbb{R}$, M be a fixed family of finite-measure sets in $\mathbb{R}$, which we will call the net. We define average $\bar{f}(t, M)$ of function $f(x)$ with respect to the net M as follows,

$$\bar{f}(t, M) = \sup_{\substack{e\in M \\ |e|>t}} \frac{1}{|e|} \left| \int_e f(x)d\mu \right|,$$

where supremum is taken over all sets $e \in M$ such that $|e| \stackrel{def}{=} \mu(e) > t, t \in (0, \infty)$. In the case of $\sup\{|e| : e \in M\} = \alpha < \infty$ we set $\bar{f}(t, M) = 0$ for $t > \alpha$.

Definition 1 *The net space $N_{p,q}(M)$, $0 < p, q \le \infty$ is the set of functions f such that*

$$\|f\|_{N_{p,q}(M)} = \left(\int_0^\infty \left(t^{1/p} \bar{f}(t, M)\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

if $q < \infty$, or

$$\|f\|_{N_{p,\infty}(M)} = \sup_{t>0} t^{\frac{1}{p}} \bar{f}(t, M) < \infty$$

if $q = \infty$

The net spaces were introduced by E.D. Nursultanov in [1]. Properties and applications of the net spaces in harmonic analysis were considered in [1]-[4]. In this paper we consider spaces close to the net spaces, study their properties and applications.

Definition 2 *Let $\alpha \in (0, 1]$, $0 < p, q \le +\infty$. We will say that a μ-measurable function $f \in N_{p,q;\alpha}$, if the following functional*

$$\|f\|_{N_{p,q;\alpha}} := \begin{cases} \left(\int_0^{+\infty} (t^{\frac{1}{p}} \widetilde{f}(t, \alpha))^q \frac{dt}{t} \right)^{\frac{1}{q}} & for \;\; 0 < p < \infty \;\; and \; 0 < q < \infty, \\ \sup_t t^{\frac{1}{p}} \widetilde{f}(t, \alpha) & for \;\; 0 < p \le \infty, \;\; q = \infty, \end{cases}$$

is finite, where

$$\widetilde{f}(t, \alpha) = \sup_{y\ge t} \frac{1}{y^\alpha} \left| \int_0^y \frac{f(s)}{(y - s)^{1-\alpha}} ds \right|.$$

Note that if $\alpha = 1$, then these spaces coincide with the net spaces $N_{p,q}(M)$, where $M = \{[t,\infty] : [t,\infty] \subseteq [0,\infty]\}$. Analogical discrete spaces were considered in [5]. Throughout this paper, we denote by C a positive constant that may be different on different occasions. In addition, $T \sim S$ means that $\frac{1}{C}S \le T \le CS$.

2. Interpolation properties of the general net spaces

Now we will study the interpolation between the $N_{p,q;\alpha}$ spaces. Let (A_0, A_1) be a compatible couple of quasi-normed spaces and

$$K(t,a) = K(t,a;A_0,A_1) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1$$

be the Peetre K-functional ([7]).
The space $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, consists of all elements $a \in A_0 + A_1$ for which the functional

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \begin{cases} \left(\int_0^\infty (t^{-\theta}K(t,a))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup\limits_{0<t<\infty} t^{-\theta}K(t,a), & q = \infty, \end{cases}$$

is finite.

Theorem 1 *Let $\alpha \in (0,1]$, $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1, q \le \infty$, $0 < \theta < 1$. Then*

$$(N_{p_0,q_0;\alpha}, N_{p_1,q_1;\alpha})_{\theta,q} \hookrightarrow N_{p,q;\alpha}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Before proving theorem 1, we prove some lemmas.

Lemma 1 *If $q_1 \le q_2$, then $N_{p,q_1;\alpha} \hookrightarrow N_{p,q_2;\alpha}$.*

Proof. Let $f \in N_{p,q_1;\alpha}$. Let $q_2 = +\infty$. Then using monotonicity of the function $\widetilde{f}(t,\alpha)$, we get

$$\|f\|_{N_{p,\infty;\alpha}} = \sup_{t\in[0,\infty]} t^{\frac{1}{p}}\widetilde{f}(t,\alpha) = \sup_{t\in[0,\infty]} \left(\frac{p}{q_1}\int\limits_0^t s^{\frac{q_1}{p}-1}ds\right)^{1/q_1} \widetilde{f}(t,\alpha)$$

$$= C\sup_{t\in[0,1]} \left(\int\limits_0^t (\widetilde{f}(t,\alpha))^{q_1} s^{\frac{q_1}{p}-1}ds\right)^{1/q_1} \le C\sup_{t\in[0,\infty]} \left(\int\limits_0^t (\widetilde{f}(s,\alpha))^{q_1} s^{\frac{q_1}{p}-1}ds\right)^{1/q_1}$$

$$\le C\left(\int\limits_0^\infty (\widetilde{f}(s,\alpha)s^{\frac{1}{p}})^{q_1}\frac{ds}{s}\right)^{1/q_1} = C\|f\|_{N_{p,q_1;\alpha}}.$$

Let now $q_2 < +\infty$. Then

$$\|f\|^{q_2}_{N_{p,q_2;\alpha}} = \int\limits_0^\infty \left(s^{\frac{1}{p}}\widetilde{f}(s,\alpha)\right)^{q_2}\frac{ds}{s} = \int\limits_0^\infty \left(s^{\frac{1}{p}}\widetilde{f}(s,\alpha)\right)^{q_2-q_1}\left(s^{\frac{1}{p}}\widetilde{f}(s,\alpha)\right)^{q_1}\frac{ds}{s}.$$

From the above, we get

$$\|f\|^{q_2}_{N_{p,q_2;\alpha}} \le \int\limits_0^\infty \left(\sup_{s\in[0,\infty]} s^{\frac{1}{p}}\widetilde{f}(s,\alpha)\right)^{q_2-q_1} \left(s^{\frac{1}{p}}\widetilde{f}(s,\alpha)\right)^{q_1}\frac{ds}{s}$$
$$= \|f\|^{q_2-q_1}_{N_{p,\infty;\alpha}} \int\limits_0^\infty \left(s^{\frac{1}{p}}\widetilde{f}(s,\alpha)\right)^{q_1}\frac{ds}{s} = \|f\|^{q_2-q_1}_{N_{p,\infty;\alpha}}\|f\|^{q_1}_{N_{p,q_1;\alpha}}$$
$$\le C\|f\|^{q_2-q_1}_{N_{p,q_1;\alpha}}\|f\|^{q_1}_{N_{p,q_1;\alpha}} = C\|f\|^{q_2}_{N_{p,q_1;\alpha}}.$$

Lemma 2 *Let $\alpha\in(0,1]$, $0<p,q\le+\infty$. Then*

$$\|f\|_{N_{p,q;\alpha}} \sim \left(\sum_{k=-\infty}^{\infty}\left(2^{\frac{k}{p}}\widetilde{f}(2^k,\alpha)\right)^q\right)^{\frac{1}{q}} \quad \textit{for } q<\infty$$

and

$$\|f\|_{N_{p,\infty;\alpha}} \sim \sup_{k\in\mathbb{Z}} 2^{\frac{k}{p}}\widetilde{f}(2^k,\alpha).$$

Proof. We will prove in the case when $q<\infty$. The case $q=\infty$ is proved in the same way. We have

$$\|f\|_{N_{p,q;\alpha}} = \left(\int_0^\infty \left(t^{\frac{1}{p}}\widetilde{f}(t,\alpha)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} = \left(\sum_{k=-\infty}^{\infty}\int\limits_{2^k}^{2^{k+1}}\left(t^{\frac{1}{p}}\widetilde{f}(t,\alpha)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}.$$

The necessary equivalence follows from

$$\left(2^{\frac{k+1}{p}}\widetilde{f}(2^{k+1},\alpha)\right)^q 2^{-\frac{q}{p}}\ln 2 \le \int_{2^k}^{2^{k+1}}\left(t^{\frac{1}{p}}\widetilde{f}(t,\alpha)\right)^q\frac{dt}{t} \le \left(2^{\frac{k}{p}}\widetilde{f}(2^k,\alpha)\right)^q 2^{\frac{q}{p}}\ln 2.$$

Proof of theorem 1. By the embeddings $N_{p_i,q_i;\alpha}\hookrightarrow N_{p_i,\infty;\alpha}$, $i=0,1$ (see lemma 1) it is enough to prove

$$(N_{p_0,\infty;\alpha},N_{p_1,\infty;\alpha})_{\theta,q}\hookrightarrow N_{p,q;\alpha}.$$

Let $t\in(0,\infty)$, $s\in[0,\infty]$, $f\in(N_{p_0,\infty;\alpha},N_{p_1,\infty;\alpha})_{\theta,q}$ such that $f=f_0+f_1$ is any decomposition with $f_i\in N_{p_i,\infty;\alpha}$, $(i=0,1)$. Then

$$\widetilde{f}(s,\alpha) = \sup_{y\ge s}\frac{1}{y^\alpha}\left|\int_0^y\frac{f(t)}{(y-t)^{1-\alpha}}dt\right| = \sup_{y\ge s}\frac{1}{y^\alpha}\left|\int_0^y\frac{f_0(t)+f_1(t)}{(y-t)^{1-\alpha}}dt\right|$$
$$\le \sup_{y\ge s}\frac{1}{y^\alpha}\left|\int_0^y\frac{f_0(t)}{(y-t)^{1-\alpha}}dt\right| + \sup_{y\ge s}\frac{1}{y^\alpha}\left|\int_0^y\frac{f_1(t)}{(y-t)^{1-\alpha}}dt\right|$$
$$= \widetilde{f_0}(s,\alpha)+\widetilde{f_1}(s,\alpha).$$

Denote $v(t) = t^{\frac{p_1 p_0}{p_1 - p_0}}$, $t \in (0,\infty)$. We have

$$\begin{aligned}
\sup_{v(t)\ge s} s^{\frac{1}{p_0}} \widetilde{f}(s,\alpha) &\le \sup_{v(t)\ge s} s^{\frac{1}{p_0}} \widetilde{f}_0(s,\alpha) + \sup_{v(t)\ge s} s^{\frac{1}{p_0}} \widetilde{f}_1(s,\alpha) \\
&= \sup_{v(t)\ge s} s^{\frac{1}{p_0}} \widetilde{f}_0(s,\alpha) + \sup_{v(t)\ge s} s^{\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} \widetilde{f}_1(s,\alpha) s^{\frac{1}{p_1}} \\
&\le \sup_{v(t)\ge s} s^{\frac{1}{p_0}} \widetilde{f}_0(s,\alpha) + \sup_{v(t)\ge s} (v(t))^{\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} \widetilde{f}_1(s,\alpha) s^{\frac{1}{p_1}} \\
&= \sup_{v(t)\ge s} s^{\frac{1}{p_0}} \widetilde{f}_0(s,\alpha) + t \sup_{v(t)\ge s} s^{\frac{1}{p_1}} \widetilde{f}_1(s,\alpha) \\
&\le \sup_{s\in[0,1]} s^{\frac{1}{p_0}} \widetilde{f}_0(s,\alpha) + t \sup_{s\in[0,1]} s^{\frac{1}{p_1}} \widetilde{f}_1(s,\alpha).
\end{aligned}$$

Hence,

$$\sup_{v(t)\ge s} s^{\frac{1}{p_0}} \widetilde{f}(s,\alpha) \le K(t,f;N_{p_0,\infty;\alpha},N_{p_1,\infty;\alpha}).$$

Thus, for $0 < q \le \infty$ we get

$$\begin{aligned}
\|f\|_{\theta,q}^q &:= \|f\|^q_{(N_{p_0,\infty;\alpha},N_{p_1,\infty;\alpha})_{\theta,q}} = \int_0^\infty \left(t^{-\theta} K(t,f;N_{p_0,\infty;\alpha},N_{p_1,\infty;\alpha})\right)^q \frac{dt}{t} \\
&\ge \int_0^\infty \left(t^{-\theta} \sup_{v(t)\ge s} s^{\frac{1}{p_0}} \widetilde{f}(s,\alpha)\right)^q \frac{dt}{t} \\
&= C \int_0^\infty \left(y^{-\theta\frac{p_1-p_0}{p_1p_0}} \sup_{y\ge s\ge 0} s^{\frac{1}{p_0}} \widetilde{f}(s,\alpha)\right)^q \frac{dy}{y} \\
&= C \sum_{r=-\infty}^{\infty} \int_{2^{-(r+1)}}^{2^{-r}} \left(y^{-\theta\frac{p_1-p_0}{p_1p_0}} \sup_{y\ge s\ge 0} s^{\frac{1}{p_0}} \widetilde{f}(s,\alpha)\right)^q \frac{dy}{y}.
\end{aligned}$$

By the monotonicity of the integrand we get

$$\begin{aligned}
\|f\|_{\theta,q}^q &\ge C \sum_{r=-\infty}^{\infty} \left(2^{r\theta\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} \sup_{2^{-(r+1)}\ge s\ge 0} s^{\frac{1}{p_0}} \widetilde{f}(s,\alpha)\right)^q \int_{2^{-(r+1)}}^{2^{-r}} \frac{dy}{y} \\
&= C \sum_{r=-\infty}^{\infty} \left(2^{r\theta\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} \sup_{2^{-(r+1)}\ge s\ge 0} s^{\frac{1}{p_0}} \widetilde{f}(s,\alpha)\right)^q \\
&= C \sum_{r=-\infty}^{\infty} \left(2^{-r\left(\frac{(1-\theta)}{p_0}+\frac{\theta}{p_1}\right)} 2^{\frac{r}{p_0}} \sup_{2^{-(r+1)}\ge s\ge 0} s^{\frac{1}{p_0}} \widetilde{f}(s,\alpha)\right)^q.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|f\|_{\theta,q}^q &\ge C \sum_{r=-\infty}^{\infty} \left(2^{-\frac{r}{p}} 2^{\frac{r}{p_0}} 2^{-\frac{r+1}{p_0}} \widetilde{f}(2^{-(r+1)},\alpha)\right)^q = C \sum_{r=-\infty}^{\infty} \left(2^{-\frac{r}{p}} 2^{-\frac{1}{p_0}} \widetilde{f}(2^{-(r+1)},\alpha)\right)^q \\
&\ge C \sum_{r=-\infty}^{\infty} \left(2^{-\frac{r}{p}} 2^{-\frac{1}{p_0}} \widetilde{f}(2^{-r},\alpha)\right)^q = C \sum_{r=-\infty}^{\infty} \left(2^{-\frac{r}{p}} \widetilde{f}(2^{-r},\alpha)\right)^q \ge C\|f\|^q_{N_{p,q;\alpha}}.
\end{aligned}$$

This completes the proof of theorem 1.

3. General net spaces and Fourier transform

Let f be a μ-measurable function on $\mathbb{R}$, then by f^* we denote of the nonincreasing rearrangement of f,

$$f^*(t) = \inf\{\sigma : \mu\{x \in \mathbb{R} : |f(x)| > \sigma\} \leq t\}.$$

Definition 3 *Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then Lorentz space $L_{p,q}(\mathbb{R})$ is the set of μ-measurable functions f for which, the functional*

$$\|f\|_{L_{p,q}} := \begin{cases} \left(\int\limits_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{for } 0 < p < \infty \text{ and } 0 < q < \infty, \\ \sup\limits_{t \geq 0} t^{\frac{1}{p}} f^*(t) & \text{for } 0 < p \leq \infty \text{ and } q = \infty. \end{cases}$$

is finite.

Theorem 2 *Let $\alpha \in (0,1]$, $p \in \left(1, \frac{1}{1-\alpha}\right]$, $0 < q \leq \infty$. Let also f be a mesurable function on $[0,\infty)$ with the cosine transform $\widehat{f}(t) = \int\limits_0^\infty f(x) \cos xt \, dx$. Then the following inequality*

$$\|\widehat{f}\|_{N_{p',q;\alpha}} \leq C\|f\|_{L_{p,q}} \tag{2}$$

holds, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Before proving theorem 2, we provide some auxiliary lemmas.

Lemma 3 (Hardy-Littlewood) *[6, p. 44] Let f and g are μ-measurable functions on $(0,\infty)$, then*

$$\int_0^\infty |f(x)g(x)|dx \leq \int_0^\infty f^*(t)g^*(t)dt. \tag{1}$$

Lemma 4 *Let $\alpha \in (0,1]$, $x, y \geq 0$. Then*

$$\left|\int_0^y \frac{\cos xs}{(y-s)^{1-\alpha}} ds\right| \leq C \min\left(y^\alpha, \frac{1}{x^\alpha}\right).$$

Proof. By elementary calculations we get

$$\left|\int_0^y \frac{\cos xs}{(y-s)^{1-\alpha}} ds\right| \leq \int_0^y \frac{1}{(y-s)^{1-\alpha}} ds = \frac{y^\alpha}{\alpha}.$$

On the other hand

$$\begin{aligned} \int_0^y \frac{\cos xs}{(y-s)^{1-\alpha}} ds &= \int_0^y \frac{\cos x(y-z)}{z^{1-\alpha}} dz \\ &= \cos xy \int_0^y \frac{\cos xz}{z^{1-\alpha}} dz + \sin xy \int_0^y \frac{\sin xz}{z^{1-\alpha}} dz \\ &= \cos xy \int_0^{xy} \frac{\cos t}{\left(\frac{t}{x}\right)^{1-\alpha}} \frac{dt}{x} + \sin xy \int_0^{xy} \frac{\sin t}{\left(\frac{t}{x}\right)^{1-\alpha}} \frac{dt}{x} \\ &= \frac{1}{x^\alpha} \left(\cos xy \int_0^{xy} \frac{\cos t}{t^{1-\alpha}} dt + \sin xy \int_0^{xy} \frac{\sin t}{t^{1-\alpha}} dt\right). \end{aligned}$$

Hence,

$$\left|\int_0^y \frac{\cos xs}{(y-s)^{1-\alpha}}ds\right| \le C\frac{1}{x^\alpha}.$$

Proof of theorem 2. Using lemma 4, we get

$$\begin{aligned}\|\widehat{f}\|_{N_{p',\infty;\alpha}} &= \sup_{t\ge 0} t^{\frac{1}{p'}}\widetilde{\widehat{f}}(t,\alpha) = \sup_{t\ge 0} t^{\frac{1}{p'}}\sup_{t\le y}\frac{1}{y^\alpha}\left|\int_0^y \frac{\widehat{f}(s)}{(y-s)^{1-\alpha}}ds\right| \\ &= \sup_{t\ge 0} t^{\frac{1}{p'}}\sup_{t\le y}\frac{1}{y^\alpha}\left|\int_0^y\int_0^\infty \frac{f(x)\cos xs}{(y-s)^{1-\alpha}}dxds\right| \\ &= \sup_{t\ge 0} t^{\frac{1}{p'}}\sup_{t\le y}\frac{1}{y^\alpha}\left|\int_0^\infty f(x)\int_0^y \frac{\cos xs}{(y-s)^{1-\alpha}}dsdx\right| \\ &\le \sup_{y\ge 0}\frac{1}{y^{\alpha-\frac{1}{p'}}}\left|\int_0^\infty f(x)\int_0^y \frac{\cos xs}{(y-s)^{1-\alpha}}dsdx\right| \\ &\le \sup_{y\ge 0}\frac{1}{y^{\alpha-\frac{1}{p'}}}\int_0^\infty |f(x)|\left|\int_0^y \frac{\cos xs}{(y-s)^{1-\alpha}}ds\right|dx \\ &\le C\sup_{y\ge 0}\frac{1}{y^{\alpha-\frac{1}{p'}}}\int_0^\infty |f(x)|\min\left(y^\alpha,\frac{1}{x^\alpha}\right)dx.\end{aligned}$$

Let $y>0$ and $\frac{1}{x}<y$. From $p\le\frac{1}{1-\alpha}$ it follows that $\alpha\ge\frac{1}{p'}$. Then

$$\begin{aligned}\frac{1}{y^{\alpha-\frac{1}{p'}}}\min\left(y^\alpha,\frac{1}{x^\alpha}\right) &= \frac{1}{y^{\alpha-\frac{1}{p'}}}\frac{1}{x^\alpha} = \frac{1}{(xy)^{\alpha-\frac{1}{p'}}}\frac{1}{x^{\frac{1}{p'}}} \\ &\le x^{-\frac{1}{p'}} = x^{\frac{1}{p}-1}.\end{aligned}$$

Now let $\frac{1}{x}\ge y$. Then

$$\frac{1}{y^{\alpha-\frac{1}{p'}}}\min\left(y^\alpha,\frac{1}{x^\alpha}\right) = \frac{1}{y^{\alpha-\frac{1}{p'}}}y^\alpha = y^{\frac{1}{p'}}\le x^{-\frac{1}{p'}}.$$

By the Hardy-Litllewood inequality (1) we obtain

$$\|\widehat{f}\|_{N_{p',\infty;\alpha}}\le C\int_0^\infty |f(x)|x^{\frac{1}{p}-1}dx\le C\|f\|_{L_{p,1}}.$$

Using the interpolation theorem for the Lorentz spaces $L_{p,q}$ (see [7, p. 113]) and theorem 1 we get inequality (2).

Remark 1 *Analogical estimates of norms of the Fourier transforms were obtained in [1, theorem 2] and [5, theorem 4].*

4. Conclusion

The obtained results demonstrate that the $N_{p,q;\alpha}$ spaces have properties which similar to properties of the net spaces. These results could be used in harmonic analysis. In particular, these results could be used for the studying of the properties of the Fourier transforms on functional spaces.

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References

[1] *Nursultanov E.D.* Setevye prostranstva i neravenstva tipa Hardy-Littlewooda // Mat. sbornik. - 1998. - V. 189. - №3. - P. 83-102.

[2] *Nursultanov E.D.* Setevye prostranstva i preobrazovanie Fourier // Dokl. RAN. - 1998. - V. 361. - №5. - P. 597-599.

[3] *Nursultanov E.D.* Interpolation properties of some anisotropic spaces and Hardy-Littlewood type inequalities // East J. Approx. - 1998. - V. 4. - №2. - P. 243-275.

[4] *Nursultanov E., Tikhonov S.* Net spaces and boundedness of integral operators // J. Geom. Anal. - 2011. - V. 21. - №4. - P. 950-981.

[5] *Dyachenko M., Kankenova A., Nursultanov E.* On Summability of Fourier Coefficients of Functions From Lebesgue Space // J. Math. Anal. Appl. - 2014. - V. 419. - №2. - P. 959-971.

[6] *Bennet C., Sharpley R.* Interpolation of Operators. - San Diego: Academic Press, 1988.

[7] *Berg Y., Lofstrom Y.* Interpolyacionnye prostranstva. Vvedenie. - Moskva: Mir, 1980. - 264 p.